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OPTIMALITY AND DUALITY DEFINED BY THE CONCEPT OF TEMPERED FRACTIONAL UNIVEX FUNCTIONS IN MULTI-OBJECTIVE OPTIMIZATION

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Abstract. In this paper, we purpose the concept of tempered Univex functions by utilizing a tempered fractional difference-differential operator type Caputo. This instruction indicates a new class of these functions in some optimal problems by exemplifying the settings on the modified formula. We call it the class of tempered fractional Univex functions. Our study is based on the strong, weak, converse, and strict converse duality propositions. A Multi-objective optimal problem includes the new process is disentangled.

1. Introduction

In 1989, Dunkl imposed a difference-differential operator [1] setting on some Euclidean space and realizing the commutative law for a differentiable function on $\mathbb{R}^n$. This operator can be employed in various parts in pure mathematics, such as Lee algebra, Clifford algebra and complex analysis. In 1998, Rosler and Voit acquired into consideration this operator to adapt the tool of the Markov Processes [2]. Unevenly, these operators can be expected as a simplification of the partial derivatives and various constructions of operators like the Laplace operator, the Fourier transform, and the Hermite polynomials. Also, these operators convoluted in famous processes such as the Brownian motion and the Cauchy processes.

Recently, this operator and its some simplifications have improved significant care in many fields of mathematics and physics. They shield a helpful method in the study of special functions and they are closely related with definite demonstrations of degenerate affine Hecke algebras. Furthermore, Dunkl operator is obviously convoluted in the algebraic explanation of definite devotedly resolvable quantum multi-body systems. It can be used to identify the generalized method of the heat equation, which is called the Dunkl heat equation. It can be recommended to adapt the idea of moments of probability measures on $\mathbb{R}^n$. Our goal is to generalize the Dunkl operator in view of the tempered fractional calculus and propose it to simplify the class of non-linear Univex functions. This class typically appears in many non-linear multi-objective problems. The benefit of exploiting the Dunkle operator is that this operator deals with multi-dimensional spaces. Moreover, the author extend it to the complex plane and provided a modified differential-difference Dunkl operator in the open unit disk. The study was in the field of geometric function theory [3].

Fractional calculus is the most important branch of mathematical analysis, because it refers to the non-linearity studies in all science. The most famous operators are the Riemann-Liouville, Caputo (continuous operators) and Grunwald-Letnikov (discrete operator) (see [4], [5]). It has been presented the fractional calculus discoveries usage in many categories of science and engineering, containing fluid flow, diffusive transport theory, electrical networks, electromagnetic theory, probability and statistics. The tempered fractional diffusion idea was established in statistics. This idea has demonstrated useful applications in geophysics and finance [6]. Moreover, it applied to introduce a fractional multi-objective function in optimal control [7].

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In this study, we aim to generalize the concept of tempered Univex functions by utilizing a tempered fractional differential-difference operator (Dunkl operator), based on different types of fractional calculus. This study gives us a new class of these functions in some optimal problems by illustrating conditions on the generalized functions. We call it the class of tempered fractional Univex functions. The strong, weak, converse, and strict converse duality theorems are proposed. The main tool employed in the analysis is based on the tempered Caputo operator.

2. Handling

We need the following concepts in the sequel of the article:

2.1. **Dunkl operator.** Suppose the two column vectors $x = (x_1, ..., x_n)$ and $v = (v_1, ..., v_n) \in \mathbb{R}^n$ with their dot product $x.v = x^T.v = \sum_{i=1}^n v_i x_i$. The operator through the hyper-plane is defined by

$$\sigma_v\, x \ = x - 2\,\frac{v.x}{v.v} v.$$

In a matrix form, we have

$$\sigma_v \ = \ I - 2\,\frac{vv^T}{v^T v}.$$

Note that σ_v can be represented by a symmetric matrix. The Dunkl operator is formulated by:

$$\mathfrak{D}_i \phi(x) = \frac{\partial}{\partial x_i}\phi(x) + \sum_{v\in R_+} k_v \frac{(1-\sigma_v)\phi(x)}{v.x} v_i, \quad i = 1, ..., n,$$

where v_i is the i-th component of $v, 1 \le i \le n$, $x \in \mathbb{R}^n$. , and ϕ smooth function on $\mathbb{R}^n$. When $k_v = 0$, then we have

$$\mathfrak{D}_i \phi(x) = \frac{\partial}{\partial x_i}\phi(x).$$

One of the outcomes of these operators is satisfying

$$\mathfrak{D}_i(\mathfrak{D}_j \phi(x)) = \mathfrak{D}_j(\mathfrak{D}_i \phi(x)). \tag{2.1}$$

Moreover, the operator achieves the product

$$\mathfrak{D}_i[\phi(x)\psi(x)] \ = \psi(x)\mathfrak{D}_i\phi(x) + \phi(x)\mathfrak{D}_i\psi(x).$$

Note that, if ϕ is a polynomial of degree n, then $(1-\sigma_v)\phi(x)/v.x$ is a polynomial of degree $n-1$. Moreover, the path of the Dunkl process onto a subset of $\mathbb{R}^n$ is collected by the set

$$\mathfrak{S} = \{x \in \mathbb{R}^n : x.v > 0, \quad \forall\, v \in R_+\}.$$

Finally, Dunkl processes are formulated as the Markov processes which achieve the Dunkl heat equation

$$\frac{\partial}{\partial t} - \frac{1}{2}\sum_{i=1}^{n} \mathfrak{D}_i^2 = 0.$$

2.2. **Fractional calculus.** The Cauchy formula for frequent integration, to be specific as follows:

$$(I^n\phi)(\chi) = \frac{1}{(n-1)!}\int_0^{\chi} (\chi - t)^{n-1}\phi(t)\, dt,$$

drives in an explicit way to a generalization for real n. Utilizing the gamma function to take off the discrete nature of the factorial function allows us a natural candidate for fractional usage of the integral operator.

$$(I^\alpha\phi)(\chi) = \frac{1}{\Gamma(\alpha)}\int_0^{\chi} (\chi - t)^{\alpha-1}\phi(t)\, dt$$

This operator is well-defined and it is represented to the classical fractional calculus, which is called the Riemann-Liouville fractional integral operator. It is straightforward to show that the integral operator achieves the semi-group property of fractional differ-integral operators

$$(I^\alpha)(I^\beta\phi)(\chi) = (I^\beta)(I^\alpha\phi)(\chi) = (I^{\alpha+\beta}\phi)(\chi) = \frac{1}{\Gamma(\alpha+\beta)}\int_0^{\chi} (\chi - t)^{\alpha+\beta-1}\phi(t)\, dt.$$

Corresponding to the above fractional integral operator and for a general function $\phi(\chi)$ and $0 < \alpha < 1$, the complete fractional derivative is defined as follows:

$$D^{\alpha}\phi(\chi) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\chi}\int_{0}^{\chi}\frac{\phi(t)}{(\chi-t)^{\alpha}}\,dt.$$

For the fractional power $\alpha < 1$, since the gamma function is sloppy for values whose real part is a negative integer with the imaginary part is equal to zero, it is important to employ the fractional derivative after the integer derivative has been accurate. For example,

$$D^{5/4}\phi(\chi) = D^{1/4}\Big(D^{1}\phi(\chi)\Big) = D^{1/4}\Big(\frac{d}{d\chi}\phi(\chi)\Big).$$

In general, calculating n-th order derivative over the integral of order $(n-\alpha)$, is given by the formula

$${}_{a}D_{\chi}^{\alpha}\phi(\chi) = \frac{d^{n}}{d\chi^{n}}\,{}_{a}D_{\chi}^{-(n-\alpha)}\phi(\chi) = \frac{d^{n}}{d\chi^{n}}\,{}_{a}I_{\chi}^{n-\alpha}\phi(\chi).$$

The Riemann-Liouville calculus admits a fast converge, historical property, natural generalization and wide applications in almost all science. One of the most property of this calculus is as follows:

$$D^{\alpha}\chi^{m} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}\chi^{m-\alpha}, \qquad m \geq 0.$$

There is another fractional differential operator called the Caputo operator, which is defined as follows:

$${}_{a}^{C}D_{\chi}^{\alpha}\phi(\chi) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{\chi}\frac{\phi^{(n)}(\tau)\,d\tau}{(\chi-\tau)^{\alpha+1-n}}.$$

This type of fractional differential operator is applied to find the solutions of fractional differential equations with initial conditions.

2.2.1. *Tempered fractional calculus.* The Riemann-Liouville tempered fractional derivative is formed as follows [9]

$$\hat{D}^{\alpha,\lambda}\phi(\chi) = e^{-\lambda\chi}D^{\alpha}\big(e^{\lambda\chi}\phi(\chi)\big) - \lambda^{\alpha}\phi(\chi), \quad \lambda \geq 0.$$

And the The Caputo tempered fractional derivative is formed as follows

$${}_{a}^{C}\hat{D}^{\alpha,\lambda}\phi(\chi) = \hat{D}^{\alpha,\lambda}[\phi(\chi) - \sum_{i=0}^{n-1}\frac{\chi_{i}}{i!}\phi^{(i)}(0)], \quad \lambda \geq 0,$$

where $\phi^{(i)}$ is the derivative of order i, and n is the upper integer value less than α. When $\lambda = 0$, the above equation reduces to the usual formula of the Caputo operator.

2.3. **Tempered-Dunkl operator.** Based on the Caputo tempered fractional calculus, assume that the fractional partial derivative is denoted by ${}_{a}^{C}\hat{D}^{\alpha,\lambda}$. Then for $x \in \mathbb{R}^{n}$we receive the tempered Dunkl operator as follows:

$$\mathfrak{D}_{i}^{\alpha,\lambda}\phi(x) = {}_{a}^{C}\hat{D}_{i}^{\alpha,\lambda}\phi(x) + \sum_{i=1,\, v\in R_{+}}^{n} k_{v}\frac{(1-\sigma_{v})^{\alpha}\phi(x)}{v.x}v_{i}, \tag{2.2}$$

$$\Big(i = 1, ..., n, \quad 0 < \alpha < 1\Big),$$

where ${}_{a}^{C}\hat{D}_{i}^{\alpha,\lambda}$ is denoted the fractional derivative with respect the component x_{i}, R_{+} is a positive subsystem, satisfying for all $u \in R_{+}$, $u.v > 0$. Dunkl operators in the direction of $y \in \mathbb{R}^{n}$ is defined as follows:

$$\mathfrak{D}^{\alpha,\lambda}\phi(x) = \sum_{i=1}^{n} y_{i}\mathfrak{D}_{i}^{\alpha,\lambda}\phi(x).$$

Our aim is to include the generalized tempered Dunkl operator in a class of fractional stochastic differential equations and to study the behavior of the solutions. We have the following properties of the new operator:

Proposition 2.1 Let $\phi(x)$ be analytic function converging in the interval $(0,\rho]$ with the approximate form

$$\phi(x)=\sum_{m=0}^{\infty}a_m\,x^{m+\lambda},\quad \lambda>-1.$$

Then

$$\mathfrak{D}_i^{\alpha,\lambda}\Big(\mathfrak{D}_j^{\beta,\lambda}\phi(x)\Big)=\mathfrak{D}_j^{\beta,\lambda}\Big(\mathfrak{D}_i^{\alpha,\lambda}\phi(x)\Big),\quad \alpha,\beta\in(0,1),\ i=1,...,n.$$

Proof. In view of Theorem 3 in [3] and (2.1) , we have

$$\begin{aligned}\mathfrak{D}_i^{\alpha,\lambda}\Big(\mathfrak{D}_j^{\beta,\lambda}\phi(x)\Big) &= \mathfrak{D}_i^{\alpha,\lambda}\Big(\frac{\partial^{\beta}}{\partial x_j^{\beta}}\phi(x)+\sum_{v\in R_+}k_v\frac{(1-\sigma_v)^{\beta}\phi(x)}{v.x}v_j\Big)\\ &= \mathfrak{D}_j^{\beta,\lambda}\Big(\frac{\partial^{\alpha}}{\partial x_i^{\alpha}}\phi(x)+\sum_{v\in R_+}k_v\frac{(1-\sigma_v)^{\alpha}\phi(x)}{v.x}v_i\Big)\\ &= \mathfrak{D}_j^{\beta,\lambda}\Big(\mathfrak{D}_i^{\alpha,\lambda}\phi(x)\Big).\end{aligned}$$

Proposition 2.2 Let ϕ and ψ be power functions in x. Then

$$\mathfrak{D}_i^{\alpha}[\phi(x)\psi(x)]=\psi(x)\mathfrak{D}_i^{\alpha}\phi(x)+\phi(x)\mathfrak{D}_i^{\alpha}\psi(x).$$

Proof. By applying the fractional generalization of the Leibniz rule of the Caputo derivative [8]

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}[\phi(x)\psi(x)]=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)}\partial^{\alpha-k}\phi(x)\partial^{k}\psi(x),$$

we conclude the desire result.

2.4. **Tempered Univex function.** In this subsection, we generalize the concept of tempered Univex function, by using the fractional calculus. Let Ω be a nonempty subset of $\mathbb{R}^n,\eta:\Omega\times\Omega\to\mathbb{R}^m$, ξ be an arbitrary point of Ω and $h:\Omega\to\mathbb{R}^m,\phi:\mathbb{R}^m\to\mathbb{R}$.

Definition 2.1 A differential function h is said to be a tempered fractional univex function of order $\alpha\in(0,1)$ in the direction of $\xi\in\Omega$ if for all $x\in\Omega$, we have

$$\eta(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,h(x)\le\phi\Big(h(x)-h(\xi)\Big),$$

where

$$\mathfrak{D}^{\alpha,\lambda}\,h(x)=\sum_{i=1}^{n}\xi_i\mathfrak{D}_i^{\alpha,\lambda}h(x),\quad \xi=(\xi_1,...,\xi_n).$$

The advantage of using the tempered fractional Dunkl operator, is that can be acted on multi-dimensional Euclidean spaces as well as it can be defined a parametric family of deformations of the polynomial . Therefor, it can be employed in non-linear multi-objective problem

$$\begin{aligned}&Minimize\quad \Psi(x)=\big(\psi_1(x),...,\psi_m(x)\big)\\ &subject\,to\quad \Theta(x)\le 0,\end{aligned}\tag{2.3}$$

where $\Psi:\Omega\to\mathbb{R}^m$ and $\Theta:\Omega\to\mathbb{R}^p$ and 0 is the zero vector in $\mathbb{R}^p$. The function $\Psi(x)$ can be applied in various studies. It can be considered as a utility function over some set of needs (goods), cost function of production presented a fixed quantity produced, growth function and others.

Definition 2.2 A point $\xi\in\Lambda:=\{x\in\Omega:\Theta(x)\le 0\}$ is said to be an efficient solution of (2.3), if there exists no $x\in\Lambda$, such that $\Psi(x)\le\Psi(\xi)$. And it is called a weak efficient solution if $\Psi(x)<\Psi(\xi)$.

Next, we define a new class of fractional Univex function for the problem (2.3), we denote this class by : $(\alpha,\rho,\eta,\vartheta)$ as follows:

Definition 2.3 The couple (Ψ, Θ) is called $(\alpha, \lambda, \rho, \eta, \vartheta)$-type univex at $\xi \in \Omega$ if for all $x \in \Lambda$ such that
$$\eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(x) + \rho_1\|\vartheta(x,\xi)\|^2 \leq \phi_1\Big(\Psi(x) - \Psi(\xi)\Big)$$
and
$$\eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Theta(x) + \rho_2\|\vartheta(x,\xi)\|^2 \leq -\phi_2\Big(\Theta(x) - \Theta(\xi)\Big),$$
where $\eta_1 : \Omega \times \Omega \to \mathbb{R}^m$, $\eta_2 : \Omega \times \Omega \to \mathbb{R}^p$, $\vartheta : \Omega \times \Lambda \to \mathbb{R}$, $\phi_1 : \mathbb{R}^m \to \mathbb{R}$, $\phi_2 : \mathbb{R}^p \to \mathbb{R}$ and $\rho_1, \rho_2 \in \mathbb{R}$.

We have the following facts:

Remark 2.1

- If $\phi_1\Big(\Psi(x) - \Psi(\xi)\Big) \leq 0 \Rightarrow \eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(x) \leq -\rho_1\|\vartheta(x,\xi)\|^2$ and $\phi_2\Big(\Psi(x) - \Psi(\xi)\Big) \geq 0 \Rightarrow \eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(x) \leq -\rho_2\|\vartheta(x,\xi)\|^2$. Then the couple (Ψ, Θ) is called weak pseudo-quasi $(\alpha, \lambda, \rho, \eta, \vartheta)$-type univex at $\xi \in \Omega$.
- If $\phi_1\Big(\Psi(x) - \Psi(\xi)\Big) \leq 0 \Rightarrow \eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(x) < -\rho_1\|\vartheta(x,\xi)\|^2$ and $\phi_2\Big(\Psi(x) - \Psi(\xi)\Big) \geq 0 \Rightarrow \eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(x) < -\rho_2\|\vartheta(x,\xi)\|^2$. Then the couple (Ψ, Θ) is called strong pseudo-quasi $(\alpha, \lambda, \rho, \eta, \vartheta)$-type univex at $\xi \in \Omega$.

3. Results

In this section, we investigate some sufficient optimality conditions for a point to be an efficient solution of (2.3) under the tempered $(\alpha, \lambda, \rho, \eta, \vartheta)$-type Univex.

Theorem 3.1. *Let ξ be an initial solution of the multi-objective problem (2.3) and c_1 and c_2 be two non-negative constants such that*

(A) $\Theta(\xi) = 0$;

(B) $c_1\big(\eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(x)\big) + c_2\big(\eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Theta(x)\big) \geq 0$;

(C) The couple (Ψ, Θ) is a strong (or weak) pseudo-quasi $(\alpha, \lambda, \rho, \eta, \vartheta)$-type univex at $\xi \in \Omega$;

(D) $u \leq 0 \in \mathbb{R}^m \Rightarrow \phi_1(u) \leq 0$ *and* $v \geq 0 \in \mathbb{R}^p \Rightarrow \phi_2(v) \geq 0$;

(E) $c_1\rho_1 + c_2\rho_2 \geq 0$.

Then ξ is an efficient solution of (2.3).

Proof. Suppose that ξ is not an efficient solution of (2.3), then there exists $x \in \Lambda$ such that $\Psi(x) \leq \Psi(\xi)$. By the assumptions (A) and (D), we have
$$\phi_1(\Psi(x) - \Psi(\xi)) \leq 0, \quad \textit{and} \quad \phi_2(\Theta(\xi)) \geq 0. \tag{3.1}$$
In view of the assumption (C), we get
$$c_1\big(\eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(x)\big) < -c_1\rho_1\|\vartheta(x,\xi)\|^2 \tag{3.2}$$
and
$$c_2\big(\eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Theta(x)\big) \leq -c_2\rho_2\|\vartheta(x,\xi)\|^2. \tag{3.3}$$
Summing the above inequalities and utilizing (E), we conclude that
$$\begin{aligned} c_1\big(\eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(x)\big) + c_2\big(\eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Theta(x)\big) &< -\big(c_1\rho_1 + c_2\rho_2\big)\|\vartheta(x,\xi)\|^2 \\ &\leq 0, \end{aligned}$$
which contradicts the assumption (B). Hence, ξ is an efficient solution of (2.3). This completes the proof.

Theorem 3.2. *If the following conditions are satisfied:*

(A) ξ is a weakly efficient solution of (2.3);

(B) Θ is continuous in ξ;

(C) The functions Ψ and Θ are fractional tempered Univex functions of order $\alpha \in (0,1)$, $\lambda \geq 0$ in the direction of $\xi \in \Lambda$. Moreover, for some $\bar{x} \in \Lambda$, we have $\Theta(\bar{x}) < 0$.

Then there are two constants $c_1 \geq 0$ *and* $c_2 \geq 0$ *such that*

$$c_1\big(\eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(x)\big) + c_2\big(\eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Theta(x)\big) \geq 0,$$
$$\Big(x\in\Omega,\ c_2\Theta(\xi)=0,\ \eta_1:\Omega\times\Omega\to\mathbb{R}^m,\ \eta_2:\Omega\times\Omega\to\mathbb{R}^p\Big).$$

Proof. Our aim is to show that the system

$$\eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(x)<0, \quad \eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Theta(x)<0,$$

has no solution for $x\in\Omega$. Let the system has a solution $y\in\Omega$. By the assumption (A), we have

$$\Psi(\xi+\epsilon_1 y)<\Psi(\xi) \quad and \quad \Theta(\xi+\epsilon_2 y)<\Theta(\xi),$$

for sufficient small arbitrary constants $\epsilon_1,\,\epsilon_2>0$. Now, we let $\bar{x}:=\xi+\epsilon_2 y$; which implies that $\bar{x}\in\Lambda\cap N_{\epsilon_2}(\xi)$ thus by (B) and (C), we have $\Theta(\xi+\epsilon_2 y)=\Theta(\bar{x})<0$; which contradicts (A), where ξ is a weak solution. Therefore, the above inequalities are non-negative. Hence, in view of (C) these are two constants c_1 and c_2 satisfy the inequality

$$c_1\big(\eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(x)\big) + c_2\big(\eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Theta(x)\big) \geq 0,$$

with the property $c_2\Theta(\xi)=0$. This completes the proof.

Next, we consider the dual problem of (2.3) as follows:

$$\begin{aligned} &Max \quad \Psi(\chi)=\big(\psi_1(\chi),...,\psi_m(\chi)\big)\\ &subject\,to \quad c_1\big(\eta_1(x,\chi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(x)\big)+c_2\big(\eta_2(x,\chi).\mathfrak{D}^{\alpha,\lambda}\,\Theta(x)\big)\geq 0,\\ &\qquad\qquad\quad c_2\Theta(\chi)\geq 0, \end{aligned} \tag{3.4}$$

where $\chi\in\Omega$, c_1 and c_2 be two non negative constants.

Theorem 3.3. *Let* x,χ *be initial solutions of the multi-objective problems* (2.3) *and* (3.4) *respectively. If*

(A) The couple (Ψ,Θ) *is a strong (or weak) pseudo-quasi* $(\alpha,\lambda,\rho,\eta,\vartheta)$*-type univex at* $\xi\in\Omega$*;*
(B) $u\leq 0\in\mathbb{R}^m\Rightarrow\phi_1(u)\leq 0$ *and* $v\geq 0\in\mathbb{R}^p\Rightarrow\phi_2(v)\geq 0$*;*
(C) $c_1\rho_1+\rho_2\geq 0$*;*
then $\Psi(x)\nleq\Psi(\chi)$*.*

Proof. Suppose that $\Psi(x)\leq\Psi(\chi)$. Since $c_1\rho_1+\rho_2\geq 0$ then by (B), we obtain

$$\phi_1(\Psi(x)-\Psi(\chi))\leq 0$$
$$\phi_2(\Theta(\chi))\geq 0.$$

In virtue of the assumption (A) the above inequalities yield

$$\big(\eta_1(x,\chi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(\chi)\big)<-\rho_1\|\vartheta(x,\chi)\|^2$$
$$\big(\eta_2(x,\chi).\mathfrak{D}^{\alpha,\lambda}\,\Theta(\chi)\big)\leq-\rho_2\|\vartheta(x,\chi)\|^2,$$

consequently, we obtain

$$c_1\big(\eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(x)\big)<-c_1\rho_1\|\vartheta(x,\chi)\|^2$$

and

$$c_2\big(\eta_2(x,\chi).\mathfrak{D}^{\alpha,\lambda}\,\Theta(x)\big)\leq-\rho_2\|\vartheta(x,\chi)\|^2.$$

Summing the above inequalities and utilizing (C), we conclude that

$$\begin{aligned} c_1\big(\eta_1(x,\chi).\mathfrak{D}^{\alpha,\lambda}\,\Psi(\chi)\big)+c_2\big(\eta_2(x,\chi).\mathfrak{D}^{\alpha,\lambda}\,\Theta(\chi)\big) &< -\big(c_1\rho_1+\rho_2\big)\|\vartheta(x,\chi)\|^2\\ &\leq 0, \end{aligned}$$

which contradicts the assumption (C). This completes the proof.

Theorem 3.4. *Let* x_0 *and* χ_0 *be initial solution for the problems* (2.3) *and* (3.4) *respectively. If* $\Psi(x_0)=\Psi(\chi_0)$ *then the (weak or strong) duality problems* (2.3) *and* (3.4) *has efficient solutions* x_0 *and* χ_0 *respectively.*

Proof. Suppose that x_0 is not efficient for (2.3), then for some $x \in \Lambda$

$$\Psi(x) \leq \Psi(x_0) = \Psi(\chi_0),$$

which contradicts weak (strong) duality theorems as χ_0 is initial solution for (3.4). Therefore, x_0 is efficient for (2.3). Similarly χ_0 is efficient solution for (3.4). Hence the proof.

Theorem 3.5. *Let χ_0 be an initial solution of the multi-objective problem (3.4) and c_1 and c_2 be two non negative constants such that*

(A) The couple (Ψ, Θ) is a strong (or weak) pseudo-quasi $(\alpha, \lambda, \rho, \eta, \vartheta)$-type univex at $\xi \in \Omega$;
(B) $u \leq 0 \in \mathbb{R}^m \Rightarrow \phi_1(u) \leq 0$ and $v \geq 0 \in \mathbb{R}^p \Rightarrow \phi_2(v) \geq 0$;
(C) $c_1\rho_1 + \rho_2 \geq 0$.

Then χ_0 is an efficient solution of (3.4).

Proof. Suppose that χ_0 is not an efficient solution of (3.4), then there exists $x_0 \in \Lambda$ such that $\Psi(x_0) \leq \Psi(\chi_0)$. Now going on as in Theorem 3.3, we have a contradiction. Hence, χ_0 is an efficient solution of (3.4).

Theorem 3.6. *Let x_0, χ_0 be initial solutions of the multi-objective problems (2.3) and (3.4) respectively. If*

(A) $\Psi(x_0) \leq \Psi(\chi_0)$;
(B) The couple (Ψ, Θ) is a strong (or weak) pseudo-quasi $(\alpha, \lambda, \rho, \eta, \vartheta)$-type univex at $\xi \in \Omega$;
(C) $u \leq 0 \in \mathbb{R}^m \Rightarrow \phi_1(u) \leq 0$ and $v \geq 0 \in \mathbb{R}^p \Rightarrow \phi_2(v) \geq 0$;
(D) $c_1\rho_1 + \rho_2 \geq 0$;

then $x_0 = \chi_0$.

Proof. Suppose that $x_0 \neq \chi_0$. Since χ_0 is an initial solution for (3.4) then by (A) and (C), we have

$$\phi_1(\Psi(x_0) - \Psi(\chi_0)) \leq 0$$
$$\phi_2(\Theta(\chi_0)) \geq 0.$$

In virtue of the assumption (B) the above inequalities imply that

$$\big(\eta_1(x_0, \chi_0).\mathfrak{D}^{\alpha,\lambda}\, \Psi(\chi_0)\big) < -\rho_1 \|\vartheta(x_0, \chi_0)\|^2$$
$$\big(\eta_2(x_0, \chi_0).\mathfrak{D}^{\alpha,\lambda}\, \Theta(\chi_0)\big) \leq -\rho_2 \|\vartheta(x_0, \chi_0)\|^2,$$

which on summing yields

$$\begin{aligned} c_1\big(\eta_1(x_0, \chi_0).\mathfrak{D}^{\alpha,\lambda}\, \Psi(\chi_0)\big) + c_2\big(\eta_2(x_0, \chi_0).\mathfrak{D}^{\alpha,\lambda}\, \Theta(\chi_0)\big) &< -\big(c_1\rho_1 + \rho_2\big)\|\vartheta(x_0, \chi_0)\|^2 \\ &\leq 0, \end{aligned}$$

which contradicts to initially of χ_0. Then we obtain $x_0 = \chi_0$. This completes the proof.

4. Simulation

In this section, we illustrate a simulation to show how the tempered fractional calculus is effected on the multi-objective functions.

Let $\Psi, \Theta : \mathbb{R} \to \mathbb{R}^2$ such that

$$\Psi(x) = \Big(x^2, x^3\Big); \quad \Theta(x) = \Big(x, x^2\Big).$$

Our aim is to show that the couple (Ψ, Θ) is $(\alpha, \lambda, \rho, \eta, \vartheta)$-type univex at $\xi \in [0, 1]$. To determine the fractional Dunkl operator on these functions, we shall introduce three cases depending on the value of k_v for $v = 1$.

4.1. **Case (i)** $\lambda = 0,\ k_v = 0$**.** The tempered fractional Dunkl operator acts on the functions Ψ and Θ as follows:

$$\mathfrak{D}^{\alpha,\lambda}\Psi(x) = \Big(\frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{2-\alpha},\ \frac{\Gamma(4)}{\Gamma(4-\alpha)}x^{3-\alpha}\Big);\quad \mathfrak{D}^{\alpha,\lambda}\Theta(x) = \Big(\frac{\Gamma(2)}{\Gamma(2-\alpha)}x^{1-\alpha},\ \frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{2-\alpha}\Big).$$

Now, by letting

$$\eta_{1,2}(x,\xi) = \Big(\frac{x-\xi}{2},\ \frac{x-\xi}{2}\Big),\quad \xi = 0,$$

we have

$$\eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\Psi(x) = \frac{x^{3-\alpha}}{\Gamma(3-\alpha)} + \frac{3x^{4-\alpha}}{\Gamma(4-\alpha)};\ \eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\Theta(x) = \frac{x^{2-\alpha}}{2\Gamma(2-\alpha)} + \frac{x^{3-\alpha}}{\Gamma(3-\alpha)}.$$

Consider $\rho_1 = \rho_2 = 1$, $x \in [0,1]$ and $\vartheta(x,\xi) = x^2 - \xi$, therefore, we obtain

$$\|\vartheta(x,\xi)\|^2 = x^4,\quad \xi = 0.$$

It is clear that

$$\Psi(\xi) = \Psi(0) = (0,0);\quad \Theta(\xi) = \Theta(0) = (0,0),$$

then by assuming

$$\phi_1\Big(\Psi(x) - \Psi(\xi)\Big) = 5x,\quad \phi_2\Big(\Theta(x) - \Theta(\xi)\Big) = -5x,\quad x \in [0,1],$$

we conclude that

$$\begin{aligned}\eta_1(x,\xi).\mathfrak{D}^{\alpha}\,\Psi(x) + \rho_1\|\vartheta(x,\xi)\|^2 &= \frac{x^{3-\alpha}}{\Gamma(3-\alpha)} + \frac{3x^{4-\alpha}}{\Gamma(4-\alpha)} + x^4 \\ &< 5x,\quad x \in [0,1] \\ &= \phi_1\Big(\Psi(x) - \Psi(\xi)\Big)\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Theta(x) + \rho_2\|\vartheta(x,\xi)\|^2 &= \frac{x^{2-\alpha}}{2\Gamma(2-\alpha)} + \frac{x^{3-\alpha}}{\Gamma(3-\alpha)} + x^4 \\ &< 5x,\quad x \in [0,1] \\ &= -\phi_2\Big(\Theta(x) - \Theta(\xi)\Big)\end{aligned} \tag{4.2}$$

Hence, the couple (Ψ,Θ) is $(\alpha,\lambda,\rho,\eta,\vartheta)$-type univex at $\xi \in [0,1]$. Table 1 shows that for various values of $\alpha \in (0,1)$, the outcomes yield the tempered fractional univexty of the couple (Ψ,Θ).

TABLE 1. Fractional multi-objective function, $k_v = 0$

(α)	Eq. (4.1)	Eq. (4.2)
0.25	1.6	1.9
0.5	2.6	2.4
0.75	3.1	3.2

To apply the conditions of Theorem 3.1, we assume that $c_1 = c_2 = 1$; thus, we have $c_1\rho_1 + c_2\rho_2 = 2 > 0$ with the inequalities (4.1) and (4.2). This leads that all the conditions of Theorem 3.1 are achieved and hence, $\xi = 0$ is an efficient solution. Note that if we let $\phi_1(Y) = 3Y$ and $\phi_2(Y) = -3Y$, the couple (Ψ,Θ) is not $(\alpha,\lambda,\rho,\eta,\vartheta)$-type univex at $\xi \in [0,1]$.

4.2. **Case (ii)** $\lambda = 0,\ k_v = 1$**.** To evaluate the tempered fractional Dunkl operator, a calculation implies that

$$\sigma_{x^2} = x^2 - 2\frac{v.x^2}{v.v} = -x^2,\quad \sigma_{x^3} = -x^3.$$

Therefore, one can attain

$$\eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\Psi(x) = \frac{x^{3-\alpha}}{\Gamma(3-\alpha)} + \frac{x(2x^2)^{\alpha}}{2} + \frac{3x^{4-\alpha}}{\Gamma(4-\alpha)} + \frac{x(2x^3)^{\alpha}}{2}$$

and

$$\eta_2(x,\xi).\mathfrak{D}^{\alpha}\Theta(x) = \frac{x^{2-\alpha}}{2\Gamma(2-\alpha)} + \frac{x(2x)^{\alpha}}{2} + \frac{x^{3-\alpha}}{\Gamma(3-\alpha)} + \frac{x(2x^2)^{\alpha}}{2}.$$

Table 2 shows the evaluation of the tempered fractional multi-objective functions for different values of α.

TABLE 2. Fractional multi-objective function, $k_v = 1$

(α)	Eq. (4.1)	Eq. (4.2)
0.25	2.7	2.9
0.5	5	3.8
0.75	4.7	4.8

Thus, we conclude that the conditions of Theorem 3.1 are satisfied when $c_1 = c_2 = 1$; such that $c_1\rho_1 + c_2\rho_2 = 2 > 0$ with the inequalities (4.1) and (4.2). Consequently, we obtain $\xi = 0$ is an efficient solution.

4.3. **Case (iii)** $\lambda = 0$, $k_v = 2$**.** By applying (2.2), we have

$$\eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\Psi(x) = \frac{x^{3-\alpha}}{\Gamma(3-\alpha)} + x(2x^2)^{\alpha} + \frac{3x^{4-\alpha}}{\Gamma(4-\alpha)} + x(2x^3)^{\alpha}$$

and

$$\eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\Theta(x) = \frac{x^{2-\alpha}}{2\Gamma(2-\alpha)} + x(2x)^{\alpha} + \frac{x^{3-\alpha}}{\Gamma(3-\alpha)} + x(2x^2)^{\alpha}.$$

Table 3 shows the evaluation of the tempered fractional multi-objective functions for different values of α. It is clear that the couple (Ψ,Θ) is not $(\alpha,\lambda,\rho,\eta,\vartheta)$-type univex at $\xi \in [0,1]$. It is of $(\alpha,\lambda,\rho,\eta,\vartheta)$-type univex at $\xi \in [0,1]$, when $\alpha \in (0, 0.25]$. Hence, Theorem 3.1 can be applied only for this value of α.

TABLE 3. Fractional multi-objective function, $k_v = 2$

(α)	Eq. (4.1)	Eq. (4.2)
0.25	3.5	4.1
0.5	5.4	5.2
0.75	6.4	5.5

4.4. **Case (iv)** $\lambda = 1$, $k_v = 0$**.** The tempered fractional Dunkl operator acts on the functions Ψ and Θ as follows:

$$\mathfrak{D}^{\alpha,\lambda}\Psi(x) = \Big(\frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{2-\alpha} + x^2e^x,\ \frac{\Gamma(4)}{\Gamma(4-\alpha)}x^{3-\alpha} + x^3e^x\Big);$$

$$\mathfrak{D}^{\alpha,\lambda}\Theta(x) = \Big(\frac{\Gamma(2)}{\Gamma(2-\alpha)}x^{1-\alpha} + xe^x,\ \frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{2-\alpha} + x^2e^x\Big).$$

Now, by letting

$$\eta_{1,2}(x,\xi) = \Big(\frac{x-\xi}{2}, \frac{x-\xi}{2}\Big), \quad \xi = 0,$$

we have

$$\eta_1(x,\xi).\mathfrak{D}^{\alpha,\lambda}\Psi(x) = \frac{x^{3-\alpha}}{\Gamma(3-\alpha)} + \frac{x^3e^x}{2} + \frac{3x^{4-\alpha}}{\Gamma(4-\alpha)} + \frac{x^4e^x}{2};$$

$$\eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\Theta(x) = \frac{x^{2-\alpha}}{2\Gamma(2-\alpha)} + \frac{x^2e^x}{2} + \frac{x^{3-\alpha}}{\Gamma(3-\alpha)} + \frac{x^3e^x}{2}.$$

Consider $\rho_1 = \rho_2 = 1$, $x \in [0,1]$ and $\vartheta(x,\xi) = x^2 - \xi$, therefore, we obtain

$$\|\vartheta(x,\xi)\|^2 = x^4, \quad \xi = 0.$$

It is clear that

$$\Psi(\xi) = \Psi(0) = (0,0); \quad \Theta(\xi) = \Theta(0) = (0,0),$$

then by assuming

$$\phi_1\Big(\Psi(x)-\Psi(\xi)\Big)=7x, \quad \phi_2\Big(\Theta(x)-\Theta(\xi)\Big)=-7x, \quad x\in[0,1],$$

we conclude that

$$\begin{aligned}\eta_1(x,\xi).\mathfrak{D}^{\alpha}\,\Psi(x)+\rho_1\|\vartheta(x,\xi)\|^2 &= \frac{x^{3-\alpha}}{\Gamma(3-\alpha)}+\frac{x^3e^x}{2}+\frac{3x^{4-\alpha}}{\Gamma(4-\alpha)}+\frac{x^4e^x}{2}+x^4\\ &< 7x, \quad x\in[0,1]\\ &= \phi_1\Big(\Psi(x)-\Psi(\xi)\Big)\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\eta_2(x,\xi).\mathfrak{D}^{\alpha,\lambda}\,\Theta(x)+\rho_2\|\vartheta(x,\xi)\|^2 &= \frac{x^{2-\alpha}}{2\Gamma(2-\alpha)}+\frac{x^2e^x}{2}+\frac{x^{3-\alpha}}{\Gamma(3-\alpha)}+\frac{x^3e^x}{2}+x^4\\ &< 7x, \quad x\in[0,1]\\ &= -\phi_2\Big(\Theta(x)-\Theta(\xi)\Big)\end{aligned} \tag{4.4}$$

Hence, the couple (Ψ,Θ) is $(\alpha,\lambda,\rho,\eta,\vartheta)$-type univex at $\xi\in[0,1]$. Table 1 shows that for various values of $\alpha\in(0,1)$, the outcomes yield the tempered fractional univexty of the couple (Ψ,Θ).

TABLE 4. , $k_v=0,\lambda=1$

$(\alpha,1)$	Eq. (4.3)	Eq. (4.4)
0.25	4.3	4.6
0.5	5.4	5.1
0.75	5.8	5.9

To apply the conditions of Theorem 3.1, we assume that $c_1=c_2=1$; thus, we have $c_1\rho_1+c_2\rho_2=2>0$ with the inequalities (4.3) and (4.4). This leads that all the conditions of Theorem 3.1 are achieved and hence, $\xi=0$ is an efficient solution. Note that if we let $\phi_1(Y)=5Y$ and $\phi_2(Y)=-5Y$, the couple (Ψ,Θ) is not $(\alpha,\lambda,\rho,\eta,\vartheta)$-type univex at $\xi\in[0,1]$. Also, the case $\lambda=1$ and $k_v=1$ does not imply the univex function when $\phi_1(Y)=7Y$ and $\phi_2(Y)=-7Y$.

5. Conclusion

This effort is generalized, for the first time, two important concepts in science. The Dunkl tempered fractional operator and the tempered Univex function, by utilizing the Caputo tempered fractional differential operator. These two generalizations are combined to deliver the fractional multi-objective problems. We studied the duality cases by minimize and maximize the desired function in the $\mathbb{R}^n$. Simulation is provided to apply the existing solutions. It has been found that the fractional case converges to the ordinary case. These problems can be employed in many studies not only in mathematics, but also in the economy; such as the utility function the cost function and the entropy function.

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