International Journal of Analysis and Applications
ISSN 2291-8639
Volume 15, Number 1 (2017), 57-61
http://www.etamaths.com

ABOUT HEINZ MEAN INEQUALITIES

DAESHIK CHOI*

Abstract. We present some inequalities related to Heinz means. Among them, we will provide an inequality involving Heinz means and Heron means, which is reverse to the one found by Bhatia.

1. Introduction

Throughout the paper, $\mathcal{B}$ stands for the set of all bounded linear operators on a Hilbert space $\mathcal{H}$ and $\mathcal{B}^+$ denotes the subset of $\mathcal{B}$ consisting of positive invertible operators. For self-adjoint operators A, B in $\mathcal{B}$, $A \geq B$ implies that $A - B$ is positive semidefinite.

For $0 \leq v \leq 1$, the Heinz mean $H_v(a, b)$ of positive numbers a, b is defined by

$$H_v(a,b) = \frac{1}{2}(a^{1-v}b^v + a^v b^{1-v}).$$

It is easy to see that $H_v(a, b)$, as a function of v, attains its minimum at $v = 1/2$ and its maximum at $v = 0$ or $v = 1$. Thus

$$\sqrt{ab} \leq H_v(a,b) \leq \frac{a+b}{2} \tag{1.1}$$

holds for all $\leq v \leq 1$.

For $A, B \in \mathcal{B}^+$ and $0 \leq v \leq 1$, the v-weighted arithmetic mean $A\nabla_v B$ and geometric mean $A\sharp_v B$ are defined, respectively, by

$$\begin{aligned} A\nabla_v B &= (1-v)A + vB, \\ A\sharp_v B &= A^{1/2}(A^{-1/2}BA^{-1/2})^v A^{1/2}. \end{aligned}$$

For convenience of notation, we write $A\nabla_{1/2}B$ as $A\nabla B$ and $A\sharp_{1/2}B$ as $A\sharp B$. The Heinz operator mean of $A, B \in \mathcal{B}^+$ is defined by

$$H_v(A,B) = \frac{1}{2}(A\sharp_v B + A\sharp_{1-v} B)$$

for $0 \leq v \leq 1$. The operator mean inequalities corresponding to (1.1) are

$$A\sharp_v B \leq H_v(A,B) \leq A\nabla_v B, \tag{1.2}$$

which are easily derived by the operator monotonicity of continuous functions, which states that if f is a real valued continuous function defined on the spectrum of a self-adjoint operator A, then $f(t) \geq 0$ for every t in the spectrum of A implies that $f(A)$ is a positive operator. We refer to [2–4] for more results related to Heinz inequalities.

Using the Taylor series of hyperbolic functions, Bhatia [1] and Liang and Shi [5,6] derived interesting Heinz operator inequalities. In this paper, we will improve their results using a simple but useful lemma. In particular, we note the following inequality [1]:

$$H_v(a,b) \leq F_{(2v-1)^2}(a,b),\ \forall a,b > 0,\ 0 \leq v \leq 1, \tag{1.3}$$

where

$$F_\alpha(a,b) = (1-\alpha)\sqrt{ab} + \alpha\frac{a+b}{2}$$

Received 8th May, 2017; accepted 11th July, 2017; published 1st September, 2017.
2010 *Mathematics Subject Classification.* 47A63.
Key words and phrases. Heinz mean; Heron mean; positive definite operators; operator inequalities.

are Heron means of a, b. As mentioned in [1], there is no inequality reverse to (1.3) in the sense that

$$F_\alpha(a,b) \le H_v(a,b)$$

for all $a, b > 0$, $0 < \alpha < 1$, and $0 < v < \frac{1}{2}$. However, we will present a kind of reverse inequality to (1.3) (see Theorem 2.2 or (2.9)).

2. Improvements of Heinz means

The following is the main lemma in this paper.

Lemma 2.1. *For $c > 1$ and $\rho \in \mathbb{R}$, we define φ by*

$$\varphi(x) = \frac{c^x - c^{-x}}{x^\rho}$$

for $x > 0$. Then,

1. *if $\rho \le 1$, φ is increasing on $(0, \infty)$, and*
2. *if $\rho > 1$, then there exists $x_\rho > 0$ such that φ is decreasing on $(0, x_\rho)$ and increasing on (x_ρ, ∞). If $t = t_\rho$ is the root of the equation $\frac{t+1}{2(t-1)} \ln t = \rho$, then $x_\rho = \frac{\ln t_\rho}{2 \ln c}$ and*

$$\min_{x>0} \varphi(x) = \varphi(x_\rho) = \left(\frac{\ln c}{\rho} \cdot \frac{t_\rho + 1}{t_\rho - 1}\right)^\rho (t_\rho^{1/2} - t_\rho^{-1/2}).$$

Proof. Let $f(t) = \frac{t+1}{2(t-1)} \ln t$ for $t > 1$. By direct computation, we have

$$\begin{aligned} x^{\rho+1} c^x \varphi'(x) &= x(c^{2x}+1)\ln c - \rho(c^{2x} - 1), \\ &= \frac{1}{2}(s+1)\ln s - \rho(s-1), \\ &= (s-1)(f(s) - \rho), \end{aligned} \tag{2.1}$$

where $s = c^{2x} > 1$. Simple algebra shows that $\lim_{t\to 1} f(t) = 1$ and that f is strictly increasing on $(1, \infty)$. Thus if $\rho \le 1$,

$$x^{\rho+1} c^x \varphi'(x) = (s-1)(f(s) - \rho) > (s-1)(1-\rho) \ge 0$$

for any $x > 0$, which implies that φ is increasing on $(0, \infty)$. Meanwhile, if $\rho > 1$, let $t_\rho > 1$ be the (unique) zero of $f(t) = \rho$. Then for $x_\rho = \frac{\ln t_\rho}{2 \ln c}$, (2.1) says that $\varphi'(x) < 0$ on $(0, x_\rho)$, $\varphi'(x) > 0$ on (x_ρ, ∞), and the minimum value of φ on $(0, \infty)$ is

$$\varphi(x_\rho) = \frac{c^{x_\rho} - c^{-x_\rho}}{x_\rho^\rho} = \left(\frac{2\ln c}{\ln t_\rho}\right)^\rho (t_\rho^{1/2} - t_\rho^{-1/2}) = \left(\frac{\ln c}{\rho} \cdot \frac{t_\rho+1}{t_\rho - 1}\right)^\rho (t_\rho^{1/2} - t_\rho^{-1/2}),$$

where the last equation follows from $\frac{2}{\ln t_\rho} = \frac{t_\rho + 1}{\rho(t_\rho - 1)}$. □

Heinz means $H_v(a,b)$ or $H_v(A,B)$ were defined for $0 \le v \le 1$, but we don't restrict v to be in the interval $[0,1]$ in the following theorem. Lemma 2.1 with $\rho = 1$ is used below.

Theorem 2.1. *For $A, B \in \mathcal{B}^+$ and $r, s, t \in \mathbb{R}$ with $0 < |1-2r| \le |1-2s| \le |1-2t|$, we have*

$$\begin{aligned} H_s(A,B) &\ge \left(1 - \frac{(1-2s)^2}{(1-2r)^2}\right) A\sharp B + \frac{(1-2s)^2}{(1-2r)^2} H_r(A,B), \\ H_s(A,B) &\le \left(1 - \frac{(1-2s)^2}{(1-2t)^2}\right) A\sharp B + \frac{(1-2s)^2}{(1-2t)^2} H_t(A,B). \end{aligned}$$

Proof. Let $a, b > 0$ and

$$f(x) = \begin{cases} \left(H_{(1-x)/2}(a,b) - \sqrt{ab}\right)/x^2, & x \in \mathbb{R}\backslash\{0\} \\ \frac{1}{8}\left(\ln \frac{a}{b}\right)^2 \sqrt{ab}, & x = 0 \end{cases}.$$

Letting $c = (ab^{-1})^{1/4}$, we have

$$f(x) = \frac{\sqrt{ab}}{2} \cdot \frac{c^{2x} + c^{-2x} - 2}{x^2} = \frac{\sqrt{ab}}{2}\left(\frac{c^x - c^{-x}}{x}\right)^2.$$

Without loss of generality, we assume $c > 1$. Since f is even on $(-\infty, \infty)$ and increasing on $(0, \infty)$ by Lemma 2.1, we have

$$f(1-2r) \le f(1-2s) \le f(1-2t)$$

for $r, s, t \in \mathbb{R}$ with $0 < |1-2r| \le |1-2s| \le |1-2t|$, which can be written as

$$\frac{H_r(a,b) - \sqrt{ab}}{(1-2r)^2} \le \frac{H_s(a,b) - \sqrt{ab}}{(1-2s)^2} \le \frac{H_t(a,b) - \sqrt{ab}}{(1-2t)^2}$$

or equivalently,

$$\begin{aligned} H_s(a,b) &\ge \left(1 - \frac{(1-2s)^2}{(1-2r)^2}\right)\sqrt{ab} + \frac{(1-2s)^2}{(1-2r)^2} H_r(a,b), \\ H_s(a,b) &\le \left(1 - \frac{(1-2s)^2}{(1-2t)^2}\right)\sqrt{ab} + \frac{(1-2s)^2}{(1-2t)^2} H_t(a,b). \end{aligned}$$

By the operator monotonicity of continuous functions, we get the desired operator inequalities. □

Remark 2.1. *The second inequality in Theorem 2.1 is shown in [5, Theorem 2.1] with* $0 \le s, t \le 1$.

Now we use Lemma 2.1 with $\rho \ge 1$ below.

Theorem 2.2. *For* $A, B \in \mathcal{B}^+$ *and* $0 \le s \le 1$*, we have*

$$H_s(A,B) \le \left(1 - (1-2s)^2\right) A \sharp B + (1-2s)^2 A \nabla B. \tag{2.2}$$

For $\rho > 1$*, let* $t_\rho > 1$ *be the root of the equation* $\frac{t+1}{2(t-1)} \ln t = \rho$.

(1) *If* $A > t_\rho^2 B$ *or* $B > t_\rho^2 A$*, then*

$$H_s(A,B) \ge \left(1 + \alpha_\rho |1-2s|^{2\rho} (2 \ln t_\rho)^{2\rho}\right) A \sharp B \tag{2.3}$$

where

$$\alpha_\rho = \left(\frac{t_\rho + 1}{4\rho(t_\rho - 1)}\right)^{2\rho} \left(\frac{t_\rho + t_\rho^{-1}}{2} - 1\right).$$

(2) *If* $t_\rho^{-2} B \le A \le t_\rho^2 B$*, then*

$$H_s(A,B) \ge \left(1 - |1-2s|^{2\rho}\right) A \sharp B + |1-2s|^{2\rho} A \nabla B. \tag{2.4}$$

Proof. First, we will show the following:

$$H_{(1-x)/2}(a,b) \le \left(1 - x^2\right)\sqrt{ab} + x^2 \frac{a+b}{2}, \tag{2.5}$$

$$H_{(1-x)/2}(a,b) \ge \begin{cases} \sqrt{ab}\left(1 + \alpha_\rho |x|^{2\rho} |\ln a - \ln b|^{2\rho}\right), & \text{if } t_\rho < \mu_{a,b} \\ \left(1 - |x|^{2\rho}\right)\sqrt{ab} + |x|^{2\rho} \frac{a+b}{2}, & \text{if } t_\rho \ge \mu_{a,b} \end{cases} \tag{2.6}$$

for $-1 \le x \le 1$, where $\mu_{a,b} = \max\left\{\sqrt{\frac{a}{b}}, \sqrt{\frac{b}{a}}\right\}$. Since $H_{(1-x)/2}(a,b) = H_{(1+x)/2}(a,b)$ and $H_{1/2}(a,b) = \sqrt{ab}$, we may assume $x > 0$. For $\rho \ge 1$, define f_ρ by

$$f_\rho(x) = \frac{H_{(1-x)/2}(a,b) - \sqrt{ab}}{x^{2\rho}}$$

for $x > 0$. Letting $c = (ab^{-1})^{1/4}$, we have

$$f_\rho(x) = \frac{\sqrt{ab}}{2} \cdot \frac{c^{2x} + c^{-2x} - 2}{x^{2\rho}} = \frac{\sqrt{ab}}{2} \left(\frac{c^x - c^{-x}}{x^\rho}\right)^2.$$

Without loss of generality, we assume $c > 1$. By Lemma 2.1,

$$\begin{aligned} f_1(x) &\le \frac{\sqrt{ab}}{2}\left(c - c^{-1}\right)^2 \\ &= \frac{\sqrt{ab}}{2}\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} - 2\right) = \frac{a+b}{2} - \sqrt{ab}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} f_\rho(x) &\ge \frac{\sqrt{ab}}{2}(\ln c)^{2\rho}\left(\frac{t_\rho + 1}{\rho(t_\rho - 1)}\right)^{2\rho}(t_\rho + t_\rho^{-1} - 2) \\ &= \sqrt{ab}(\ln a - \ln b)^{2\rho}\alpha_\rho \end{aligned} \tag{2.8}$$

for $\rho > 1$. (2.5) follows from (2.7). Using the same notation as in Lemma 2.1, we know

$$\varphi(x) = \frac{c^x - c^{-x}}{x^\rho} \ge \varphi(x_\rho)$$

for all $x > 0$. Here we consider x with $|x| \le 1$. Then we can bound φ as follows:

$$\varphi(x) \ge \begin{cases} \varphi(x_\rho), & \text{if } x_\rho < 1 \\ \varphi(1), & \text{if } x_\rho \ge 1 \end{cases}.$$

Since

$$x_\rho < 1 \iff t_\rho < c^2 = \sqrt{\frac{a}{b}}$$

and $f_\rho(1) = \frac{a+b}{2} - \sqrt{ab}$, we can improve (2.8) as

$$f_\rho(x) \ge \begin{cases} \sqrt{ab}(\ln a - \ln b)^{2\rho}\alpha_\rho, & \text{if } t_\rho < \sqrt{\frac{a}{b}} \\ \frac{a+b}{2} - \sqrt{ab}, & \text{if } t_\rho \ge \sqrt{\frac{b}{a}} \end{cases}$$

which implies (2.6).

We get (2.2) from (2.5) by the operator monotonicity of continuous functions. Meanwhile, since

$$t_\rho < \mu_{a,b} \iff a > t_\rho^2 b \text{ or } b > t_\rho^2 a,$$

if $t_\rho < \mu_{a,b}$, then $|\ln a - \ln b| \ge 2\ln t_\rho$ and

$$H_s(a,b) \ge \left(1 + \alpha_\rho|1-2s|^{2\rho}|2\ln t_\rho|^{2\rho}\right)\sqrt{ab}$$

from the first inequality of (2.6). On the other hand, if $t_\rho \ge \mu_{a,b}$, that is , if $a \le t_\rho^2 b$ and $b \le t_\rho^2 a$, then

$$H_s(a,b) \ge \left(1 - |1-2s|^{2\rho}\right)\sqrt{ab} + |1-2s|^{2\rho}\frac{a+b}{2}$$

from the second inequality of (2.6). Finally, (2.3) and (2.4) follow from the operator monotonicity of continuous functions. □

Remark 2.2. *In the proof of Theorem 2.2, we showed that*

$$H_s(a,b) \ge \begin{cases} \left(1 + \alpha_\rho|1-2s|^{2\rho}|\ln a - \ln b|^{2\rho}\right)\sqrt{ab}, & \text{if } t_\rho < \mu_{a,b} \\ \left(1 - |1-2s|^{2\rho}\right)\sqrt{ab} + |1-2s|^{2\rho}\frac{a+b}{2}, & \text{if } t_\rho \ge \mu_{a,b} \end{cases} \tag{2.9}$$

for any $\rho > 1$. The above inequality improves the known relation $H_s(a,b) \ge \sqrt{ab}$ considerably. Note that the minimum value of the right hand side of (2.9), as a function in s, is $\sqrt{ab}$ (when $s = 1/2$). Figure 1 shows the graphs of the both sides of (2.9) as functions in $s \in [0,1]$ for some values of a, b, where $\rho = 1.1$ and $t_\rho = 3.0237$.

The following corollary also improves the Heinz mean - Geometric mean inequality:

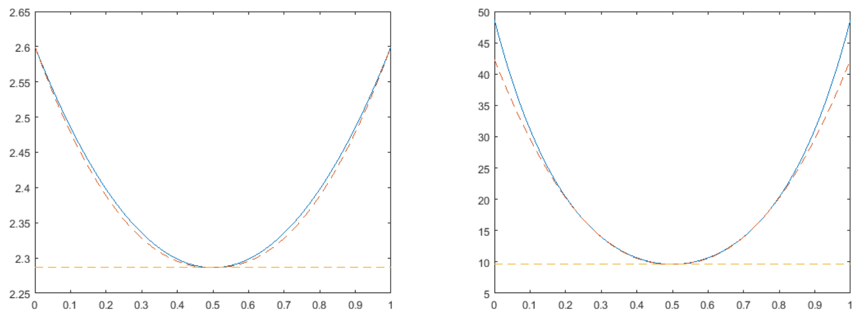
$$H_s(a,b) \ge \sqrt{ab},\ a,b > 0$$

and

$$H_s(A,B) \ge A\sharp B,\ A,B \in \mathcal{B}^+$$

under a condition.

FIGURE 1. The graphs of $H_s(a,b)$ (solid curves) and the right hand side (dotted lines) of (2.9) as functions in $s \in [0,1]$, where $\rho = 1.1$ and $t_\rho = 3.0237$; $a = 3.8390, b = 1.3615, \mu_{a,b} = 1.6792$ on the left figure and $a = 0.9575, b = 96.4889, \mu_{a,b} = 10.0385$ on the right figure. The horizontal dotted lines denote $\sqrt{ab}$ which is the minimum value of the two functions.



Corollary 2.1. *For $0 \le s \le 1$ and $a, b > 0$, we have*

$$H_s(a,b) \ge \left(1 + \frac{1}{8}(1-2s)^2(\ln \frac{a}{b})^2\right)\sqrt{ab}. \tag{2.10}$$

For $0 \le s \le 1$ and $A, B \in \mathcal{B}^+$ with either $B \ge \alpha A$ or $A \ge \alpha B$ for a real number $\alpha \ge 1$, we have

$$H_s(A,B) \ge \left(1 + \frac{1}{8}(1-2s)^2(\ln \alpha)^2\right) A\sharp B. \tag{2.11}$$

Proof. It is easily shown that $\alpha_\rho \to \frac{1}{8}$ and $t_\rho \to 1$ as $\rho \to 1$. Thus (2.10) follows from the first inequality of (2.9).

To show (2.11), it suffices to consider the case $B \ge \alpha A$, since $H_s(A,B) = H_s(B,A)$ and $A\sharp B = B\sharp A$. Letting $a = 1$ and assuming $b \ge \alpha \ge 1$ in (2.10), we get

$$\frac{1}{2}(b^s + b^{1-s}) \ge \left(1 + \frac{1}{8}(1-2s)^2(\ln \alpha)^2\right)\sqrt{b}. \tag{2.12}$$

Thus if $B \ge \alpha A$, then for $X = A^{-1/2}BA^{-1/2}$ we have

$$\frac{1}{2}(X^s + X^{1-s}) \ge \left(1 + \frac{1}{8}(1-2s)^2(\ln \alpha)^2\right)X^{1/2}$$

from (2.12). Multiplying each side of the above inequality by $A^{1/2}$ on its left- and right-hand sides, we get (2.11). □

REFERENCES

[1] R. Bhatia, Interpolating the arithmetic–geometric mean inequality and its operator version, Lin. Alg. and its Appl. 413 (2006) 355–363.
[2] R. Bhatia, C. Davis, More matrix forms of the arithmetic–geometric mean inequality, SIAM J. Matrix Anal. Appl. 14 (1993) 132–136.
[3] F. Kittaneh, M. Krnic, N. Lovricevic, and J. Pecaric, Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators, Publ. Math. Debrecen. 80 (3-4) (2012), 465– 478.
[4] F. Kittaneh and M. Krnic, Refined Heinz operator inequalities, Linear Multilinear Algebra, 61 (8) (2013), 1148–1157.
[5] J. Liang and G. Shi, Refinements of the Heinz operator inequalities, Linear Multilinear Algebra, 63 (7) (2015), 1337–1344.
[6] J.Liang and G. Shi, Some means inequalities for positive operators in Hilbert spaces, J. Inequal. Appl. 2017 (2017), Art. ID 14.

DEPT. OF MATHEMATICS AND STATISTICS, SOUTHERN ILLINOIS UNIVERSITY EDWARDSVILLE, BOX 1653, EDWARDSVILLE, IL 62026, USA

*CORRESPONDING AUTHOR: math.dchoi@gmail.com