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COSINE INTEGRALS FOR THE CLAUSEN FUNCTION AND ITS FOURIER SERIES EXPANSION

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Abstract. In a recent work, on taking into account certain finite sums of trigonometric functions I have derived exact closed-form results for some non-trivial integrals, including $\int_0^\pi \sin(k\,\theta)\,\mathrm{Cl}_2(\theta)\,d\theta$, where k is a positive integer and $\mathrm{Cl}_2(\theta)$ is the Clausen function. There in that paper, I pointed out that this integral has the form of a Fourier coefficient, which suggest that its cosine version $\int_0^\pi \cos(k\,\theta)\,\mathrm{Cl}_2(\theta)\,d\theta$, $k \ge 0$, is worthy of consideration, but I could only present a few conjectures at that time. Here in this note, I derive exact closed-form expressions for this integral and then I show that they can be taken as Fourier coefficients for the series expansion of a periodic extension of $\mathrm{Cl}_2(\theta)$. This yields new closed-form results for a series involving harmonic numbers and a partial derivative of a generalized hypergeometric function.

1. Introduction

In its more general form, the Fourier series expansion of a periodic real function $f(x)$ of period L is conventionally written as (see, e.g., Sec. 4.2 of Ref. [6])

$$S[f(x)] := \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \, \cos\!\left(\frac{2\pi k x}{L}\right) + b_k \, \sin\!\left(\frac{2\pi k x}{L}\right) \right], \tag{1.1}$$

where

$$a_k = \frac{2}{L} \int_{x_0}^{x_0+L} f(x)\, \cos(2\pi k x/L)\, dx \,, \quad k \ge 0\,, \tag{1.2a}$$

$$b_k = \frac{2}{L} \int_{x_0}^{x_0+L} f(x)\, \sin(2\pi k x/L)\, dx \,, \quad k > 0\,, \tag{1.2b}$$

are the Fourier coefficients and x_0 is an arbitrary constant (often taken as 0). As is well-known, if $f(x)$ satisfies the Dirichlet conditions then this series converges to $f(x)$ at all points of continuity of $f(x)$ and to the average of $f(x)$ taken at the lateral limits of x if it is a point of finite discontinuity. In fact, the *periodicity condition* is irrelevant for pointwise convergence in the finite domain $[x_0, x_0 + L]$, as shown by Connon in Ref. [2], which is important for the Fourier expansion of non-periodic functions using periodic extensions.

In a very recent work, by taking into account certain finite sums involving trigonometric functions at rational multiples of π, I have derived exact closed-form expressions for some non-trivial integrals [5]. Among them, I showed in Theorem 6 of Ref. [5] that

$$\frac{2}{\pi} \int_0^\pi \sin(k\,\theta)\, \mathrm{Cl}_2(\theta)\, d\theta = \frac{1}{k^2} \tag{1.3}$$

holds for every integer $k > 0$. Here, $\mathrm{Cl}_2(\theta) := \Im\left\{\mathrm{Li}_2\left(e^{i\,\theta}\right)\right\}$ is the Clausen function, $\mathrm{Li}_2(z) := \sum_{n=1}^\infty z^n/\,n^2$, $|z| \le 1$, being the dilogarithm function [4, Sec. 1.1]. Clausen himself proved in Ref. [1] that $\mathrm{Cl}_2(\theta) = -\int_0^\theta \ln|2\,\sin(t/2)|\,dt$, which is known as the *Clausen integral* [3, Sec. 4.1]. Since the

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integral in Eq. (1.3) resembles that of Fourier coefficient b_k in Eq. (1.2b), then a natural follow-up is the investigation of the corresponding *cosine* integral, i.e.

$$A_k := \frac{2}{\pi} \int_0^{\pi} \cos(k\,\theta)\,\mathrm{Cl}_2(\theta)\,d\theta\,, \quad k \ge 0\,. \tag{1.4}$$

However, there in Eqs. (25)–(30) of Ref. [5] I could only conjecture, based upon strong numerical evidence, a few simple results for small values of k. They of course suggest a pattern, but there in Ref. [5] I could not find it out.

In this note, I make use of a well-known series expansion for $\mathrm{Cl}_2(\theta)$ to derive closed-form expressions for A_k, one for $k=0$ and another for $k>0$. I then use these results to obtain a Fourier series for a suitable periodic extension of $\mathrm{Cl}_2(\theta)$, which yields new closed-form results.

2. Cosine integrals of Clausen function

In what follows, we shall make use of a well-known series representation for $\mathrm{Cl}_2(\theta)$.

Lemma 1 (Clausen series for $\mathrm{Cl}_2(\theta)$)**.** *The trigonometric series* $\sum_{n=1}^{\infty} \dfrac{\sin(n\,\theta)}{n^2}$ *converges to* $\mathrm{Cl}_2(\theta)$ *for all* $\theta \in \mathbb{R}$.

Proof. This series representation of $\mathrm{Cl}_2(\theta)$ remounts to Clausen's original work (1832) [1], but, for completeness, let us present a proof based on Fourier series. In Theorem 3 of Ref. [7], a recent note on Fourier series by Zhang, it is shown that, given a real function $f(x)$ integrable on $[0,L]$ and such that $f(x) = -f(L-x)$ for all $x \in (L/2, L]$, if $f(x)$ is an odd function in $(-L,L)$, then

$$f(x) = \sum_{n=1}^{\infty} c_{2n}\,\sin\!\left(\frac{2n\pi x}{L}\right) \tag{2.1}$$

for all $x \in [-L,L]$ where $f(x)$ is a continuous function. Here,

$$c_{2n} = \frac{4}{L}\int_0^{L/2} f(t)\,\sin\!\left(\frac{2n\pi t}{L}\right)\,dt\,. \tag{2.2}$$

Since $\mathrm{Cl}_2(\theta)$ is an odd function which is continuous (thus integrable) on $(-2\pi, 2\pi)$ and $\mathrm{Cl}_2(\theta) = -\mathrm{Cl}_2(2\pi-\theta)$ [3, Secs. 4.2 and 4.3], then the convergence of $\sum_{n=1}^{\infty} \sin(n\,\theta)/n^2$ to $\mathrm{Cl}_2(\theta)$ follows by taking $L = 2\pi$ in Zhang's theorem and noting that $c_{2n} = 1/n^2$, as seen in Eq. (1.3). Finally, the periodicity of $\mathrm{Cl}_2(\theta)$, as established in Sec. 4.2 of Ref. [3], extends the convergence to all $\theta \in \mathbb{R}$. □

Let us begin our main results with the integral A_k for $k=0$.

Theorem 1 (Integral A_0)**.** *The exact closed-form result*

$$A_0 := \frac{2}{\pi}\int_0^{\pi} \mathrm{Cl}_2(\theta)\,d\theta = \frac{7}{2}\,\frac{\zeta(3)}{\pi}\,,$$

where $\zeta(3) := \sum_{n=1}^{\infty} 1/n^3$ *is the Apéry's constant, holds.*

Proof. From Lemma 1, one has

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi} \mathrm{Cl}_2(\theta)\,d\theta &= \frac{2}{\pi}\int_0^{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2}\,d\theta = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\int_0^{\pi}\sin(n\theta)\,d\theta}{n^2}\\ &= -\,\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(n\theta)\,\big|_0^{\pi}}{n^3} = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^3} = \frac{4}{\pi}\sum_{\text{odd}}\frac{1}{n^3}\,,\end{aligned} \tag{2.3}$$

where the last sum takes only the odd values of n into account. The interchange of the integral and the series is allowed because this series converges absolutely. Since $\zeta(3) = \sum_{\text{odd}} 1/n^3 + \sum_{\text{even}} 1/n^3$ and $\sum_{\text{even}} 1/n^3 = \sum_{m=1}^{\infty} 1/(2m)^3 = \frac{1}{8}\,\zeta(3)$, then $\sum_{\text{odd}} 1/n^3 = \frac{7}{8}\,\zeta(3)$. □

Now, let us derive a general result valid for all integrals A_k, $k>0$. For this, it will be useful to define $h_n := \sum_{\ell=1}^{n} 1/(2\ell-1)$, n being a positive integer, which is the *odd analogue* of the harmonic number $H_n := \sum_{\ell=1}^{n} 1/\ell$. Since $h_{\lceil n/2\rceil} = H_n - \frac{1}{2}\,H_{\lfloor n/2\rfloor}$, it is easy to rewrite any expression containing h_n in terms of the usual harmonic numbers.

Theorem 2 (Integral A_k, $k > 0$)**.** *Let* A_k *be the integral defined in Eq.* (1.4)*. The exact closed-form result*

$$A_k = \begin{cases} \dfrac{2}{\pi} \, \dfrac{\ln 4 - 2\, h_{\lfloor k/2 \rfloor} - 1/k}{k^2} \, , & k \text{ odd} \\ -\dfrac{4}{\pi} \, \dfrac{h_{k/2}}{k^2} \, , & k \text{ even} \, , \end{cases}$$

holds for all integers $k > 0$.

Proof. From Lemma 1, one has

$$A_k = \frac{2}{\pi} \int_0^{\pi} \cos{(k\theta)} \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2} \, d\theta = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\int_0^{\pi} \cos{(k\theta)} \, \sin(n\theta) \, d\theta}{n^2} \, , \tag{2.4}$$

where k is a positive integer. On applying the trigonometric identity $\sin\alpha \, \cos\beta = \frac{1}{2} \, [\sin{(\alpha+\beta)} + \sin{(\alpha-\beta)}]$ to the last integral, one finds

$$I_{kn} := \int_0^{\pi} \cos{(k\theta)} \, \sin(n\theta) \, d\theta = \frac{1}{2} \int_0^{\pi} \{ \sin{[(n+k)\theta]} + \sin{[(n-k)\theta]} \} \, d\theta \, . \tag{2.5}$$

For $n = k$, the above integral reduces to $I_{nn} = \int_0^{\pi} \cos(n\theta) \, \sin(n\theta) \, d\theta = \frac{1}{2} \int_0^{\pi} \sin(2n\theta) \, d\theta = -\cos{(2n\theta)}/(2n) \, |_0^{\pi} = 0$. For all $n \neq k$, one has

$$\begin{aligned} I_{kn} &= -\frac{1}{2} \left\{ \frac{\cos{[(n+k)\,\theta]}}{n+k} + \frac{\cos{[(n-k)\,\theta]}}{n-k} \right\} \Bigg|_0^{\pi} \\ &= -\frac{1}{2} \left\{ \frac{\cos{[(n+k)\,\pi]} - 1}{n+k} + \frac{\cos{[(n-k)\,\pi]} - 1}{n-k} \right\} \\ &= -\frac{1}{2} \left[\frac{(-1)^{n+k} - 1}{n+k} + \frac{(-1)^{n-k} - 1}{n-k} \right] . \end{aligned} \tag{2.6}$$

Therefore, $A_k = \frac{2}{\pi} \sum_{n=1}^{\infty} I_{kn}/\, n^2$ expands to

$$A_k = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\frac{1 - (-1)^{n+k}}{n+k} + \frac{1 - (-1)^{n-k}}{n-k} \right] \tag{2.7}$$

and, since $1 - (-1)^{n \pm k} = 0$ whenever n and k have the same parity (i.e., when they are both odd or even numbers), whereas $1 - (-1)^{n \pm k} = 2$ when n and k have opposite parities, then

$$\begin{aligned} A_k = \frac{1}{\pi} \sum_n{}' \, \frac{1}{n^2} \left[\frac{2}{n+k} + \frac{2}{n-k} \right] &= \frac{2}{\pi} \sum_n{}' \, \frac{1}{n^2} \, \frac{2n}{n^2 - k^2} \\ &= \frac{4}{\pi} \sum_n{}' \, \frac{1}{n} \, \frac{1}{n^2 - k^2} \, , \end{aligned} \tag{2.8}$$

where $\sum{}'$ means a sum over n values with the opposite parity with respect to k. Explicitly,

$$A_k = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m\,(4m^2 - k^2)} \, , \quad k \text{ odd} \, , \tag{2.9}$$

and

$$A_k = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)\,[(2m-1)^2 - k^2]} \, , \quad k \text{ even} \, . \tag{2.10}$$

For odd values of k, the substitution $k = 2p - 1$, $p > 0$, in Eq. (2.9) yields

$$\frac{\pi}{2} \, A_{2p-1} = \sum_{m=1}^{\infty} \frac{1}{m\,[4m^2 - (2p-1)^2]} \, . \tag{2.11}$$

This series can be written in terms of the digamma function $\psi(x) := \frac{d}{dx} \ln\Gamma(x)$, where $\Gamma(x) := \int_0^{\infty} t^{\,x-1} \, e^{-t} \, dt$ is the classical gamma function. From a well-known series representation for $\psi(x)$, namely [8, Sec. 8.362]

$$\psi(x) = -\,\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+x} \right), \tag{2.12}$$

one finds, after some algebra,

$$\frac{\pi}{2}\, A_{2p-1} = -\,\frac{\psi(3/2-p) + \psi(p+1/2) + 2\,\gamma}{2\,(2p-1)^2}\,, \tag{2.13}$$

where $\gamma := \lim_{n\to\infty} (H_n - \ln n)$ is the Euler's constant. From Eq. (3) in Ref. [8, Sec. 8.366], one knows that

$$\psi\left(\frac{1}{2} \pm p\right) = -\,\gamma - \ln 4 + 2\,h_p\,, \tag{2.14}$$

which, together with

$$\psi\left(\frac{3}{2} - p\right) = \psi\left(\frac{1}{2} - p\right) + \frac{1}{\frac{1}{2} - p}\,, \tag{2.15}$$

which promptly follows from $\psi(x+1) = \psi(x) + 1/x$ [8, Sec. 8.365], reduces Eq. (2.13) to

$$\frac{\pi}{2}\, A_{2p-1} = \frac{\ln 4 - 1/(2p-1) - 2\,h_{p-1}}{(2p-1)^2}\,, \tag{2.16}$$

which is equivalent to Eq. (2.9). The special value $\psi(1/2) = -\,\gamma - \ln 4$, as stated in Ref. [8, Sec. 8.366], is required in the derivation of Eq. (2.14).

For even values of k, substitute $k = 2p$ in Eq. (2.10). This leads to

$$\frac{\pi}{4}\, A_{2p} = \sum_{m=1}^{\infty} \frac{1}{(2m-1)\,[(2m-1)^2 - 4p^2]}\,. \tag{2.17}$$

The series representation of $\psi(x)$ given in Eq. (2.12) then leads to

$$\frac{\pi}{4}\, A_{2p} = -\,\frac{\psi(1/2-p) + \psi(p+1/2) + 2\,\gamma + 2\,\ln 4}{16\,p^2}\,. \tag{2.18}$$

On taking Eq. (2.14) into account, one finds, after some algebra,

$$\frac{\pi}{4}\, A_{2p} = -\,\frac{h_p}{(2\,p)^2}\,, \tag{2.19}$$

which completes the proof. □

As expected, this theorem shows that all conjectures stated at the end of Ref. [5] are indeed true.

3. Fourier series for an even periodic extension of Clausen function

Now, let us examine the Fourier cosine series whose coefficients are the A_k expressions derived above.

Theorem 3. *The series*

$$\frac{A_0}{2} + \sum_{k=1}^{\infty} A_k\, \cos{(k\,\theta)}\,,$$

where A_0 *and* A_k *are the coefficients derived in our Theorems 1 and 2, respectively, converges to* $\mathrm{Cl}_2(\theta)$ *for all* $\theta \in [0,\pi]$ *and to* $-\,\mathrm{Cl}_2(\theta)$ *when* $\theta \in (\pi, 2\pi]$*, thus yielding a continuous even function on* $[-2\pi, 2\pi]$*. This convergence can be extended to all* $\theta \in \mathbb{R}$*.*

Proof. Let $g(\theta)$ be a real function defined in the interval $[-2\pi, 2\pi]$ as follows:

$$g(\theta) := \begin{cases} +\mathrm{Cl}_2(\theta)\,, & \theta \in [-2\pi, -\pi) \text{ or } \theta \in [0, \pi] \\ -\mathrm{Cl}_2(\theta)\,, & \theta \in [-\pi, 0) \text{ or } \theta \in (\pi, 2\pi]\,. \end{cases} \tag{3.1}$$

Since $\mathrm{Cl}_2(\theta)$ is a continuous odd function, it is clear that $g(\theta)$ is a continuous *even* function in the interval $[-2\pi, 2\pi]$. In Theorem 4 of Zhang's paper [7], it is shown that, given a real function $f(x)$

integrable on $[0, L]$ and such that $f(x) = f(L-x)$ for all $x \in (L/2, L]$, if $f(x)$ is an *even* function in $(-L, L)$, then the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n} \, \cos\!\left(\frac{2n\pi x}{L}\right), \tag{3.2}$$

where

$$a_{2n} = \frac{4}{L} \int_0^{L/2} f(t) \, \cos\!\left(\frac{2n\pi t}{L}\right) dt \,, \quad n \geq 0 \,, \tag{3.3}$$

converges to $f(x)$ for all $x \in [-L, L]$ where $f(x)$ is a continuous function. The absence of the term $a_0/2$ in Theorem 4 of Ref. [7] is corrected here in our Eq. (3.2). Since the function $g(\theta)$ defined in Eq. (3.1) is an *even* function which is continuous (thus integrable) on $(-2\pi, 2\pi)$ and $g(\theta) = g(2\pi - \theta)$, then the convergence of the series $A_0/2 + \sum_{k=1}^{\infty} A_k \, \cos{(k\,\theta)}$ to $g(\theta)$ follows by taking $L = 2\pi$ in Zhang's theorem and noting that $a_{2n} = \frac{2}{\pi} \int_0^{\pi} g(\theta) \, \cos{(n\theta)} \, d\theta = \frac{2}{\pi} \int_0^{\pi} \mathrm{Cl}_2(\theta) \, \cos{(n\theta)} \, d\theta$ are just the coefficients A_0 and A_n derived in our Theorems 1 and 2, respectively. Finally, since this cosine series converges to an even *periodic extension* of $\mathrm{Cl}_2(\theta)$, with a period 2π, then its convergence to $g(\theta)$ can be extended to all $\theta \in \mathbb{R}$. □

Interestingly, new closed-form results can be deduced directly from Theorem 3. For instance, on taking $\theta = 0$ (or π), one finds

Corollary 1. *The following closed-form result holds:*

$$\sum_{p=1}^{\infty} \frac{h_{p-1}}{(2p-1)^2} = \frac{\pi^2}{8} \, \ln 2 - \frac{7}{16} \, \zeta(3) \,.$$

On taking $\theta = \pi/2$ in Theorem 3, a less obvious expression arises which can be written in terms of the regularized hypergeometric function

$${}_p\widetilde{F}_q\!\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \; ; \, z \right) := \frac{{}_pF_q\!\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \; ; \, z \right)}{\prod_{j=1}^{q} \Gamma\left(b_j\right)} \,, \tag{3.4}$$

where

$${}_pF_q\!\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \; ; \, z \right) := \sum_{n=0}^{\infty} \frac{\left(a_1\right)_n \, \ldots \, \left(a_p\right)_n}{\left(b_1\right)_n \, \ldots \, \left(b_q\right)_n} \, \frac{z^n}{n!} \tag{3.5}$$

is the generalized hypergeometric series. As usual, $\left(a\right)_n := a\,(a+1)\,\ldots\,(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. By convention, $\left(a\right)_0 = 1$.

Corollary 2 (A special value for $\theta = \pi/2$)**.** *The following closed-form result holds:*

$${}_4\widetilde{F}'_3\!\left(\begin{matrix} 1, 1, 1, 3/2 \\ 2, 2, 3/2 \end{matrix} \; ; \, -1 \right) = \frac{\zeta(2)\,(\gamma + \ln 4) + 7\,\zeta(3) - 4\pi\,G}{\sqrt{\pi}} \,, \tag{3.6}$$

where $\zeta(2) := \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ *and* $G := \sum_{n=0}^{\infty} (-1)^n/(2n+1)^2$ *is the Catalan's constant. Here, the prime symbol (′) indicates a partial derivative with respect to* b_3*.*

As shown below, this result can be written in terms of the corresponding generalized hypergeometric function. Interestingly, this yields a nice closed-form result which, to the author knowledge, is not found in literature.

Corollary 3 (Corresponding generalized hypergeometric function)**.** *The following closed-form result holds:*

$${}_4F'_3\!\left(\begin{matrix} 1, 1, 1, 3/2 \\ 2, 2, 3/2 \end{matrix} \; ; \, -1 \right) = \frac{\pi^2}{6} + \frac{7}{2} \, \zeta(3) - 2\pi\,G \,. \tag{3.7}$$

Proof. In a shortened notation, Eq. (3.4) reads

$${}_p\widetilde{F}_q\!\left(\vec{a}, \vec{b}; z\right) = \frac{{}_pF_q\!\left(\vec{a}, \vec{b}; z\right)}{\prod_{j=1}^{q} \Gamma(b_j)}$$

where $\vec{a}$ and $\vec{b}$ denote the arrays of coefficients $[1,1,1,3/2]$ and $[2,2,3/2]$, respectively. This implies that

$$\frac{\partial}{\partial b_3}\,{}_4\widetilde{F}_3\Big(\vec{a},\vec{b};-1\Big) = \frac{1}{\prod_{j\neq 3}\Gamma(b_j)}\,\frac{\partial}{\partial b_3}\left[\frac{{}_4F_3\Big(\vec{a},\vec{b};-1\Big)}{\Gamma(b_3)}\right]$$

$$= \frac{1}{\Gamma(b_1)\,\Gamma(b_2)}\left[\frac{{}_4F'_3\Big(\vec{a},\vec{b};-1\Big)}{\Gamma(b_3)} - {}_4F_3\Big(\vec{a},\vec{b};-1\Big)\,\frac{\Gamma'(b_3)}{\Gamma^2(b_3)}\right]$$

$$= \frac{1}{\Gamma^2(2)}\left[\frac{{}_4F'_3\Big(\vec{a},\vec{b};-1\Big)}{\Gamma(3/2)} - {}_4F_3\Big(\vec{a},\vec{b};-1\Big)\,\frac{\Gamma'(3/2)}{\Gamma^2(3/2)}\right]. \tag{3.8}$$

Since $\Gamma(1+x)=x\,\Gamma(x)$, then $\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}/\,2$, which reduces the last expression, above, to

$${}_4\widetilde{F}'_3\Big(\vec{a},\vec{b};-1\Big) = \frac{{}_4F'_3\Big(\vec{a},\vec{b};-1\Big)}{\sqrt{\pi}/2} - {}_4F_3\Big(\vec{a},\vec{b};-1\Big)\,\frac{\psi(3/2)}{\sqrt{\pi}/2}$$

$$= 2\,\frac{{}_4F'_3(\vec{a},\vec{b};-1)}{\sqrt{\pi}} - 2\,{}_4F_3(\vec{a},\vec{b};-1)\,\frac{\psi(3/2)}{\sqrt{\pi}}\,. \tag{3.9}$$

Note that, for all positive integers n, $\Gamma(n)=(n-1)!$ (in particular, $\Gamma(2)=1!=1$). The proof completes by substituting the result in Corollary 2, together with the special values $\psi(3/2)=\psi(1/2)+1/(1/2)=-\gamma-\ln 4+2$ and ${}_4F_3(\vec{a},\vec{b};-1)=\pi^2/\,12$, in Eq. (3.9). □

The closed-form result in Corollary 3 has been conjectured by Ancarani and the author in a recent discussion, by following an entirely different approach, but we could not find a formal proof at that time.

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