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EXISTENCE OF SOLUTIONS AND ULAM STABILITY FOR CAPUTO TYPE SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATIONS OF ORDER $\alpha \in (2,3)$

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Abstract. We study initial value problems of sequential fractional differential equations and inclusions involving a Caputo type differential operator of the form: $\left({}^C D^{\alpha}_{a+} + \lambda_1 \, {}^C D^{\alpha-1}_{a+} + \lambda_2 \, {}^C D^{\alpha-2}_{a+}\right)$, where $\alpha \in (2,3)$ and $\lambda_i (i=1,2)$ are nonzero constants. Several existence and uniqueness results are accomplished by means of fixed point theorems. Sufficient conditions for Ulam stability of the given problem are also presented. Examples are constructed for the illustration of obtained results. Then we investigate the inclusions case of the problem at hand. An initial value problem for coupled sequential fractional differential equations is also discussed.

1. Introduction

Fractional calculus is a generalization of the classical differentiation and integration to arbitrary non-integer order. The idea of fractional calculus has been a subject of interest not only among mathematicians but also among physicists and engineers. They have used it effectively to improve the mathematical modelling of several phenomena occurring in scientific and engineering disciplines such as viscoelasticity [1], electrochemistry [2], electromagnetism [3], biology ([4], [5]), control ([6], [7], [14]), diffusion process ([8], [9], [10]), economics [11], chaotic systems ([12], [13]), variational problems [15] etc.

The mathematical models involving fractional order derivatives are more realistic and practical than the classical models as they help to trace the history of the associated phenomena. Also, the enriched material on theoretical aspects and analytic/numerical methods for solving fractional order models attracts the modelers. During the last decade, many researchers have focused on the existence of solutions for initial and boundary value problems of fractional differential equations see ([16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26]) and the references cited therein.

The stability theory of fractional differential systems needs more investigations than the one for classical differential systems, since fractional derivatives are nonlocal and have weak singular kernels. For recent development on the stability of fractional differential systems, for instance, see ([27], [28], [29]) and the references cited therein.

The Ulam type stabilities [30] for fractional differential systems are quite significant in realistic problems, numerical analysis, biology and economics. For details and examples, we refer the reader to the works ([31], [32], [33]).

In this paper, we investigate the existence of solutions for an initial value problem of sequential fractional differential equations given by

$$\begin{cases} \left({}^C D^{\alpha}_{a+} + \lambda_1 \, {}^C D^{\alpha-1}_{a+} + \lambda_2 \, {}^C D^{\alpha-2}_{a+}\right) x(t) = f(t, x(t)), \, \alpha \in (2,3), \, t \in J, \\ x^{(k)}(a) = b_k, k = 0, 1, 2, \end{cases} \tag{1.1}$$

where ${}^C D^{\alpha}_{a}$ denote the Caputo fractional derivative of order α, λ_1 and λ_2 are nonzero constants, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and $J = [a, T]$, $T > a \geq 0$.

The rest of the paper is organized as follows. In Section 2, we recall some preliminary concepts and prove an auxiliary lemma, which plays a fundamental role in defining the fixed point problem associated with the problem at hand. Existence results and illustrative examples are presented in

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Section 3, while Ulam type stability for the given problem is discussed in Section 4. An existence result for the multivalued (inclusions) case of the problem (1.1) is proved in Section 5. An initial value problem for coupled sequential fractional differential equations is formulated and investigated in Section 6.

2. Preliminaries

Let us first recall some basic notions of fractional calculus ([16], [17]).

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as*

$$I_{a+}^{\alpha}h(t) = \int_a^t \frac{(t-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds,$$

provided the integral exists, and $I_{a+}^{0}h(t) = h(t)$.

Definition 2.2. *The Caputo derivative of fractional order $\alpha > 0$ is defined as*

$${}^{C}D_{a+}^{\alpha}h(t) = \int_a^t \frac{(t-s)^{n-\alpha-1}h^{(n)}(s)}{\Gamma(n-\alpha)}ds,\ \ n-1<\alpha<n, n=[\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Let $C(J,\mathbb{R})$ be a Banach space of all continuous real valued functions defined on J endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t\in J\}$, $C^n(J,\mathbb{R})$ be a Banach space of all n times continuously differentiable on J. By $AC(J,\mathbb{R})$, we denote the space of functions which are absolutely continuous on J, and by $AC^n(J,\mathbb{R})$, the space of functions f which have continuous derivatives up to order $n-1$ on J such that $f^{(n)} \in AC(J,\mathbb{R})$.

Here we remark that the fractional integral I_{a+}^{α} is bounded operator on $C(J,\mathbb{R})$ (see Lemma 2.8 [16]), and the fractional derivative ${}^{C}D_{a+}^{\alpha}h$ exists almost everywhere if $h \in AC^n(J,\mathbb{R})$ (see Theorem 2.1 [16]). Notice that $C^1(J,\mathbb{R}) \subseteq AC(J,\mathbb{R}) \subseteq C(J,\mathbb{R})$; in general, $C^n(J,\mathbb{R}) \subseteq AC^n(J,\mathbb{R}) \subseteq C^{n-1}(J,\mathbb{R})$. Therefore, the fractional derivative ${}^{C}D_{a+}^{\alpha}h$ is continuous for any $h \in C^n(J,\mathbb{R})$ (see Theorem 2.2 [16]).

Lemma 2.1. *([16]) Let $x \in C^n(J,\mathbb{R})$ (or $AC^n(J,\mathbb{R})$), $f \in C(J,\mathbb{R})$ (or $AC(J,\mathbb{R})$), and $c_i \in \mathbb{R}$, $i = 0,1,2,...,n-1$. Then*

$$\begin{aligned} &{}^{C}D_{a+}^{\alpha}\left(I_{a+}^{\alpha}f(t)\right) = f(t),\\ &I_{a+}^{\alpha}({}^{C}D_{a+}^{\alpha}x(t)) = x(t) + \sum_{i=0}^{n-1} c_i\,(t-a)^i,\\ &{}^{C}D_{a+}^{\alpha}x(t) = 0 \text{ implies that } x(t) = \sum_{i=0}^{n-1} c_i\,(t-a)^i. \end{aligned}$$

Consider the linear variant problem

$$\begin{cases} \left({}^{C}D_{a+}^{\alpha} + \lambda_1\ {}^{C}D_{a+}^{\alpha-1} + \lambda_2\ {}^{C}D_{a+}^{\alpha-2}\right)x(t) = g(t), t\in J \\ x^{(k)}(a) = b_k, k = 0,1,2. \end{cases} \tag{2.1}$$

Lemma 2.2. *Let $x \in C^3(J,\mathbb{R})$, $g \in C(J,\mathbb{R})$, and $\lambda_1^2 = 4\lambda_2$, then the linear problem (2.1) is equivalent to the integral equation*

$$\begin{aligned} x(t) \;=\; & b_0 + b_1\left(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_1}{\lambda_2}e^{-\frac{\lambda_1}{2}(t-a)} - (t-a)\,e^{-\frac{\lambda_1}{2}(t-a)}\right)\\ & + b_2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_2}e^{-\frac{\lambda_1}{2}(t-a)} - \frac{2}{\lambda_1}(t-a)\,e^{-\frac{\lambda_1}{2}(t-a)}\right)\\ & + \frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)}\int_a^t\int_a^s (t-s)\,(s-r)^{\alpha-3}\,e^{\frac{\lambda_1}{2}s}g(r)drds. \end{aligned} \tag{2.2}$$

Proof. Applying the fractional integral operator $I_{a+}^{\alpha-2}$ to sequential fractional differential equation in (2.1), we get

$$x^{(2)}(t)+\lambda_1 x^{(1)}(t)+\lambda_2 x(t)=I_{a+}^{\alpha-2}g(t)+c_2. \tag{2.3}$$

Using the initial conditions in (2.3) leads to $c_2=b_2+\lambda_1 b_1+\lambda_2 b_0$. Now, let $y(t)=e^{\frac{\lambda_1}{2}t}x(t)$, then $y^{(1)}(t)=e^{\frac{\lambda_1}{2}t}x^{(1)}(t)+\frac{\lambda_1}{2}e^{\frac{\lambda_1}{2}t}x(t)$, and $y^{(2)}(t)=e^{\frac{\lambda_1}{2}t}x^{(2)}(t)+\lambda_1 e^{\frac{\lambda_1}{2}t}x^{(1)}(t)+\frac{\lambda_1^2}{4}e^{\frac{\lambda_1}{2}t}x(t)$. Substituting these values in (2.3), we get

$$y^{(2)}(t)=e^{\frac{\lambda_1}{2}t}I_{a+}^{\alpha-2}g(t)+(b_2+\lambda_1 b_1+\lambda_2 b_0)\,e^{\frac{\lambda_1}{2}t}. \tag{2.4}$$

Integrating equation (2.4) twice from a to t, we obtain

$$\begin{aligned} y(t) &= y(a)+y^{(1)}(a)\,(t-a)\\ &\quad+(b_2+\lambda_1 b_1+\lambda_2 b_0)\left(\frac{1}{\lambda_2}e^{\frac{\lambda_1}{2}t}-\frac{1}{\lambda_2}e^{\frac{\lambda_1}{2}a}-\frac{2}{\lambda_1}e^{\frac{\lambda_1}{2}a}\,(t-a)\right)\\ &\quad+\frac{1}{\Gamma(\alpha-2)}\int_a^t\int_a^s (t-s)\,(s-r)^{\alpha-3}\,e^{\frac{\lambda_1}{2}s}g(r)drds, \end{aligned}$$

which, on account of $y(t)=e^{\frac{\lambda_1}{2}t}x(t)$, yields

$$\begin{aligned} x(t) &= e^{-\frac{\lambda_1}{2}(t-a)}x(a)+\left(x^{(1)}(a)+\frac{\lambda_1}{2}x(a)\right)(t-a)\,e^{-\frac{\lambda_1}{2}(t-a)}\\ &\quad+(b_2+\lambda_1 b_1+\lambda_2 b_0)\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_2}e^{-\frac{\lambda_1}{2}(t-a)}-\frac{2}{\lambda_1}(t-a)\,e^{-\frac{\lambda_1}{2}(t-a)}\right)\\ &\quad+\frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)}\int_a^t\int_a^s (t-s)\,(s-r)^{\alpha-3}\,e^{\frac{\lambda_1}{2}s}g(r)drds. \end{aligned} \tag{2.5}$$

Making use of the initial conditions in (2.5) and rearranging the terms we get (2.2). Conversely, applying the fractional operator $\left({}^{C}D_{a+}^{\alpha}+\lambda_1\,{}^{C}D_{a+}^{\alpha-1}+\lambda_2\,{}^{C}D_{a+}^{\alpha-2}\right)$ to the integral equation (2.2) and using Lemma 2.1, we obtain the problem (2.1). This completes the proof. □

If g has a maximum $g_{\max}$ on J, then the integral term in equation (2.2) has upper bounds

$$\frac{2g_{\max}\,(T-a)^{\alpha-1}}{|\lambda_1|\,\Gamma(\alpha-1)}\left|1-e^{-\frac{\lambda_1}{2}(t-a)}\right|<\begin{cases}\frac{2g_{\max}(T-a)^{\alpha-1}}{|\lambda_1|\Gamma(\alpha-1)},\lambda_1>0\\ \frac{2g_{\max}(T-a)^{\alpha-1}}{|\lambda_1|\Gamma(\alpha-1)}\left(1+e^{-\frac{\lambda_1}{2}(T-a)}\right),\lambda_1<0\end{cases} \tag{2.6}$$

for any $t\in J$. Therefore, in the next sections, we prefer to use the upper bound $\frac{2g_{\max}(T-a)^{\alpha-1}}{|\lambda_1|\Gamma(\alpha-1)}\left(1+e^{-\frac{\lambda_1}{2}(T-a)}\right)$ for each nonzero λ_1.

3. Existence Theorems

We establish sufficient conditions for existence of solutions to problem (1.1) using different types of fixed point theorems.

In view of Lemma 2.2, we transform the initial value problem (1.1) into an operator equation as

$$\begin{aligned} \Psi x(t) &= b_0+b_1\left(\frac{\lambda_1}{\lambda_2}-\frac{\lambda_1}{\lambda_2}e^{-\frac{\lambda_1}{2}(t-a)}-(t-a)\,e^{-\frac{\lambda_1}{2}(t-a)}\right)\\ &\quad+b_2\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_2}e^{-\frac{\lambda_1}{2}(t-a)}-\frac{2}{\lambda_1}(t-a)\,e^{-\frac{\lambda_1}{2}(t-a)}\right)\\ &\quad+\frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)}\int_a^t\int_a^s (t-s)\,(s-r)^{\alpha-3}\,e^{\frac{\lambda_1}{2}s}f(r,x(r))drds \end{aligned} \tag{3.1}$$

where $x\in C(J,\mathbb{R})$. Note that λ_2 is nonnegative for all values of λ_1 as $\lambda_1^2=4\lambda_2$.

If the operator Ψ has a fixed point in $C(J,\mathbb{R})$, then the problem (1.1) has this fixed point as a solution.

Lemma 3.1. *The operator $\Psi: C(J,\mathbb{R})\to C(J,\mathbb{R})$ given by (3.1) is completely continuous.*

Proof. Obviously, continuity of the operator Ψ follows from the continuity of the function f. Let $\mathcal{U}$ be a bounded proper subset of $C(J,\mathbb{R})$, then for any $t\in J$, and $x\in\mathcal{U}$, there exists a positive constant L such that $|f(t,x(t))|\leq L$. Accordingly, (2.6) yields

$$\begin{aligned}
|\Psi x(t)| \leq\ & |b_0|+|b_1|\left(\frac{|\lambda_1|}{\lambda_2}+\frac{|\lambda_1|}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)}+(T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right)\\
&+|b_2|\left(\frac{1}{\lambda_2}+\frac{1}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)}+\frac{2}{|\lambda_1|}\,(T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right)\\
&+\frac{2L\,(T-a)^{\alpha-1}}{|\lambda_1|\,\Gamma(\alpha-1)}\left(1+e^{-\frac{\lambda_1}{2}(T-a)}\right),
\end{aligned}$$

which implies that Ψ is a bounded operator on $\mathcal{U}\subset C(J,\mathbb{R})$. Furthermore, for $a<t_1<t_2<T$, we have

$$\begin{aligned}
&|(\Psi x)(t_2)-(\Psi x)(t_1)|\\
\leq\ & \frac{|b_1\lambda_1|}{\lambda_2}\left|e^{-\frac{\lambda_1}{2}(t_1-a)}-e^{-\frac{\lambda_1}{2}(t_2-a)}\right|\\
&+|b_1|\left|(t_1-a)\,e^{-\frac{\lambda_1}{2}(t_1-a)}-(t_2-a)\,e^{-\frac{\lambda_1}{2}(t_2-a)}\right|\\
&+\frac{|b_2|}{\lambda_2}\left|e^{-\frac{\lambda_1}{2}(t_1-a)}-e^{-\frac{\lambda_1}{2}(t_2-a)}\right|\\
&+\frac{2\,|b_2|}{|\lambda_1|}\left|(t_1-a)\,e^{-\frac{\lambda_1}{2}(t_1-a)}-(t_2-a)\,e^{-\frac{\lambda_1}{2}(t_2-a)}\right|\\
&+\left|\frac{e^{-\frac{\lambda_1}{2}t_2}}{\Gamma(\alpha-2)}\int_a^{t_2}\int_a^{s}(t_2-s)\,(s-r)^{\alpha-3}\,e^{\frac{\lambda_1}{2}s}f(r,x(r))drds\right.\\
&\left.-\frac{e^{-\frac{\lambda_1}{2}t_1}}{\Gamma(\alpha-2)}\int_a^{t_1}\int_a^{s}(t_1-s)\,(s-r)^{\alpha-3}\,e^{\frac{\lambda_1}{2}s}f(r,x(r))drds\right|\\
\leq\ & \frac{|b_1\lambda_1|}{\lambda_2}\left|e^{-\frac{\lambda_1}{2}(t_1-t_2)}-1\right|e^{-\frac{\lambda_1}{2}(t_2-a)}\\
&+|b_1|\left|(t_1-t_2)\,e^{-\frac{\lambda_1}{2}(t_1-t_2)}+(t_2-a)\left(e^{-\frac{\lambda_1}{2}(t_1-t_2)}-1\right)\right|e^{-\frac{\lambda_1}{2}(t_2-a)}\\
&+\frac{|b_2|}{\lambda_2}\left|e^{-\frac{\lambda_1}{2}(t_1-t_2)}-1\right|e^{-\frac{\lambda_1}{2}(t_2-a)}\\
&+\frac{|b_2|}{\lambda_2}\left|(t_1-t_2)\,e^{-\frac{\lambda_1}{2}(t_1-t_2)}+(t_2-a)\left(e^{-\frac{\lambda_1}{2}(t_1-t_2)}-1\right)\right|e^{-\frac{\lambda_1}{2}(t_2-a)}\\
&+\frac{Le^{-\frac{\lambda_1}{2}t_1}e^{-\frac{\lambda_1}{2}(t_2-t_1)}}{\Gamma(\alpha-2)}\int_a^{t_1}\int_a^{s}(t_2-t_1)\,(s-r)^{\alpha-3}\,e^{\frac{\lambda_1}{2}s}drds\\
&+\frac{Le^{-\frac{\lambda_1}{2}t_1}\left(e^{-\frac{\lambda_1}{2}(t_2-t_1)}-1\right)}{\Gamma(\alpha-2)}\int_a^{t_1}\int_a^{s}(t_1-s)\,(s-r)^{\alpha-3}\,e^{\frac{\lambda_1}{2}s}drds\\
&+\frac{Le^{-\frac{\lambda_1}{2}t_1}e^{-\frac{\lambda_1}{2}(t_2-t_1)}}{\Gamma(\alpha-2)}\int_{t_1}^{t_2}\int_a^{s}(t_2-s)\,(s-r)^{\alpha-3}\,e^{\frac{\lambda_1}{2}s}drds,
\end{aligned}$$

which tends to zero independently of x as $t_2\to t_1$. This implies that Ψ is equicontinuous on J. In consequence, it follows by the Arzela-Ascoli theorem that the operator Ψ is completely continuous. This completes the proof. □

Next we recall the Schauder's fixed-point theorem ([34]).

Theorem 3.1. *If $\mathcal{U}$ is a closed , bounded, convex subset of a Banach space $\mathcal{X}$ and the mapping $\Delta:\mathcal{U}\to\mathcal{U}$ is completely continuous, then Δ has a fixed point in $\mathcal{U}$.*

Theorem 3.2. *Let $|f(t,x(t))|\leq L$. Then there exists a solution of the problem (1.1) on J.*

Proof. It is a direct consequence of Lemma 3.1 if $\mathcal{U}$ is taken to be a closed, bounded, convex subset of $C(J, \mathbb{R})$. □

Theorem 3.3. *Assume that there exists a constant K such that*

$$\lim_{x \to 0} \frac{f(t,x)}{x} = K, t \in J,$$

then the problem (1.1) has a solution on J.

Proof. By the given assumption, it follows that $|f(t,x(t))| \leq (1+K)\ |x(t)|$, whenever $|x(t)| < \delta$, for a fixed number $\delta > 0$. Therefore, we can define a subset $\mathcal{U}$ as

$$\mathcal{U} = \{x \in C(J, \mathbb{R}) : |x(t)| \leq \delta, t \in J\}.$$

Clearly $\mathcal{U}$ is a closed, bounded and convex subset of $C(J,\mathbb{R})$. If $x \in \mathcal{U}$, then $|f(t,x(t))| < \delta\,(1+K)$, for any $t \in J$. On the other hand, the operator $\Psi : \mathcal{U} \to \mathcal{U}$ defined by (3.1) is completely continuous by Lemma 3.1. Therefore, by Schauder's fixed point theorem 3.1, the problem (1.1) has a solution. This completes the proof. □

The next result is based on Krasnoselskii's fixed point theorem ([34]) .

Theorem 3.4. *Let $\mathcal{M}$ be a closed convex and nonempty subset of a Banach space $C(J,\mathbb{R})$. Let Θ, and Φ be the operators such that*
(i) $\Theta u + \Phi v \in \mathcal{M}$ whenever $u, v \in \mathcal{M}$;
(ii) Θ is compact and continuous;
(iii) Φ is a contraction.
Then there exists $x \in \mathcal{M}$ such that $x = \Theta x + \Phi x$.

Theorem 3.5. *Assume that*

- (H_1)**:** *For any $t \in J$, and $x, y \in \mathbb{R}$, there exists a positive constants C such that $|f(t,x) - f(t,y)| \leq C\,|x-y|$.*
- $(H2)$**:** *For any $t \in J$, and $x \in \mathbb{R}$, there exists $\mu \in C(J, \mathbb{R}^+)$ such that $|f(t,x)| \leq \mu(t)$.*
- (H_3)**:** *$\omega < 1$, where*

$$\omega = \frac{2C\,(T-a)^{\alpha-1}}{|\lambda_1|\,\Gamma(\alpha-1)} \left(1 + e^{-\frac{\lambda_1}{2}(T-a)}\right).$$

Then, the problem (1.1) has at least one solution on J.

Proof. Define a set $\mathcal{B}_r = \{x \in C(J,\mathbb{R}) : \|x\| \leq r\}$, where r is a positive real number satisfying the inequality

$$\begin{aligned} r \quad \geq \quad & |b_0| + |b_1| \left(\frac{|\lambda_1|}{\lambda_2} + \frac{|\lambda_1|}{\lambda_2} e^{-\frac{\lambda_1}{2}(T-a)} + (T-a)\, e^{-\frac{\lambda_1}{2}(T-a)} \right) \\ & + |b_2| \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_2} e^{-\frac{\lambda_1}{2}(T-a)} + \frac{2}{|\lambda_1|}\,(T-a)\, e^{-\frac{\lambda_1}{2}(T-a)} \right) \\ & + \frac{2L\,(T-a)^{\alpha-1}}{|\lambda_1|\,\Gamma(\alpha-1)} \left(1 + e^{-\frac{\lambda_1}{2}(T-a)}\right). \end{aligned}$$

Introduce operators Θ and Φ on $\mathcal{B}_r$ as

$$\begin{aligned} (\Theta x)\,(t) \quad = \quad & b_0 + b_1 \left(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_1}{\lambda_2} e^{-\frac{\lambda_1}{2}(t-a)} - (t-a)\, e^{-\frac{\lambda_1}{2}(t-a)} \right) \\ & + b_2 \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_2} e^{-\frac{\lambda_1}{2}(t-a)} - \frac{2}{\lambda_1}\,(t-a)\, e^{-\frac{\lambda_1}{2}(t-a)} \right), \end{aligned}$$

and

$$(\Phi x)(t) = \frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)} \int_a^t \int_a^s (t-s)\,(s-r)^{\alpha-3}\, e^{\frac{\lambda_1}{2}s} f(r, x(r))\,dr\,ds.$$

For $x, y \in \mathcal{B}_r,\ t \in J$, using (H_2), we find that

$$\begin{aligned} |\Theta x(t)+\Phi y(t)| &\leq |b_0|+|b_1|\left(\frac{|\lambda_1|}{\lambda_2}+\frac{|\lambda_1|}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)}+(T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right)\\ &\quad+|b_2|\left(\frac{1}{\lambda_2}+\frac{1}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)}+\frac{2}{|\lambda_1|}(T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right)\\ &\quad+\frac{2\,\|\mu\|\,(T-a)^{\alpha-1}}{|\lambda_1|\,\Gamma(\alpha-1)}\left(1+e^{-\frac{\lambda_1}{2}(T-a)}\right).\end{aligned}$$

Thus, $\Theta x+\Phi y \in \mathcal{B}_r$. By (H_1), for $x, y \in \mathcal{B}_r, t \in J$, we have

$$|(\Phi x)(t)-(\Phi y)(t)| \leq \frac{2C\,(T-a)^{\alpha-1}}{|\lambda_1|\,\Gamma(\alpha-1)}\left(1+e^{-\frac{\lambda_1}{2}(T-a)}\right)\|x-y\|,$$

that is, $\|(\Phi x)-(\Phi y)\| \leq \omega\,\|x-y\|$. Since $\omega<1$ by (H_3), Φ is a contraction.

The operator Θ is continuous and is uniformly bounded, since

$$\begin{aligned} \|\Theta x\| &\leq |b_0|+|b_1|\left(\frac{|\lambda_1|}{\lambda_2}+\frac{|\lambda_1|}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)}+(T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right)\\ &\quad+|b_2|\left(\frac{1}{\lambda_2}+\frac{1}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)}+\frac{2}{|\lambda_1|}(T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right).\end{aligned}$$

As in the proof of Lemma 3.1, it can be shown that Θ is equicontinuous and relatively compact on $\mathcal{B}_r$. Hence, by the Arzela-Ascoli theorem, Θ is compact on $\mathcal{B}_r$. Thus all the assumptions of Theorem 3.4 are satisfied. Therefore, the problem (1.1) has at least one solution on J. This completes the proof. □

Our next result deals with the uniqueness of solutions for the problem (1.1) and is based on the Banach contraction principle.

Theorem 3.6. *Assume that (H_1) and (H_3) hold. Then there exists a unique solution for the problem (1.1) on J.*

Proof. Let $\sup_{t\in J}|f(t,0)|=A$, and $r \geq (1-\beta)^{-1}\gamma$, where

$$\begin{aligned} \gamma &= |b_0|+|b_1|\left(\frac{|\lambda_1|}{\lambda_2}+\frac{|\lambda_1|}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)}+(T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right)\\ &\quad+|b_2|\left(\frac{1}{\lambda_2}+\frac{1}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)}+\frac{2}{|\lambda_1|}(T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right)\\ &\quad+\frac{2A\,(T-a)^{\alpha-1}}{|\lambda_1|\,\Gamma(\alpha-1)}\left(1+e^{-\frac{\lambda_1}{2}(T-a)}\right).\end{aligned}$$

Then we show that $\Psi\mathcal{B}_r \subset \mathcal{B}_r$, where $\mathcal{B}_r=\{x \in C(J,\mathbb{R}) : \|x\| \leq r\}$. This follows by the following estimate

$$\begin{aligned} |\Psi x(t)| &\leq |b_0|+|b_1|\left(\frac{|\lambda_1|}{\lambda_2}+\frac{|\lambda_1|}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)}+(T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right)\\ &\quad+|b_2|\left(\frac{1}{\lambda_2}+\frac{1}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)}+\frac{2}{|\lambda_1|}(T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right)\\ &\quad+\frac{2A\,(T-a)^{\alpha-1}}{|\lambda_1|\,\Gamma(\alpha-1)}\left(1+e^{-\frac{\lambda_1}{2}(T-a)}\right)\\ &\quad+\frac{2C\,(T-a)^{\alpha-1}}{|\lambda_1|\,\Gamma(\alpha-1)}\left(1+e^{-\frac{\lambda_1}{2}(T-a)}\right)\|x\|\\ &\leq (1-\beta)\,r+\beta r=r,\end{aligned}$$

for any $x \in \mathcal{B}_r$. Moreover, for $x,\ y \in C(J,\mathbb{R})$ and for each $t \in J$, we can obtain

$$|(\Psi x)(t)-(\Psi y)(t)| \leq \frac{2C\,(T-a)^{\alpha-1}}{|\lambda_1|\,\Gamma(\alpha-1)}\left(1+e^{-\frac{\lambda_1}{2}(T-a)}\right)\|x-y\|,$$

which, by taking the norm on J and using the assumption (H_3), yields that Ψ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof. □

Our last existence theorems is based on Leray-Schauder degree theorem [34]. For that we set the notations:

$$\begin{aligned} F &= |b_0| + |b_1| \left(\frac{|\lambda_1|}{\lambda_2} + \frac{|\lambda_1|}{\lambda_2} e^{-\frac{\lambda_1}{2}(T-a)} + (T-a)\, e^{-\frac{\lambda_1}{2}(T-a)} \right) \\ &+ |b_2| \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_2} e^{-\frac{\lambda_1}{2}(T-a)} + \frac{2}{|\lambda_1|} (T-a)\, e^{-\frac{\lambda_1}{2}(T-a)} \right) \\ &+ \frac{2E\,(T-a)^{\alpha-1}}{|\lambda_1|\,\Gamma(\alpha-1)} \left(1 + e^{-\frac{\lambda_1}{2}(T-a)} \right), \end{aligned} \tag{3.2}$$

$$G = \frac{2D\,(T-a)^{\alpha-1}}{|\lambda_1|\,\Gamma(\alpha-1)} \left(1 + e^{-\frac{\lambda_1}{2}(T-a)} \right) < 1. \tag{3.3}$$

Theorem 3.7. *Assume that there exist constants D and E such that $|f(t,x)| \le D\,|x| + E$, for $t \in J$, $x \in \mathbb{R}$. Then there exists a solution for the problem (1.1) on J.*

Proof. Define a ball $\mathcal{B}_R = \{x \in C(J,\mathbb{R}) : \|x\| < R\}$ for some positive real number R which will be determined later. We show that $\Psi : \overline{\mathcal{B}_R} \to C(J,\mathbb{R})$ satisfies

$$0 \notin (I - \lambda\Psi)\,(\partial\mathcal{B}_R)\,,$$

for any $x \in \partial\mathcal{B}_R$, and $\lambda \in [0,1]$, where $\partial\mathcal{B}_R$ denotes the boundary set of $\mathcal{B}_R$. Define the homotopy

$$h_\lambda(x) = H(\lambda, x) = x - \lambda\Psi x, x \in C(J,\mathbb{R}),\ \lambda \in [0,1].$$

Then, by Lemma 3.1, h_λ is completely continuous. Let I denote the identity operator, then the homotopy invariance and normalization properties of topological degrees imply that

$$\begin{aligned} deg(h_\lambda, \mathcal{B}_R, 0) &= deg((I - \lambda\Psi)\,, \mathcal{B}_R, 0) = deg(h_1, \mathcal{B}_R, 0) \\ &= deg(h_0, \mathcal{B}_R, 0) = deg(I, \mathcal{B}_R, 0) = 1, \end{aligned}$$

since $0 \in \mathcal{B}_R$. By the nonzero property of the Leray-Schauder degree, $h_1(x) = x - \Psi x = 0$ for at least one $x \in \mathcal{B}_R$. To find R, we assume that $x(t) = \lambda\Psi x(t)$ for some $\lambda \in [0,1]$ and for all $t \in J$. Then, using the given assumption together with (3.2) and (3.3), we get

$$\begin{aligned} |x(t)| &= |\lambda\Psi x(t)| \le |b_0| + |b_1| \left(\frac{|\lambda_1|}{\lambda_2} + \frac{|\lambda_1|}{\lambda_2} e^{-\frac{\lambda_1}{2}(T-a)} + (T-a)\, e^{-\frac{\lambda_1}{2}(T-a)} \right) \\ &\quad + |b_2| \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_2} e^{-\frac{\lambda_1}{2}(T-a)} + \frac{2}{|\lambda_1|} (T-a)\, e^{-\frac{\lambda_1}{2}(T-a)} \right) \\ &\quad + \frac{2\,(T-a)^{\alpha-1}}{|\lambda_1|\,\Gamma(\alpha-1)} \left(1 + e^{-\frac{\lambda_1}{2}(T-a)} \right) (D\,\|x\| + E) \\ &\le F + G\,\|x\|\,, \end{aligned}$$

which implies that

$$\|x\| \le \frac{F}{1-G}.$$

The value of $R = \frac{F-G+1}{1-G} > \|x\|$ is sufficient for applicability of Leray-Schauder degree theorem. This completes the proof. □

Example 3.1. *Consider the following nonlinear fractional boundary value problem*

$$\begin{cases} \left({}^C D^{2.1}_{0+} - 2\,{}^C D^{1.1}_{0+} + 4\,{}^C D^{0.1}_{0+} \right) x(t) = f(t, x(t)),\ t \in (0,1), \\ x(0) = x'(0) = x''(0) = 1. \end{cases} \tag{3.4}$$

Here $\alpha = 2.1$, $b_0 = b_1 = b_2 = 1$.

(a): *For the illustration of Theorem 3.3, let us take*

$$f(t,x(t)) = \frac{\sin(x)+x}{8}. \tag{3.5}$$

Obviously $lim_{x\to 0}\frac{f(t,x)}{x} = 1/4 = K$. *Thus the conclusion of Theorem 3.3 applies to the problem* (3.4) *with* $f(t,x(t))$ *given by (3.5).*

(b): *In order to explain Theorem 3.5, we consider*

$$f(t,x(t)) = \frac{1}{\sqrt{36+t^2}}\frac{|x|}{(1+|x|)} + \frac{1}{12}. \tag{3.6}$$

With the given data, it is easy to verify that the conditions (H_1) *and* (H_2) *hold true with* $C = \frac{1}{6}$ *and* $\mu(t) = \frac{1}{\sqrt{36+t^2}} + \frac{1}{12}$, *while* (H_3) *is satisfied with* $\omega = \frac{1+e}{6\Gamma(1.1)} < 1$. *Thus all the conditions of Theorem 3.5 are satisfied. Therefore there exists at least one solution for the problem* (3.4) *with* $f(t,x(t))$ *given by (3.6).*

(c) *We illustrate Theorem 3.6 with the aid of the following nonlinear function*

$$f(t,x(t)) = \frac{1}{\sqrt{49+t^2}}\tan^{-1}(x) + \frac{1}{14}. \tag{3.7}$$

Clearly, in this case, $C = \frac{1}{7}$ *and* $\omega = \frac{1+e}{7\Gamma(1.1)} < 1$. *Thus, by the conclusion of Theorem 3.6, the problem* (3.4) *with* $f(t,x(t))$ *given by (3.7) has a unique solution on* $[0,1]$.

(d) *Let us consider the following nonlinear function to demonstrate the application of Theorem 3.7*

$$f(t,x(t)) = \frac{1}{\sqrt{25+t}}\sin(x) + \frac{|x|}{3(1+|x|)} + \frac{1}{3}. \tag{3.8}$$

With the given values, it is found that $D = 1/5$, $E = 2/3$, *and* $G = \frac{1+e}{5\Gamma(1.1)} < 1$. *Thus, by Theorem 3.7, there exists a solution for the problem* (3.4) *with* $f(t,x(t))$ *given by (3.8).*

4. Ulam Stability

Here we investigate the Ulam stability criteria for the problem (1.1) via its equivalent integral equation

$$\begin{aligned} y(t) &= b_0 + b_1\left(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_1}{\lambda_2}e^{-\frac{\lambda_1}{2}(t-a)} - (t-a)\,e^{-\frac{\lambda_1}{2}(t-a)}\right) \\ &\quad + b_2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_2}e^{-\frac{\lambda_1}{2}(t-a)} - \frac{2}{\lambda_1}(t-a)\,e^{-\frac{\lambda_1}{2}(t-a)}\right) \\ &\quad + \frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)}\int_a^t\int_a^s (t-s)\,(s-r)^{\alpha-3}\,e^{\frac{\lambda_1}{2}s} f(r,y(r))drds. \end{aligned} \tag{4.1}$$

If $y \in C^3(J,\mathbb{R})$ and $f: J\times\mathbb{R}\to\mathbb{R}$ is continuous function, then the nonlinear operator $\Upsilon: C(J,\mathbb{R}) \to C(J,\mathbb{R})$ given by

$$\Upsilon y(t) = \left({}^C D^{\alpha}_{a+} + \lambda_1\ {}^C D^{\alpha-1}_{a+} + \lambda_2\ {}^C D^{\alpha-2}_{a+}\right) y(t) - f(t,y(t))$$

is continuous.

Definition 4.1. *The system (1.1) is Ulam-Hyers stable if there exists a real number* $c > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $y \in C^{(3)}(J,\mathbb{R})$,

$$\|\Upsilon y\| \le \epsilon, t \in J, \tag{4.2}$$

then there exists a solution $x \in C(J,\mathbb{R})$ of (1.1) satisfying the inequality:

$$\|x-y\| \le c\epsilon_1,\ t\in J,$$

where ϵ_1 is a positive real number depending on ϵ.

Definition 4.2. *The system (1.1) is generalized Ulam-Hyers stable if there exists* $\sigma \in C(\mathbb{R}^+,\mathbb{R}^+)$ *such that for each solution* $y \in C^{(3)}(J,\mathbb{R})$ *of (1.1) there exists a solution* $x \in C(J,\mathbb{R})$ *of (1.1) with*

$$|x(t)-y(t)| \le \sigma(\epsilon), t \in J.$$

Definition 4.3. *The system (1.1) is Ulam-Hyers-Rassias stable with respect to $\phi \in C(J, \mathbb{R}^+)$ if there exists a real number $c > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C^{(3)}(J, \mathbb{R})$ of (1.1),*

$$|\Upsilon y(t)| \leq \epsilon \phi(t), t \in J, \tag{4.3}$$

then there exists a solution $x \in C(J, \mathbb{R})$ of (1.1) with

$$|x(t) - y(t)| \leq c\epsilon_1 \phi(t), t \in J,$$

where ϵ_1 is a positive real number depending on ϵ.

Theorem 4.1. *Assume that (H_1) and (H_3) hold. Then the system (1.1) satisfies both Ulam-Hyers and generalized Ulam-Hyers stability criteria.*

Proof. Let $x \in C(J, \mathbb{R})$ be a unique solution of (3.1) that satisfies equation (1.1) by Theorem 3.6. Let $y \in C^{(3)}(J, \mathbb{R})$ be any solution satisfying (4.2). Then, by Lemma 2.2, y satisfies the integral equation (4.1). Moreover, the equivalence in Lemma 2.2 implies the equivalence between the operators Υ and $\Psi - I$ on every solution $y \in C(J, \mathbb{R})$ that satisfies equations (1.1) and (4.1). Therefore, by the fixed point property of the operator Ψ (given by (3.1)), we have

$$\begin{aligned} |y(t) - x(t)| &= |y(t) - \Psi y(t) + \Psi y(t) - \Psi x(t)| \\ &\leq |\Psi x(t) - \Psi y(t)| + |\Psi y(t) - y(t)| \\ &\leq \omega \|x - y\| + \epsilon, \end{aligned}$$

where $\epsilon > 0$ and ω is defined in (H_3) . In consequence, it follows that

$$\|x - y\| \leq \frac{\epsilon}{1-\omega}.$$

If we let $\epsilon_1 = \frac{\epsilon}{1-\omega}$, and $c = 1$, then, the Ulam-Hyers stability condition is satisfied. More generally, for $\sigma(\epsilon) = \frac{\epsilon}{1-\omega}$, the generalized Ulam-Hyers stability condition is also satisfied. This completes the proof. □

Theorem 4.2. *Assume that (H_1) and (H_3) hold and there exists a function $\phi \in C(J, \mathbb{R}^+)$ satisfying the condition (4.3). Then the problem (1.1) is Ulam-Hyers-Rassias stable with respect to ϕ.*

Proof. Following the arguments employed in the proof of Theorem 4.1, we can obtain that

$$\|x - y\| \leq \epsilon_1 \phi(t),$$

where $\epsilon_1 = \frac{\epsilon}{1-\omega}$. his completes proof. □

As an application, the problem given by (3.4) with $f(t, x(t)) = \frac{t|x(t)|}{4(1+|x(t)|)}$ is Ulam-Hyers stable, and generalized Ulam-Hyers stable. In addition, If there exists a function $\phi \in C(J, \mathbb{R}^+)$ satisfying the condition (4.3), then the problem (3.4) with the given value of $f(t, x(t))$ is Ulam-Hyers-Rassias stable.

5. Multivalued case

In this section, we study the multivalued (inclusions) analogue of the problem (1.1) given by

$$\begin{cases} \left({}^C D_{a+}^{\alpha} + \lambda_1 \, {}^C D_{a+}^{\alpha-1} + \lambda_2 \, {}^C D_{a+}^{\alpha-2}\right) x(t) \in F(t, x(t)), \\ x^{(k)}(a) = b_k, k = 0, 1, 2, \end{cases} \tag{5.1}$$

where $F : [a, T] \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}, \alpha \in (2, 3)$, $t \in [a, T]$ and the other quantities are the same as defined in the problem (1.1).

Before proceeding for the existence result for the problem (5.1), which relies on Bohnenblust-Karlin fixed point theorem, we outline the background material for multi-valued maps [35, 36].

Let $C[a, T]$ denote a Banach space of continuous functions from $[a, T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t \in [a,T]}\{|x(t)|\}$. Let $L^1([a, T], \mathbb{R})$ be the Banach space of functions $x : [a, T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_a^T |x(t)|dt$.

A multi-valued map $H : X \to 2^X$

(a): is convex (closed) valued if $H(x)$ is convex (closed) for all $x \in X$, where $(X, \|.\|)$ is a Banach space.

(b): is bounded on a bounded set if $H(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for any bounded set $\mathbb{B}$ of X (that is, $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in H(x)\}\} < \infty$).

(c): is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $H(x_0)$ is a nonempty closed subset of X, and if for each open set $\mathbb{B}$ of X containing $H(x_0)$, there exists an open neighborhood $\mathcal{N}$ of x_0 such that $H(\mathcal{N}) \subseteq \mathbb{B}$.

(d): is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every bounded subset $\mathbb{B}$ of X.

(e): has a fixed point if there is $x \in X$ such that $x \in H(x)$.

If the multi-valued map H is completely continuous with nonempty compact values, then H is u.s.c. if and only if H has a closed graph, that is, $x_n \to x_*,\ y_n \to y_*,\ y_n \in H(x_n)$ imply $y_* \in H(x_*)$.

In the following study, $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subset of X. Furthermore, we need the following assumptions:

($\mathbf{M_1}$) Let $F : [a,T] \times \mathbb{R} \to BCC(\mathbb{R}); (t,x) \to f(t,x)$ be measurable with respect to t for each $x \in \mathbb{R}$, u.s.c. with respect to x for a.e. $t \in [a,T]$, and for each fixed $x \in \mathbb{R}$, the set $S_{F,x} := \{f \in L^1([a,T],\mathbb{R}) : f(t) \in F(t,x)$ for a.e. $t \in [a,T]\}$ is nonempty.

($\mathbf{M_2}$) For each $\rho > 0$, there exists a function $p_\rho \in L^1([a,T],\mathbb{R}^+)$ such that $\|F(t,x)\| = \sup\{|v| : v(t) \in F(t,x)\} \le p_\rho(t)$ for each $(t,x) \in [a,T] \times \mathbb{R}$ with $|x| \le \rho$, and

$$\liminf_{\rho\to+\infty} \Big(\frac{\int_a^T p_\rho(t)dt}{\rho}\Big) = \mu < \infty. \tag{5.2}$$

Next we state the known lemmas which we need in the forthcoming analysis.

Lemma 5.1. *(Bohnenblust-Karlin [37]) Let $D \subset X$ be nonempty bounded, closed, and convex. Let $H : D \to 2^X \setminus \{0\}$ be u.s.c. with closed, convex values such that $H(D) \subset D$ and $\overline{H(D)}$ is compact. Then H has a fixed point.*

Lemma 5.2. *[38] Let F be a multi-valued map satisfying the condition (M_1) and ϕ is linear continuous from $L^1(I,\mathbb{R}) \to C(I)$. Then the operator $\phi \circ S_F : C(I) \to BCC(C(I)), x \mapsto (\phi \circ S_F)(x) = \phi(S_{F,x})$ is a closed graph operator in $C(I) \times C(I)$, where I is a compact real interval.*

Theorem 3.1. Assume that (M_1) and (M_2) hold and that

$$\mu < \frac{\Gamma(\alpha-1)}{(T-a)^{\alpha-1}}, \tag{5.3}$$

where μ is given by (5.2). Then there exists at least one solution for the problem (5.1) on $[a,T]$.

Proof. In order to transform the problem (5.1) into a fixed point problem, we introduce a multi-valued map $\Omega : C[a,T] \to 2^{C[a,T]}$ given by

$$\Omega(x) = \Big\{h \in C[a,T] : h(t) = b_0 + b_1\left[\frac{\lambda_1}{\lambda_2} - \Big(\frac{\lambda_1}{\lambda_2} + (t-a)\Big)e^{-\frac{\lambda_1}{2}(t-a)}\right]$$
$$+b_2\left[\frac{1}{\lambda_2} - \Big(\frac{1}{\lambda_2} + \frac{2}{\lambda_1}(t-a)\Big)e^{-\frac{\lambda_1}{2}(t-a)}\right]$$
$$+\frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)}\int_a^t\int_a^s (t-s)(s-r)^{\alpha-3} e^{\frac{\lambda_1}{2}s} f(r,x(r))drds,\ f \in S_{F,x}\Big\}.$$

The proof will be complete once it is shown that Ω satisfies all the assumptions of Lemma 5.1. In consequence, Ω will have a fixed point, showing that the problem (5.1) has a solution.

In the first step, we show that $\Omega(x)$ is convex for each $x \in C[a,T]$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [a,T]$, we have

$$\begin{aligned} h_i(t) &= b_0 + b_1\left[\frac{\lambda_1}{\lambda_2} - \Big(\frac{\lambda_1}{\lambda_2} + (t-a)\Big)e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+b_2\left[\frac{1}{\lambda_2} - \Big(\frac{1}{\lambda_2} + \frac{2}{\lambda_1}(t-a)\Big)e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+\frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)}\int_a^t\int_a^s (t-s)(s-r)^{\alpha-3}\, e^{\frac{\lambda_1}{2}s} f_i(r,x(r))drds,\ i=1,2. \end{aligned}$$

Let $0 \le \sigma \le 1$. Then, for each $t \in [a,T]$, we have

$$\begin{aligned} &[\sigma h_1 + (1-\sigma)h_2](t) = b_0 + b_1\left[\frac{\lambda_1}{\lambda_2} - \Big(\frac{\lambda_1}{\lambda_2} + (t-a)\Big)e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+b_2\left[\frac{1}{\lambda_2} - \Big(\frac{1}{\lambda_2} + \frac{2}{\lambda_1}(t-a)\Big)e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+\frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)}\int_a^t\int_a^s (t-s)(s-r)^{\alpha-3}\, e^{\frac{\lambda_1}{2}s}[\sigma f_1 + (1-\sigma)f_2](r,x(r))drds. \end{aligned}$$

Since $S_{F,x}$ is convex (F has convex values), therefore it follows that $\sigma h_1 + (1-\sigma)h_2 \in \Omega(x)$.
Next we show that there exists a positive number ρ such that $\Omega(B_\rho) \subseteq B_\rho$, where $B_\rho = \{x \in C[a,T] : \|x\| \le \rho\}$. Clearly B_ρ is a bounded closed convex set in $C[a,T]$ for each positive constant ρ. If it is not true, then for each positive number ρ, there exists a function $x_\rho \in B_\rho, h_\rho \in \Omega(x_\rho)$ with $\|\Omega(x_\rho)\| > \rho$, and

$$\begin{aligned} h_r(t) \qquad &= b_0 + b_1\left[\frac{\lambda_1}{\lambda_2} - \Big(\frac{\lambda_1}{\lambda_2} + (t-a)\Big)e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+b_2\left[\frac{1}{\lambda_2} - \Big(\frac{1}{\lambda_2} + \frac{2}{\lambda_1}(t-a)\Big)e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+\frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)}\int_a^t\int_a^s (t-s)(s-r)^{\alpha-3}\, e^{\frac{\lambda_1}{2}s} p_\rho(r)drds,\ \text{for some } p_\rho \in S_{F,x_\rho}. \end{aligned}$$

On the other hand, in view of (A_2), we have

$$\begin{aligned} r \quad < \quad & \|\Omega(x_r)\| \le |b_0| + |b_1|\left(\frac{|\lambda_1|}{\lambda_2} + \frac{|\lambda_1|}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)} + (T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right) \\ &+|b_2|\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)} + \frac{2}{|\lambda_1|}(T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right) \\ &+\frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)}\int_a^t\int_a^s (t-s)(s-r)^{\alpha-3}\, e^{\frac{\lambda_1}{2}s} p_\rho(r)drds \\ \le \quad & |b_0| + |b_1|\left(\frac{|\lambda_1|}{\lambda_2} + \frac{|\lambda_1|}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)} + (T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right) \\ &+|b_2|\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_2}e^{-\frac{\lambda_1}{2}(T-a)} + \frac{2}{|\lambda_1|}(T-a)\,e^{-\frac{\lambda_1}{2}(T-a)}\right) \\ &+\frac{(T-a)^{(\alpha-1)}}{\Gamma(\alpha-1)}\int_a^T p_\rho(s)ds. \end{aligned}$$

Dividing both sides by ρ and taking the lower limit as $\rho \to \infty$, we find that

$$\mu \ge \frac{\Gamma(\alpha-1)}{(T-a)^{\alpha-1}},$$

which contradicts (5.3). Hence there exists a positive number ρ_1 such that $\Omega(B_{\rho_1}) \subseteq B\rho_1$.

Now we show that $\Omega(B_\rho)$ is equicontinuous. Let $a < t_1 < t_2 < T$, $x \in B_\rho$ and $h \in \Omega(x)$, then there exists $f \in S_{F,x}$ such that for each $t \in [a, T]$, we have

$$\begin{aligned} h(t) &= b_0 + b_1 \left[\frac{\lambda_1}{\lambda_2} - \Big(\frac{\lambda_1}{\lambda_2} + (t-a)\Big) e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+ b_2 \left[\frac{1}{\lambda_2} - \Big(\frac{1}{\lambda_2} + \frac{2}{\lambda_1}(t-a)\Big) e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+ \frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)} \int_a^t \int_a^s (t-s)(s-r)^{\alpha-3} e^{\frac{\lambda_1}{2}s} f(r) dr ds \Big\} \end{aligned}$$

and

$$\begin{aligned} &|h(t_2) - h(t_1)| \le \frac{|b_1 \lambda_1|}{\lambda_2} \left| e^{-\frac{\lambda_1}{2}(t_1 - t_2)} - 1 \right| e^{-\frac{\lambda_1}{2}(t_2 - a)} \\ &+ |b_1| \left| (t_1 - t_2) e^{-\frac{\lambda_1}{2}(t_1 - t_2)} + (t_2 - a) \left(e^{-\frac{\lambda_1}{2}(t_1 - t_2)} - 1 \right) \right| e^{-\frac{\lambda_1}{2}(t_2 - a)} \\ &+ \frac{|b_2|}{\lambda_2} \left| e^{-\frac{\lambda_1}{2}(t_1 - t_2)} - 1 \right| e^{-\frac{\lambda_1}{2}(t_2 - a)} \\ &+ \frac{|b_2|}{\lambda_2} \left| (t_1 - t_2) e^{-\frac{\lambda_1}{2}(t_1 - t_2)} + (t_2 - a) \left(e^{-\frac{\lambda_1}{2}(t_1 - t_2)} - 1 \right) \right| e^{-\frac{\lambda_1}{2}(t_2 - a)} \end{aligned}$$

$$\begin{aligned} &+ \frac{e^{-\frac{\lambda_1}{2}t_1} e^{-\frac{\lambda_1}{2}(t_2 - t_1)}}{\Gamma(\alpha-2)} \int_a^{t_1} \int_a^s (t_2 - t_1)(s-r)^{\alpha-3} e^{\frac{\lambda_1}{2}s} p_\rho(s) dr ds \\ &+ \frac{e^{-\frac{\lambda_1}{2}t_1} \left(e^{-\frac{\lambda_1}{2}(t_2 - t_1)} - 1 \right)}{\Gamma(\alpha-2)} \int_a^{t_1} \int_a^s (t_1 - s)(s-r)^{\alpha-3} e^{\frac{\lambda_1}{2}s} p_\rho(s) dr ds \\ &+ \frac{e^{-\frac{\lambda_1}{2}t_1} e^{-\frac{\lambda_1}{2}(t_2 - t_1)}}{\Gamma(\alpha-2)} \int_{t_1}^{t_2} \int_a^s (t_2 - s)(s-r)^{\alpha-3} e^{\frac{\lambda_1}{2}s} p_\rho(s) dr ds, \end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $t_2 \to t_1$. Thus, Ω is equi-continuous. As Ω satisfies the above three assumptions, therefore it follows by Ascoli-Arzela theorem that Ω is a compact multi-valued map.
Finally, we show that Ω has a closed graph. Let $x_n \to x_*, h_n \in \Omega(x_n)$ and $h_n \to h_*$. We will show that $h_* \in \Omega(x_*)$. By the relation $h_n \in \Omega(x_n)$, we mean that there exists $f_n \in S_{F,x_n}$ such that for each $t \in [a, T]$,

$$\begin{aligned} h_n(t) &= b_0 + b_1 \left[\frac{\lambda_1}{\lambda_2} - \Big(\frac{\lambda_1}{\lambda_2} + (t-a)\Big) e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+ b_2 \left[\frac{1}{\lambda_2} - \Big(\frac{1}{\lambda_2} + \frac{2}{\lambda_1}(t-a)\Big) e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+ \frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)} \int_a^t \int_a^s (t-s)(s-r)^{\alpha-3} e^{\frac{\lambda_1}{2}s} f_n(r) dr ds \Big\}. \end{aligned}$$

Thus we need to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [a, T]$,

$$\begin{aligned} h_*(t) &= b_0 + b_1 \left[\frac{\lambda_1}{\lambda_2} - \Big(\frac{\lambda_1}{\lambda_2} + (t-a)\Big) e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+ b_2 \left[\frac{1}{\lambda_2} - \Big(\frac{1}{\lambda_2} + \frac{2}{\lambda_1}(t-a)\Big) e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+ \frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)} \int_a^t \int_a^s (t-s)(s-r)^{\alpha-3} e^{\frac{\lambda_1}{2}s} f_*(r) dr ds \Big\}. \end{aligned}$$

Let us consider the continuous linear operator $\phi : L^1[a,T],\mathbb{R}) \to C[a,T]$ so that

$$\begin{aligned} f \mapsto \phi(f)(t) \qquad &= b_0 + b_1 \left[\frac{\lambda_1}{\lambda_2} - \Big(\frac{\lambda_1}{\lambda_2} + (t-a)\Big) e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+ b_2 \left[\frac{1}{\lambda_2} - \Big(\frac{1}{\lambda_2} + \frac{2}{\lambda_1}(t-a)\Big) e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+ \frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)} \int_a^t \int_a^s (t-s)(s-r)^{\alpha-3} e^{\frac{\lambda_1}{2}s} f(r) drds\Big\}. \end{aligned}$$

Observe that

$$\begin{aligned} &\|h_n(t) - h_*(t)\| \\ &= \int_a^t \int_a^s (t-s)(s-r)^{\alpha-3} e^{\frac{\lambda_1}{2}s} \big(f_n(r) - f_*(r)\big) drds \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, it follows by Lemma 2.2 that $\phi \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \phi(S_{F,x_n})$. Since $x_n \to x_*$, therefore, Lemma 2.2 yields

$$\begin{aligned} h_*(t) \quad = \quad & b_0 + b_1 \left[\frac{\lambda_1}{\lambda_2} - \Big(\frac{\lambda_1}{\lambda_2} + (t-a)\Big) e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+ b_2 \left[\frac{1}{\lambda_2} - \Big(\frac{1}{\lambda_2} + \frac{2}{\lambda_1}(t-a)\Big) e^{-\frac{\lambda_1}{2}(t-a)}\right] \\ &+ \frac{e^{-\frac{\lambda_1}{2}t}}{\Gamma(\alpha-2)} \int_a^t \int_a^s (t-s)(s-r)^{\alpha-3} e^{\frac{\lambda_1}{2}s} f_*(r) drds\Big\}. \end{aligned}$$

Hence, we conclude that Ω is a compact multi-valued map, u.s.c. with convex closed values. Thus, all the assumptions of Lemma 5.1 are satisfied and so by the conclusion of Lemma 5.1, Ω has a fixed point x which is a solution of problem (5.1). This completes the proof. □

6. Coupled system of equations

In this section, we study an initial value problem of coupled sequential fractional differential equations given by

$$\begin{cases} \left({}^C D_{a+}^{\alpha_1} + \lambda_{11}\ {}^C D_{a+}^{\alpha_1 - 1} + \lambda_{21}\ {}^C D_{a+}^{\alpha_1 - 2}\right) u_1(t) = f_1(t, u_1(t), u_2(t)), \\ \left({}^C D_{a+}^{\alpha_2} + \lambda_{12}\ {}^C D_{a+}^{\alpha_2 - 1} + \lambda_{22}\ {}^C D_{a+}^{\alpha_2 - 2}\right) u_2(t) = f_2(t, u_1(t), u_2(t)), \\ u_1^{(k)}(a) = b_{k1},\ u_2^{(k)}(a) = b_{k2},\ k = 0,1,2 \end{cases} \tag{6.1}$$

where $\alpha_i \in (2,3), t \in J, \lambda_{ij} \in \mathbb{R}$, $\lambda_{1j}^2 = 4\lambda_{2j}$ and $f_i : J \times \mathbb{R}^2 \to \mathbb{R}$ $(i,j = 1,2)$ are continuous function satisfying the following condition.

(S_1)**:** There exist $C_i \in \mathbb{R}^+$ such that

$$|f_i(t,u_1,u_2) - f_i(t,v_1,v_2)| \le C_i(|u_1 - v_1| + |u_2 - v_2|), u_i, v_i \in \mathbb{R}, t \in J.$$

Consider the Banach product space $Y = C(J,\mathbb{R}) \times C(J,\mathbb{R})$ of all ordered pairs (x,y) such that $x, y \in C(J,\mathbb{R})$, and equipped with the norm $\|(x,y)\|_Y = \|x\| + \|y\|$. In view of Lemma 2.2, we define an operator $\Lambda : Y \to Y$ by

$$\Lambda(u_1,u_2)(t) = \left(\Lambda_1(u_1,u_2)(t), \Lambda_2(u_1,u_2)(t)\right), t \in J,$$

where

$$\begin{aligned} &\Lambda_i(u_1,u_2)(t) = b_{0i} + b_{1i}\left(\frac{\lambda_{1i}}{\lambda_{2i}} - \frac{\lambda_{1i}}{\lambda_{2i}} e^{-\frac{\lambda_{1i}}{2}(t-a)} - (t-a)\, e^{-\frac{\lambda_{1i}}{2}(t-a)}\right) \\ &+ b_{2i}\left(\frac{1}{\lambda_{2i}} - \frac{1}{\lambda_{2i}} e^{-\frac{\lambda_{1i}}{2}(t-a)} - \frac{2}{\lambda_{1i}}(t-a)\, e^{-\frac{\lambda_{1i}}{2}(t-a)}\right) \\ &+ \frac{e^{-\frac{\lambda_{1i}}{2}t}}{\Gamma(\alpha_i - 2)} \int_a^t \int_a^s (t-s)(s-r)^{\alpha_i - 3} e^{\frac{\lambda_{1i}}{2}s} f_i(r, u_1(r), u_2(r)) drds,\ i = 1,2. \end{aligned}$$

Theorem 6.1. *Assume that* (S_1) *is satisfied. Then there exists a unique solution for the problem (6.1) on* J *whenever* $\beta_1+\beta_2<1$, *where*

$$\beta_i=\frac{2C_i\,(T-a)^{\alpha_i-1}}{|\lambda_{1i}|\,\Gamma(\alpha_i-1)}\left(1+e^{-\frac{\lambda_{1i}}{2}(T-a)}\right),\ i=1,2.$$

Proof. Let us set $\sup_{t\in J}|f_i(t,0,0)|=A_i$, and $r\geq(1-\beta_1-\beta_2)^{-1}(\gamma_1+\gamma_2)$, where

$$\begin{aligned}\gamma_i \;=\;& |b_{0i}|+|b_{1i}|\left(\frac{|\lambda_{1i}|}{\lambda_{2i}}+\frac{|\lambda_{1i}|}{\lambda_{2i}}e^{-\frac{\lambda_{1i}}{2}(T-a)}+(T-a)\,e^{-\frac{\lambda_{1i}}{2}(T-a)}\right)\\ &+|b_{2i}|\left(\frac{1}{\lambda_{2i}}+\frac{1}{\lambda_{2i}}e^{-\frac{\lambda_{1i}}{2}(T-a)}+\frac{2}{|\lambda_{1i}|}(T-a)\,e^{-\frac{\lambda_{1i}}{2}(T-a)}\right)\\ &+\frac{2A_i\,(T-a)^{\alpha_i-1}}{|\lambda_{1i}|\,\Gamma(\alpha_i-1)}\left(1+e^{-\frac{\lambda_{1i}}{2}(T-a)}\right),\ i=1,2.\end{aligned}$$

Then

$$\begin{aligned}|\Lambda_i(u_1,u_2)(t)| \;\leq\;& |b_{0i}|+|b_{1i}|\left(\frac{|\lambda_{1i}|}{\lambda_{2i}}+\frac{|\lambda_{1i}|}{\lambda_{2i}}e^{-\frac{\lambda_{1i}}{2}(T-a)}+(T-a)\,e^{-\frac{\lambda_{1i}}{2}(T-a)}\right)\\ &+|b_{2i}|\left(\frac{1}{\lambda_{2i}}+\frac{1}{\lambda_{2i}}e^{-\frac{\lambda_{1i}}{2}(T-a)}+\frac{2}{|\lambda_{1i}|}(T-a)\,e^{-\frac{\lambda_{1i}}{2}(T-a)}\right)\\ &+\frac{2A_i\,(T-a)^{\alpha_i-1}}{|\lambda_{1i}|\,\Gamma(\alpha_i-1)}\left(1+e^{-\frac{\lambda_{1i}}{2}(T-a)}\right)\\ &+\frac{2C_i\,(T-a)^{\alpha_i-1}}{|\lambda_{1i}|\,\Gamma(\alpha_i-1)}\left(1+e^{-\frac{\lambda_{1i}}{2}(T-a)}\right)(\|u_1\|+\|u_2\|)\\ \leq\;& \gamma_i+\beta_i\,(\|u_1\|+\|u_2\|)\,.\end{aligned}$$

Taking the norm of the above inequality for $t\in J$, it easily follows that $\Lambda\mathcal{B}_r\subset\mathcal{B}_r$, where $\mathcal{B}_r=\{(x,y)\in Y:\|(x,y)\|_Y\leq r\}$. Moreover, for $(u_1,u_2),(v_1,v_2)\in Y$ and for each $t\in J$, we have

$$\begin{aligned}&|\Lambda(u_1,u_2)(t)-\Lambda(v_1,v_2)(t)|\\ \leq\;& |(\Lambda_1(u_1,u_2)(t)-\Lambda_1(v_1,v_2)(t)|+|(\Lambda_2(u_1,u_2)(t)-\Lambda_2(v_1,v_2)(t)|\\ \leq\;& \frac{2C_1\,(T-a)^{\alpha_1-1}}{|\lambda_{11}|\,\Gamma(\alpha_1-1)}\left(1+e^{-\frac{\lambda_{11}}{2}(T-a)}\right)(\|u_1-v_1\|+\|u_2-v_2\|)\\ &+\frac{2C_2\,(T-a)^{\alpha_2-1}}{|\lambda_{12}|\,\Gamma(\alpha_2-1)}\left(1+e^{-\frac{\lambda_{12}}{2}(T-a)}\right)(\|u_1-v_1\|+\|u_2-v_2\|)\\ \leq\;& (\beta_1+\beta_2)\,(\|u_1-v_1\|+\|u_2-v_2\|)\,,\end{aligned}$$

which, on taking the norm for $t\in J$ and using the condition $\beta_1+\beta_2<1$, implies that Λ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof. □

Our second result is based on Leray-Schauder alternative [39].

Theorem 6.2. *(Leray–Schauder alternative) Let* $F:E\to E$ *be a completely continuous operator and*

$$\mathcal{E}(F)=\{x\in E:x=\lambda F(x),\ \textit{for some } 0<\lambda<1\}\,.$$

Then either the set $\mathcal{E}(F)$ *is unbounded, or* F *has at least one fixed point.*

In the sequel, we need the following growth condition:

(S_2)**:** There exist $K_i\in\mathbb{R}^+$ such that

$$|f_i(t,u_1,u_2)|\leq K_i(1+|u_1|+|u_2|),u_i\in\mathbb{R},t\in J,i=1,2.$$

For computational convenience, we define

$$D_i = \frac{2K_i\,(T-a)^{\alpha_i-1}}{|\lambda_{1i}|\,\Gamma(\alpha_i-1)}\left(1+e^{-\frac{\lambda_{1i}}{2}(T-a)}\right), \tag{6.2}$$

$$\begin{aligned} B_i &= D_i+|b_{0i}|+|b_{1i}|\left(\frac{|\lambda_{1i}|}{\lambda_{2i}}+\frac{|\lambda_{1i}|}{\lambda_{2i}}e^{-\frac{\lambda_{1i}}{2}(T-a)}+(T-a)\,e^{-\frac{\lambda_{1i}}{2}(T-a)}\right)\\ &\quad+|b_{2i}|\left(\frac{1}{\lambda_{2i}}+\frac{1}{\lambda_{2i}}e^{-\frac{\lambda_{1i}}{2}(T-a)}+\frac{2}{|\lambda_{1i}|}(T-a)\,e^{-\frac{\lambda_{1i}}{2}(T-a)}\right). \end{aligned} \tag{6.3}$$

Theorem 6.3. *Assume that* (S_2) *is satisfied. Then there exists at least one solution for the problem (6.1) on* J *whenever* $D_1+D_2<1$

Proof. Clearly continuity of f_i $(i=1,2)$ implies the continuity of Λ_i and hence the continuity of Λ. Let $\overline{\mathcal{U}}$ be a bounded proper subset of Y, there exist positive constants $\overline{L}_1$ and $\overline{L}_2$ such that $|f_i(t,u_1(t),u_2(t))|\leq \overline{L}_i$ for $t\in J,(u_1,u_2)\in Y$. Following the procedure of the proof in Theorem 3.1, one can show that the operator $\Lambda:Y\to Y$ is completely continuous. Let $(u_1,u_2)\in\mathcal{E}(\Lambda)$, such that $(u_1,u_2)=\lambda\Lambda(u_1,u_2)$. For any $t\in J$, $u_1(t)=\lambda\Lambda_1(u_1,u_2)$, and $u_2(t)=\lambda\Lambda_2(u_1,u_2)$. Then, using the assumption (S_2) and (6.2)-(6.3), we obtain

$$\begin{aligned} |u_i(t)| &\leq |\Lambda_i(u_1,u_2)(t)|\\ &\leq |b_{0i}|+|b_{1i}|\left(\frac{|\lambda_{1i}|}{\lambda_{i2}}+\frac{|\lambda_{1i}|}{\lambda_{2i}}e^{-\frac{\lambda_{1i}}{2}(T-a)}+(T-a)\,e^{-\frac{\lambda_{1i}}{2}(T-a)}\right)\\ &\quad+|b_{i2}|\left(\frac{1}{\lambda_{2i}}+\frac{1}{\lambda_{2i}}e^{-\frac{\lambda_{1i}}{2}(T-a)}+\frac{2}{|\lambda_{1i}|}(T-a)\,e^{-\frac{\lambda_{1i}}{2}(T-a)}\right)\\ &\quad+\frac{2K_i\,(T-a)^{\alpha_i-1}}{|\lambda_{1i}|\,\Gamma(\alpha_i-1)}\left(1+e^{-\frac{\lambda_{1i}}{2}(T-a)}\right)(1+\|u_1\|+\|u_2\|)\\ &\leq B_i+D_i\,(\|u_1\|+\|u_2\|)\,. \end{aligned}$$

Taking the norm of the above inequality for $t\in J$, it follows in a straightforward manner that

$$\|(u_1,u_2)\|\leq\frac{B_1+B_2}{1-D_1-D_2}.$$

Hence the set $\mathcal{E}(\Lambda)=\{(u_1,u_2)\in Y:(u_1,u_2)=\lambda\Lambda(u_1,u_2)\text{ for some }0<\lambda<1\}$ is bounded. By the application of Leray-Schauder alternative Theorem, we deduce that the problem (6.1) has at least one solution on J. This completes the proof. □

References

[1] F. Meral, T. Royston and R. Magin, Fractional calculus in viscoelasticity: an experimental study, Commun. Nonlinear Sci. Numer. Simul. 15 (2010), 939-945.
[2] K. Oldham, Ractional differential equations in electrochemistry, Adv. Eng. Softw. 41 (2010), 9-12.
[3] C. Lee and F. Chang, Fractional-order PID controller optimization via improved electromagnetism-like algorithm, Expert Syst. Appl. 37 (2010), 8871-8878.
[4] E. Ahmed, A. El-Sayed and H. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models, J. Math. Anal. Appl. 325 (2007), 542-553.
[5] F. Liu and K. Burrage, Novel techniques in parameter estimation for fractional dynamical models arising from biological systems, Comput. Math. Appl. 62 (2011), 822-833.
[6] G. Mophou, Optimal control of fractional diffusion equation, Comput. Math. Appl. 61 (2011), 68-78.
[7] J. Wang, Y. Zhou and W. Wei, Optimal feedback control for semilinear fractional evolution equations in Banach spaces, Syst. Control Lett. 61 (2012), 472-476.
[8] R. Gorenflo and F. Mainardi, Some recent advances in theory and simulation of fractional diffusion processes, J. Comput. Appl. Math. 229 (2009), 400-415.
[9] X. Jiang, M. Xu and H. Qi, The fractional diffusion model with an absorption term and modified Fick's law for non-local transport processes, Nonlinear Anal. Real World Appl. 11 (2010), 262-269.
[10] I. Sokolov, A. Chechkin and J. Klafter, Fractional diffusion equation for a power-law truncated Levy process, Physica A. 336 (2004), 245-251.
[11] R. Nigmatullin, T. Omay and D. Baleanu, On fractional filtering versus conventional filtering in economics, Commun. Nonlinear Sci. Numer. Simul. 15 (2010), 979-986.
[12] M. Faieghi, S. Kuntanapreeda, H. Delavari, D. Baleanu, LMI-based stabilization of a class of fractional-order chaotic systems, *Nonlinear Dynam.* 72 (2013), 301-309.

[13] F. Zhang, G. Chen, C. Li, J. Kurths, Chaos synchronization in fractional differential systems, Phil. Trans. R. Soc. A 371 (2013), 20120155.

[14] K. Balachandran, M. Matar, J. J. Trujillo, Note on controllability of linear fractional dynamical systems, J. Control Decis. 3 (2016), 267-279.

[15] O.P. Agrawal, Generalized Variational Problems and Euler-Lagrange equations, Comput. Math. Appl. 59 (2010), 1852-1864.

[16] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[17] Y. Zhou, Basic Theory of Fractional Differential Equations, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014.

[18] B. Ahmad, M. M Matar, R. P. Agarwal, Existence results for fractional differential equations of arbitrary order with nonlocal integral boundary conditions, Bound. Value Probl. 2015:220 (2015), 13 pp.

[19] B. Ahmad, M. M. Matar, S. K. Ntouyas, On general fractional differential inclusions with nonlocal integral boundary conditions, Differ. Equ. Dyn. Syst. (2016), DOI:10.1007/s12591-016-0319-5.

[20] M. Matar, On Existence of positive solution for initial value problem of nonlinear fractional differential equations of order $1 < \alpha \leq 2$, Acta Math. Univ. Comenianae, 84 (1) (2015), 51-57.

[21] L. Zhang, B. Ahmad, G. Wang, Explicit iterations and extremal solutions for fractional differential equations with nonlinear integral boundary conditions, Appl. Math. Comput. 268 (2015), 388-392.

[22] D. Qarout, B. Ahmad, A. Alsaedi, Existence theorems for semilinear Caputo fractional differential equations with nonlocal discrete and integral boundary conditions, Fract. Calc. Appl. Anal. 19 (2016), 463-479.

[23] B. Ahmad, Sharp estimates for the unique solution of two-point fractional-order boundary value problems, Appl. Math. Lett. 65 (2017), 77-82.

[24] S. Aljoudi, B. Ahmad, J.J. Nieto, A. Alsaedi, A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions, Chaos Solitons Fractals 91 (2016), 39-46.

[25] B. Ahmad, S.K. Ntouyas, A. Alsaedi, Fractional differential equations and inclusions with nonlocal generalized Riemann-Liouville integral boundary conditions, Int. J. Anal. Appl. 13 (2017), 231-247.

[26] H.M. Srivastava, Remarks on some families of fractional-order differential equations, Integral Transforms Spec. Funct. 28 (2017), 560-564.

[27] W. Deng, Smoothness and stability of the solutions for nonlinear fractional differential equations, Nonlinear Anal. 72 (2010), 1768-1777.

[28] R. W. Ibrahim, Stability of a fractional differential equation, Miskolc Math. Notes 13 (2012), 39-52.

[29] R. Agarwal, S. Hristova and D. O'Regan, A survey of Lyapunov functions, stability and impulsive Caputo fractional differential equations, Fract. Calc. Appl. Anal. 19 (2016), 290-318.

[30] N. Brillouët-Belluot, J. Brzdęk, and K. Ciepliński, On some recent developments in Ulam's type stability, Abstr. Appl. Anal. (2012), Art. ID 716936, 41 pp.

[31] J. Wang, L. Lv and Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, Electron. J. Qual. Theory Differ. Equ. 2011 (2011), Art. ID 63.

[32] S. Abbas, M. Benchohra and A. Petrusel, Ulam stability for partial fractional differential inclusions via Picard operators theory, Electron. J. Qual. Theory Differ. Equ. 2014 (2014), Art. ID 51.

[33] R.W. Ibrahim and H.A. Jalab, Existence of Ulam stability for iterative fractional differential equations based on fractional entropy, Entropy 17 (2015), 3172-3181.

[34] D.R. Smart, Fixed Point Theorems, Cambridge Tracts in Mathematics, No. 66. Cambridge University Press, London-New York, 1974.

[35] K. Deimling, Multivalued Differential Equations, De Gruyter, Berlin, 1992.

[36] S. Hu and N. Papageorgiou, Handbook of Multivalued Analysis, Vol. I. Theory. Mathematics and its Applications, 419. Kluwer Academic Publishers, Dordrecht, 1997.

[37] H. F. Bohnenblust and S. Karlin, On a theorem of Ville, In Contributions to the Theory of Games. Vol. I, pp. 155-160, Princeton Univ. Press, 1950.

[38] A. Lasota and Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, Bull. Acad. Polon. Sci. Ser.Sci. Math. Astronom. Phys. 13 (1965), 781-786.

[39] A. Granas and J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2003.

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