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A NOTE ON ABSOLUTE CESÀRO $\varphi - |C, 1; \delta; l|_k$ SUMMABILITY FACTOR

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Abstract. A positive non-decreasing sequence has been used to establish a theorem on a minimal set of sufficient conditions for an infinite series to be absolute Cesàro $\varphi - |C, 1; \delta; l|_k$ summable. For some well-known applications, suitable conditions have been applied on the presented theorem for obtaining the sub-result of the presented theorem.

1. Introduction

Let $\{s_n\}$ be a sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n$ and n^{th} mean of $\{s_n\}$ is given by t_n s.t.

$$t_n = \sum_{k=0}^{\infty} t_{nk} s_k \tag{1.1}$$

where $\{t_{nk}\}$ is the sequence of the coefficients of the matrix. If sequence of the means $\{t_n\}$ satisfied the following conditions:

$$\lim_{n \to \infty} t_n = s, \tag{1.2}$$

and

$$\sum_{n=1}^{\infty} | \ t_n - t_{n-1}| < \infty, \tag{1.3}$$

then the series $\sum_{n=0}^{\infty} a_n$ is said to be absolute summable. If τ_n represent the n^{th} $(C, 1)$ means of the sequence (na_n), then series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|C, 1|_k, k \geq 1$ [9], if

$$\sum_{n=1}^{\infty} \frac{1}{n} |\tau_n|^k < \infty. \tag{1.4}$$

If the sequence $\{\tau_n\}$ satisfied the condition:

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |\tau_n|^k < \infty, \tag{1.5}$$

then the series $\sum_{n=0}^{\infty} a_n$ is said to be summable $\varphi - |C, 1|_k$, $k \geq 1$, and if the sequence $\{\tau_n\}$ satisfied the following condition:

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} |\tau_n|^k < \infty, \tag{1.6}$$

then the series $\sum_{n=0}^{\infty} a_n$ is $\varphi - |C, 1; \delta|_k$, summable, where $k \geq 1$, $\delta \geq 0$ and (φ_n) be a sequence of positive real numbers.

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For $\varphi-|C,1;\delta;l|_k$ summability, the infinite series $\sum\limits_{n=0}^{\infty} a_n$ satisfied

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}}|\tau_n|^k<\infty \tag{1.7}$$

where $k\geq 1$, $\delta\geq 0$ and l is a real number.

Note: If we take $l=1$, then $\varphi-|C,1;\delta;l|_k$ reduces to $\varphi-|C,1,\delta|_k$, if $\varphi=n$, then $\varphi-|C,1;\delta|_k$ summability reduces to $|C,1;\delta|_k$ summability and if $\delta=0$, then $|C,1;\delta|_k$ reduces to $|C,1|_k$.

In 1972, Mazhar [8] determined the minimal set of sufficient conditions for an infinite series to be absolute $|C,1|_k$ summable. This result became an essence of many results found in previous years. In 1980, Balci [10] defined Absolute φ-summability factors and determined a very interesting result. Bor gave a number of theorems on absolute summability by a generalization of Mazhar [8] results. In 1993 [2] and 1994 [3], he used $|\bar{N},p_n|_k$ summability for enhancement of the results of Mazhar [8] and apply it on Fourier series. Özarslan [4] generalized the result of Bor [1] by a more general absolute $\varphi-|C,\alpha|_k$ summability and in [6], he used absolute matrix $\varphi-|A,\delta|_k$ summability and improve some well-known results. Concerning the $\varphi-|N,p_n|_k$ summability factors, Saxena [7] gave a general theorem for an infinite series.

2. Known results

Absolute $\varphi-|C,1|_k$ summability has been used by Özarslan [5] to establish the following theorem.

Theorem 2.1. *Let φ_n be a sequence of positive real numbers, if*

$$\lambda_m=O(1),\ m\to\infty, \tag{2.1}$$

$$\sum_{n=1}^{m} n\log n|\Delta^2\lambda_n|=O(1), \tag{2.2}$$

$$\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^k}|t_v|^k=O(\log m),\ as\ m\to\infty, \tag{2.3}$$

$$\sum_{n=v}^{m}\frac{\varphi_n^{k-1}}{n^{k+1}}=O\left(\frac{\varphi_v^{k-1}}{v^k}\right). \tag{2.4}$$

Then the infinite series $\sum a_n\lambda_n$ is $\varphi-|C,1|_k$ summable for $k\geq 1$.

3. Main results

Generalized Cesáro $\varphi-|C,1;\delta;l|_k$ summability and a positive non-decreasing sequence have been used to moderate the conditions of Özarslan [5] results for an infinite series.

Theorem 3.1. *Let (φ_n) is a sequence of positive real numbers and (μ_n) is positive non-decreasing sequence satisfying the following conditions:*

$$\lambda_m=O(1),\ m\to\infty, \tag{3.1}$$

$$\sum_{n=1}^{m} n\log n|\Delta^2\lambda_n|=O(1), \tag{3.2}$$

$$\sum_{v=1}^{m}\frac{\varphi_v^{l(k-1)}}{v^{l(k-\delta k)}}|t_v|^k=O(\log m.\mu_m)\ as\ m\to\infty, \tag{3.3}$$

$$\sum_{n=v}^{m}\frac{\varphi_n^{l(k-1)}}{n^{1+l(k-\delta k)}}=O\left(\frac{\varphi_v^{l(k-1)}}{v^{l(k-\delta k)}}\right), \tag{3.4}$$

$$n\log n\mu_n\Delta\left(\frac{1}{\mu_n}\right)=O(1). \tag{3.5}$$

Then the infinite series $\sum a_n\lambda_n/\mu_n$ is $\varphi-|C,1;\delta;l|_k$ summable for $k\geq 1$, $\delta\geq 0$ and l is a real number.

4. Proof of the Theorem

Let T_n be the n^{th} $(C,1)$ mean of the sequence $(na_n\lambda_n/\mu_n)$. The series is $\varphi-|C,1;\delta;l|_k$ summable, if

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}}|T_n|^k<\infty. \tag{4.1}$$

Applying Able's transformation, we have

$$\begin{aligned}
T_n &= \frac{1}{n+1}\sum_{v=1}^{n}\frac{va_v\lambda_v}{\mu_v}\\
&= \frac{1}{n+1}\left(\sum_{v=1}^{n-1}\left(\sum_{r=1}^{v}ra_r\right)\Delta\left(\frac{\lambda_v}{\mu_v}\right)+\left(\frac{\lambda_v}{\mu_v}\right)\sum_{v=1}^{n}va_v\right)\\
&= \frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)t_v\left(\frac{1}{\mu_v}\right)\Delta\lambda_v+\frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)t_v\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)+\frac{t_n\lambda_n}{\mu_n}\\
&= T_{n,1}+T_{n,2}+T_{n,3}.
\end{aligned}\tag{4.2}$$

Using Minkowski's inequality,

$$|T_n|^k=|T_{n,1}+T_{n,2}+T_{n,3}|^k<3^k\Big(|T_{n,1}|^k+|T_{n,2}|^k+|T_{n,3}|^k\Big). \tag{4.3}$$

In order to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}}|T_{n,r}|^k<\infty,\ \ for\ r=1,2,3. \tag{4.4}$$

By using Hölder's inequality and Abel's transformation, we have

$$\begin{aligned}
\sum_{n=2}^{m}\frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}}|T_{n,1}|^k &= \sum_{n=2}^{m}\frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)t_v\frac{\Delta\lambda_v}{\mu_v}\right|^k\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{l(k-1)}}{n^{k+l(k-\delta k)}}\left(\sum_{v=1}^{n-1}v|t_v|\frac{|\Delta\lambda_v|}{\mu_v}\right)^k\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{l(k-1)}}{n^{k+l(k-\delta k)}}\sum_{v=1}^{n-1}v\frac{|\Delta\lambda_v|}{\mu_v}|t_v|^k\left(\sum_{v=1}^{n-1}v\frac{|\Delta\lambda_v|}{\mu_v}\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{l(k-1)}}{n^{1+l(k-\delta k)}}\left(\sum_{v=1}^{n-1}v\frac{|\Delta\lambda_v|}{\mu_v}|t_v|^k\right)\\
&= O(1)\sum_{v=1}^{m}v\frac{|\Delta\lambda_v|}{\mu_v}|t_v|^k\left(\sum_{n=v}^{m}\frac{\varphi_n^{l(k-1)}}{n^{1+l(k-\delta k)}}\right)\\
&= O(1)\sum_{v=1}^{m}v\frac{|\Delta\lambda_v|}{\mu_v}|t_v|^k\frac{\varphi_v^{l(k-1)}}{v^{l(k-\delta k)}}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1}\left|\Delta\left(v\frac{|\Delta\lambda_v|}{\mu_v}\right)\right|\sum_{r=1}^{v}\frac{\varphi_r^{l(k-1)}}{r^{l(k-\delta k)}}|t_r|^k+m\frac{|\Delta\lambda_m|}{\mu_m}\sum_{r=1}^{v}\frac{\varphi_r^{l(k-1)}}{r^{l(k-\delta k)}}|t_r|^k\\
&= O(1)\sum_{v=1}^{m-1}\frac{|\Delta\lambda_v|}{\mu_v}\log v.\mu_v+\sum_{v=1}^{m-1}(v+1)\Delta\left(\frac{|\Delta\lambda_v|}{\mu_v}\right)\log v.\mu_v\\
&+ m\frac{|\Delta\lambda_m|}{\mu_m}\log m.\mu_m\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|\log v+\sum_{v=1}^{m-1}(v+1)\frac{1}{\mu_v}|\Delta^2\lambda_v|\log v.\mu_v\\
&+ \sum_{v=1}^{m-1}(v+1)|\Delta\lambda_{v+1}|\Delta\left(\frac{1}{\mu_v}\right)\log v.\mu_v+m|\Delta\lambda_m|\log m\\
&= O(1).
\end{aligned}\tag{4.5}
$$

$$
\begin{aligned}
\sum_{n=2}^{m}\frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}}|T_{n,2}|^k &= \sum_{n=2}^{m}\frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}\lambda_{v+1}(v+1)t_v\Delta\left(\frac{1}{\mu_v}\right)\right|^k\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{l(k-1)}}{n^{k+l(k-\delta k)}}\left(\sum_{v=1}^{n-1}v\lambda_{v+1}|t_v|\Delta\left(\frac{1}{\mu_v}\right)\right)^k\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{l(k-1)}}{n^{k+l(k-\delta k)}}\sum_{v=1}^{n-1}v\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)|t_v|^k\left(\sum_{v=1}^{n-1}v\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{l(k-1)}}{n^{1+l(k-\delta k)}}\left(\sum_{v=1}^{n-1}v\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)|t_v|^k\right)\\
&= O(1)\sum_{v=1}^{m-1}\left|v\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)\right|\sum_{r=1}^{v}\frac{\varphi_r^{l(k-1)}}{r^{1+l(k-\delta k)}}|t_r|^k\\
&+ m\lambda_{m+1}\Delta\left(\frac{1}{\mu_m}\right)\sum_{r=1}^{m}\frac{\varphi_r^{l(k-1)}}{r^{1+l(k-\delta k)}}|t_r|^k\\
&= O(1)\sum_{v=1}^{m-1}\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)\mu_v\log v+\sum_{v=1}^{m-1}(v+1)\Delta\left(\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)\right)\mu_v\log v\\
&+ m\lambda_{m+1}\Delta\left(\frac{1}{\mu_m}\right)\log m.\mu_m\\
&= O(1)\sum_{v=1}^{m-1}\lambda_{v+1}\log v+\sum_{v=1}^{m-1}(v+1)\Delta\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)\mu_v\log v\\
&+ \sum_{v=1}^{m-1}(v+1)\lambda_{v+2}\Delta^2\left(\frac{1}{\mu_v}\right)\mu_v\log v+m\lambda_{m+1}\log m\\
&= O(1)\ as\ m\to\infty.
\end{aligned}\tag{4.6}
$$

$$
\begin{aligned}
\sum_{n=1}^{m}\frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}}|T_{n,3}|^k &= \sum_{n=1}^{m}\frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}}\left|\frac{t_n\lambda_n}{\mu_n}\right|^k\\
&= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}}|t_n|^k\left|\sum_{v=n}^{\infty}\Delta\left(\frac{\lambda_v}{\mu_v}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{\infty}\left|\Delta\left(\frac{\lambda_v}{\mu_v}\right)\right|\sum_{n=1}^{v}\frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}}|t_n|^k\\
&= O(1)\sum_{v=1}^{\infty}\frac{1}{\mu_v}\Delta\lambda_v \log v.\mu_v+\sum_{v=1}^{\infty}\lambda_{v+1}\Delta\left(\frac{1}{\mu_v}\right)\log v.\mu_v\\
&= O(1). \qquad (4.7)
\end{aligned}
$$

Collecting (4.2) - (4.7), we have

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^{k-\delta k}}|T_n|^k<\infty. \qquad (4.8)$$

Hence proof of the theorem is completed.

5. Corollaries

Corollary 5.1. *Let (φ_n) is a sequence of positive real numbers and (μ_n) is positive non-decreasing sequence satisfying (3.1)-(3.2), (3.5) and following conditions:*

$$\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^{k-\delta k}}|t_v|^k=O(\log m.\mu_m)\ \text{as}\ m\to\infty, \qquad (5.1)$$

$$\sum_{n=v}^{m}\frac{\varphi_n^{k-1}}{n^{1+k-\delta k}}=O\left(\frac{\varphi_v^{k-1}}{v^{k-\delta k}}\right). \qquad (5.2)$$

Then the infinite series $\sum a_n\lambda_n/\mu_n$ is $\varphi-|C,1;\delta|_k$ summable for $k\geq 1$ and $\delta\geq 0$.

Proof. By using specific value $l=1$ in Theorem 3.1, we will get (5.1) and (5.2). We omit the details as the proof is similar to that of Theorem 3.1 and we use (5.1) and (5.2) instead of (3.3) and (3.4). □

Corollary 5.2. *Let (φ_n) is a sequence of positive real numbers and (μ_n) is positive non-decreasing sequence satisfying (3.1)-(3.2), (3.5) and following conditions:*

$$\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^{k}}|t_v|^k=O(\log m.\mu_m)\ \text{as}\ m\to\infty, \qquad (5.3)$$

$$\sum_{n=v}^{m}\frac{\varphi_n^{k-1}}{n^{1+k}}=O\left(\frac{\varphi_v^{k-1}}{v^{k}}\right). \qquad (5.4)$$

Then the infinite series $\sum a_n\lambda_n/\mu_n$ is $\varphi-|C,1;\delta|_k$ summable for $k\geq 1$ and $\delta\geq 0$.

Proof. By using specific value $l=1$ and $\delta=0$ in Theorem 3.1, we will get (5.3) and (5.4). We omit the details as the proof is similar to that of Theorem 3.1 and we use (5.3) and (5.4) instead of (3.3) and (3.4). □

Hence theorem 3.1 is a generalization of above Corollaries.

6. Conclusion

The aim of this research article is to formulate the problem of generalization of absolute Cesáro ($\varphi-|C,1;\delta;l|_k$, $k\geq 1$, $\delta\geq 0$ and l is a real number) summability factor of infinite series which is a motivation for the researchers, interested in theoretical studies of an infinite series. Further, this study has a number of direct applications in rectification of signals in FIR filter (Finite impulse response filter) and IIR filter (Infinite impulse response filter). In a nut shell, the absolute summability methods have vast potential in dealing with the problems based on infinite series.

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