International Journal of Analysis and Applications
ISSN 2291-8639
Volume 4, Number 2 (2014), 100-106
http://www.etamaths.com

SOME COMMON FIXED POINT THEOREMS IN GENERALIZED VECTOR METRIC SPACES

RAJESH SHRIVASTAVA[1], RAJENDRA KUMAR DUBEY[2], PANKAJ TIWARI[1,*]

ABSTRACT. In this paper we give some theorems on point of coincidence and common fixed point for two self mappings satisfying some general contractive conditions in generalized vector spaces. Our results generalize some well-known recent results in this direction.

1. INTRODUCTION AND PRELIMINARIES

In 2003, Mustafa and Sims [6] introduced a more appropriate and robust notion of a generalized metric space as follows.

Definition 1.1. [6]. Let X be a nonempty set, and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following axioms:

(1) $G(x, y, z) = 0$ if and only if $x = y = z$;
(2) $G(x, x, y) > 0$, for all $x \neq y$;
(3) $G(x, y, z) \geq G(x, x, y)$, for all $x, y, z \in X$;
(4) $G(x, y, z) = G(x, z, y) = G(z, y, x) = \cdots$ (symmetric in all three variables);
(5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$, for all $x, y, z, w \in X$.

Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is called a G-metric space.

A Riesz space is an ordered vector space and a lattice. Let $\mathbb{E}$ be a Riesz space with the positive cone $\mathbb{E}_+ = \{x \in \mathbb{E} : x \geq 0\}$. If $\{a_n\}$ is a decreasing sequence in $\mathbb{E}$ such that $\inf a_n = a$, write $a_n \downarrow a$.

Definition 1.2. The Riesz space $\mathbb{E}$ is said to be Achimedean if $\frac{1}{n} a_n \downarrow 0$ holds for every $\mathbb{E}_+$.

Definition 1.3. A sequence $\{b_n\}$ is said to be order convergent (or o-convergent) to b if there is a sequence $\{a_n\}$ in $\mathbb{E}$ satisfying $a_n \downarrow 0$ and $|b_n - b| \leq a_n$ for all n, and written $b_n \xrightarrow{o} b$ or $o - \lim b_n = b$, where $|a| = \sup\{a, -a\}$ for any $a \in \mathbb{E}$.

Definition 1.4. A sequence $\{b_n\}$ is said to be order-Cauchy (or o-Cauchy) if there exists a sequence $\{a_n\}$ in $\mathbb{E}$ such that $a_n \downarrow 0$ and $|b_n - b_{n+p}| \leq a_n$ holds for all n and p.

Definition 1.5. The Riesz space $\mathbb{E}$ is said to be $o - Cauchy$ *complete* if every $o - Cauchy$ *sequence* in $o - convergent$.

For notion and other facts regarding Riesz spaces we refer to [1].

2010 *Mathematics Subject Classification.* 47H10.
Key words and phrases. Reisz space, Generalized vector space, coincidence point, fixed point.

2. Vector G-metric spaces

In this section we introduce the following concepts and properties of Vector G-metric spaces.

Definition 2.1. Let X be a non-empty set and $\mathbb{E}$ be a Riesz space. The function $G : X \times X \times X \to \mathbb{E}$ is said to be vector $G-$metric if it is satisfying the following properties :

(VGM-1) $G(x, y, z) = 0$ if and only if $x = y = z$,
(VGM-2) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$,

for all $x, y, z, w \in X$. Also the triple $(X, G, \mathbb{E})$ is said to be vector $G-$metric space.

For arbitrary elements $x, y, z, w \in X$ of a vector $G-$metric space, the following statements are satisfied

1. $G(x, x, y) > 0$, for all $x \neq y$;
2. $G(x, y, z) \geq G(x, x, y)$, for all $x, y, z \in X$;
3. $G(x, y, z) = G(x, z, y) = G(z, y, x) = \cdots$ (symmetric in all three variables).

Example 2.2. A Riesz space $\mathbb{E}$ is a vector $G-$metric space with $G : \mathbb{E}\times\mathbb{E}\times\mathbb{E} \to \mathbb{E}$ defined by $G(x, y, z) = |x - y| + |y - z| + |z - x|$. This vector $G-$metric space is called to be absolute valued $G-$metric space on $\mathbb{E}$.

It is well known that $\mathbb{R}^2$ is a Riesz space with coordinatwise ordering defined by

$$(x_1, y_1) \leq (x_2, y_2) \Longleftrightarrow x_1 \leq x_2 \ \ and \ \ y_1 \leq y_2$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.
Again $\mathbb{R}^2$ is a Riesz space with lexicographical ordering defined by

$$(x_1, y_1) \leq (x_2, y_2) \Longleftrightarrow x_1 < x_2 \ \ or \ \ x_1 = x_2 \ \ and \ \ y_1 \leq y_2.$$

Note that $\mathbb{R}^2$ is Archimedean with coordinatwise ordering but not with lexicographical ordering.

Example 2.3. Let $G : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$G((x_1, y_1), (x_2, y_2), (x_3, y_3)) = (\alpha x^*, \beta y^*)$$

where $x^* = |x_1 - x_2| + |x_2 - x_3| + |x_3 - x_1|$ and $y^* = |y_1 - y_2| + |y_2 - y_3| + |y_3 - y_1|$ also α, β are positive real numbers. Then G is a vector $G-$metric space.

Let $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ defined by

$$G(x, y, z) = (\alpha w, \beta w)$$

where $w = |x - y| + |y - z| + |z - x|$, $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$. Then G is a vector $G-$metric space.

Definition 2.4. A sequence $\{x_n\}$ in a vector $G-$metric space $(X, G, \mathbb{E})$ vectorial $G-$convergence to some $x \in \mathbb{E}$, written $x_n \xrightarrow{G,\mathbb{E}} x$, if there is a sequence $\{a_n\}$ in $\mathbb{E}$ such that $a_n \downarrow 0$ and satisfying,

1. $G(x_n, x_n, x) \leq a_n$,
2. $G(x_n, x, x) \leq a_n$,

(3) $G(x_n, x_m, x) \leq a_n$,

for all n.

Definition 2.5. A sequence $\{x_n\}$ is called $G_E - Cauchy\ \ sequence$ whenever there exists a sequence $\{a_n\}$ in $\mathbb{E}$ such that $a_n \downarrow 0$ and $G(x_n, x_m, x_m) \leq a_n$ holds for all n and m.

Definition 2.6. A vector G-metric space is said to be complete if each $G_E - Cauchy\ \ sequence$ in X is $E - convergens$ to a limit in X.

Using the above definitions, we have the following properties.

If $x_n \xrightarrow{G,\mathbb{E}} x$, then

(1) The limit x unique,

(2) Every subsequence of $\{x_n\}$ $E-$converges to x,

(3) If also $y_n \xrightarrow{G,\mathbb{E}} y$ and $z_n \xrightarrow{G,\mathbb{E}} z$, then $G(x_n, y_n, z_n) \xrightarrow{o} G(x, y, z)$.

When $\mathbb{E} = \mathbb{R}$ then the concepts of vectorial $G_E - convergence$ and $convergence$ in $G-$metric space are the same also the concepts of $G_E - Cauchy\ \ sequence$ and $G - Cauchy\ \ sequence$ are the same.

Remark 2.7. It $\mathbb{E}$ is a Riesz space and $a \leq ka$ where $a \in \mathbb{E}_+$ $k \in [0, 1)$, then $a = 0$.

Proof. The condition $a \leq ka$ means that $-(1-k)a = ka - a \in \mathbb{E}_+$. Since $a \in \mathbb{E}_+$ and $1 - k > 0$, then also $(1-k)a \in \mathbb{E}_+$. Thus we have $(1-k)a = 0$ and $a = 0$. □

3. Main Results

Theorem 3.1. *Let X be a vector $G-$metric space with $\mathbb{E}$ is Archimedean. Suppose the mappings $S, T : X \to X$ satisfying the following conditions,*

(i) *for all $x, y, z \in X$ and $\alpha, \beta, \gamma, \delta \in [0, 1)$ such that $0 \leq \alpha + \beta + \gamma + \delta < 1$*

$$G(Tx, Ty, Tz) \leq \alpha G(Sx, Sy, Sz) + \beta G(Sx, Tx, Tx) + \gamma G(Sy, Ty, Ty) + \delta G(Sz, Tz, Tz) \tag{3.1}$$

or

$$G(Tx, Ty, Tz) \leq \alpha G(Sx, Sy, Sz) + \beta G(Sx, Sx, Tx) + \gamma G(Sy, Sy, Ty) + \delta G(Sz, Sz, Tz) \tag{3.2}$$

(ii) *$T(X) \subseteq S(X)$,*

(iii) *$T(X)$ or $S(X)$ is complete subspace of X.*

Then S and T have a unique point of coincidence in X. Moreover, if S and T are weakly compatible, then they have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X, since $T(X) \subseteq S(X)$ so we can choose a point $x_1 \in X$ such that $Sx_1 = Tx_0$. In general we can choose $Sx_{n+1} = Tx_n = y_n$ for all n.

Now, form 3.1 we have

$$\begin{aligned} G(Sx_n, Sx_{n+1}, Sx_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq (\alpha + \beta)G(Sx_{n-1}, Sx_n, Sx_n) + (\gamma + \delta)G(Sx_n, x_{n+1}, x_{n+1}) \end{aligned}$$

$$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq \frac{\alpha + \beta}{1 - (\gamma + \delta)} G(Sx_{n-1}, Sx_n, Sx_n). \tag{3.3}$$

Let $q = \frac{\alpha+\beta}{1-(\gamma+\delta)}$, then $0 \leq q < 1$ since $0 \leq \alpha + \beta + \gamma + \delta < 1$. So

$$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq qG(Sx_{n-1}, Sx_n, Sx_n). \tag{3.4}$$

Continuing in the same way, we have

$$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq q^n G(Sx_0, Sx_1, Sx_1). \tag{3.5}$$

Therefore, for all $n, m \in N, n < m$, we have by (VGM-2)

$$\begin{aligned} G(y_n, y_m, y_m) &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + ... + G(y_{m-1}, y_m, y_n) \\ &\leq (q^n + q^{n+1} + q^{n+2} + ... + q^{m-1}) G(y_0, y_1, y_1) \\ &\leq \frac{q^n}{1-q} G(y_0, y_1, y_1). \end{aligned}$$

Now, since $\mathbb{E}$ is Archimedean then $\{y_n\}$ is an $G_E-Cauchy\ \ sequence$ in X. Since the range of S contains the range of T and the range of at least one is $G_E-complete$, so there is $w \in X$ such that $Sx_n \xrightarrow{G,\mathbb{E}} w$. Hence there exists a sequence $\{a_n\} \in \mathbb{E}$ such that $a_n \downarrow 0$ and $G(Sx_n, w, w) \leq a_n$. On the other hand, we can find $u \in X$ such that $Sw = u$.

Let us show that $Tw = u$, we have

$$\begin{aligned} G(Tw, u, u) &\leq G(Tw, Tx_n, Tx_n) + G(Tx_n, u, u) \\ &\leq \alpha G(Sw, Sx_n, Sx_n) + \beta G(Sw, Tw, Tw) + (\gamma + \delta) G(Sx_n, Tx_n, Tx_n) + a_{n+1} \\ &\leq (\alpha + \beta + \gamma + \delta + 1) a_{n+1}. \end{aligned}$$

Since the infimum of the sequence on the right side of the above inequality are zero, then $Tw = u$. Therefore, w is a point of coincidence of T and S. If w_1 is another point of coincidence then there is $w_1 \in X$ with $w_1 = Tw_1 = Sw_1$. Now from 3.1 it follows that $G(w, w_1, w_1) = 0$, that is $w = w_1$.

If S and T are weakly compatible, then it is obvious that w is unique common fixed point of T and S in X.

If S and T satisfies condition 3.2, then the argument is similar to that above. However to show that the sequence $\{x_n\}$ is $G_{\mathbb{E}} - Cauchy\ \ sequence$, we start with

$$\begin{aligned} G(Sx_n, Sx_n, Sx_{n+1}) &= G(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq (\alpha + \beta + \gamma) G(Sx_{n-1}, Sx_{n-1}, Sx_n) + \delta G(Sx_n, x_{n+1}, x_{n+1}) \end{aligned}$$

$$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq \frac{\alpha + \beta + \gamma}{1 - \delta} G(Sx_{n-1}, Sx_n, Sx_n). \tag{3.6}$$

Let $q = \frac{\alpha+\beta+\gamma}{1-\delta}$, then $0 \leq q < 1$ since $0 \leq \alpha + \beta + \gamma + \delta < 1$. So

$$G(Sx_n, Sx_n, Sx_{n+1}) \leq q G(Sx_{n-1}, Sx_{n-1}, Sx_n). \tag{3.7}$$

Continuing in the same way, we have

$$G(Sx_n, Sx_n, Sx_{n+1}) \leq q^n G(Sx_0, Sx_0, Sx_1). \tag{3.8}$$

Then for all $n, m \in N, n < m$, we have by (VGM-2) we prove the remaining part of the proof. □

Corollary 3.2. *Let X be a vector $G-$metric space with $\mathbb{E}$ is Archimedean. Suppose the mappings $S, T : X \to X$ satisfying the following conditions,*

(i) *for all* $x, y, z \in X$ *and* $\alpha, \beta, \gamma, \delta \in [0, 1)$ *such that* $0 \leq \alpha + \beta + \gamma + \delta < 1$

$$G(T^m x, T^m y, T^m z) \leq \alpha G(S^m x, S^m y, S^m z) + \beta G(S^m x, T^m x, T^m x) + \gamma G(S^m y, T^m y, T^m y) + \delta G(S^m z, T^m z, T^m z)$$

or

$$G(T^m x, T^m y, T^m z) \leq \alpha G(S^m x, S^m y, S^m z) + \beta G(S^m x, S^m x, T^m x) + \gamma G(S^m y, S^m y, T^m y) + \delta G(S^m z, S^m z, T^m z)$$

(ii) $T(X) \subseteq S(X)$,

(iii) $T(X)$ *or* $S(X)$ *is complete subspace of* X.

Then S *and* T *have a unique point of coincidence in* X. *Moreover, if* S *and* T *are weakly compatible, then they have a unique common fixed point in* X, *also* T^m *and* S^m *are* $G_{\mathbb{E}}-$ *continuous at* u. .

Proof. From Theorem 3.1, we see that T^m and S^m have a unique common fixed point (say u), that is, $T^m(u) = u$. but $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u))$, so $T(u)$ is another fixed point for T^m and by uniqueness $Tu = u$.Similarly we can show that $Su = u$. □

Theorem 3.3. *Let* X *be a vector* $G-$*metric space with* $\mathbb{E}$ *is Archimedean. Suppose the mappings* $S, T : X \to X$ *satisfying the following conditions,*

(i) *for all* $x, y, z \in X$ *and* $\alpha \in [0, 1)$ *such that,*

$$G(Tx, Ty, Tz) \leq \alpha\{G(Sx, Sy, Sz), G(Sx, Tx, Tx), G(Sy, Ty, Ty), G(Sz, Tz, Tz)\} \tag{3.9}$$

or

$$G(Tx, Ty, Tz) \leq \alpha\{G(Sx, Sy, Sz), G(Sx, Sx, Tx), G(Sy, Sy, Ty), G(Sz, Sz, Tz)\} \tag{3.10}$$

(ii) $T(X) \subseteq S(X)$,

(iii) $T(X)$ *or* $S(X)$ *is complete subspace of* X.

Then S *and* T *have a unique point of coincidence in* X. *Moreover, if* S *and* T *are weakly compatible, then they have a unique common fixed point in* X, *also* T *and* S *are* $G_{\mathbb{E}}-$ *continuous at* u.

Proof. Let x_0 be an arbitrary point in X, since $T(X) \subseteq S(X)$ so we can choose a point $x_1 \in X$ such that $Sx_1 = Tx_0$. In general we can choose $Sx_{n+1} = Tx_n = y_n$ for all n.

Now, form 3.9 we have

$$\begin{aligned} G(Sx_n, Sx_{n+1}, Sx_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \alpha\{G(Sx_{n-1}, Sx_n, Sx_n), G(Sx_n, Sx_{n+1}, Sx_{n+1})\} \\ G(Sx_n, Sx_{n+1}, Sx_{n+1}) &\leq \alpha G(Sx_{n-1}, Sx_n, Sx_n) \end{aligned}$$

$$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq \alpha G(Sx_{n-1}, Sx_n, Sx_n). \tag{3.11}$$

Continuing in the same way, we have

$$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq \alpha^n G(Sx_0, Sx_1, Sx_1). \tag{3.12}$$

Therefore, for all $n, m \in N, n < m$, we have by (VGM-2)

$$\begin{aligned} G(y_n, y_m, y_m) &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + ... + G(y_{m-1}, y_m, y_n) \\ &\leq (\alpha^n + \alpha^{n+1} + \alpha^{n+2} + ... + \alpha^{m-1}) G(y_0, y_1, y_1) \\ &\leq \frac{\alpha^n}{1-\alpha} G(y_0, y_1, y_1). \end{aligned}$$

Now, since $\mathbb{E}$ is Archimedean then $\{y_n\}$ is an G_E-*Cauchy sequence* in X. Since the range of S contains the range of T and the range of at least one is G_E-*complete*, so there is $w \in X$ such that $Sx_n \xrightarrow{G,\mathbb{E}} w$. Hence there exists a sequence $\{a_n\} \in \mathbb{E}$ such that $a_n \downarrow 0$ and $G(Sx_n, w, w) \leq a_n$. On the other hand, we can find $u \in X$ such that $Sw = u$.

Let us show that $Tw = u$, we have

$$\begin{aligned} G(Tw, u, u) &\leq G(Tw, Tx_n, Tx_n) + G(Tx_n, u, u) \\ &\leq \alpha \max\{G(Sw, Sx_n, Sx_n), G(Sw, Tw, Tw), \\ &\quad G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sx_n, Sx_{n+1}, Sx_{n+1})\} + a_{n+1} \\ &\leq (\alpha + 1) a_n \end{aligned}$$

Since the infimum of the sequence on the right side of the above inequality are zero, then $Tw = u$. Therefore, w is a point of coincidence of T and S. If w_1 is another point of coincidence then there is $w_1 \in X$ with $w_1 = Tw_1 = Sw_1$. Now from 3.9 it follows that $G(w, w_1, w_1) = 0$, that is $w = w_1$.

If S and T are weakly compatible, then it is obvious that w is unique common fixed point of T and S in X.

If S and T satisfies condition 3.10, then the argument is similar to that above. However to show that the sequence $\{x_n\}$ is $G_{\mathbb{E}} -$ *Cauchy sequence*, we start with

$$\begin{aligned} G(Sx_n, Sx_n, Sx_{n+1}) &= G(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq \alpha \max\{G(Sx_{n-1}, Sx_{n-1}, Sx_n), G(Sx_n, x_{n+1}, x_{n+1})\} \\ G(Sx_n, Sx_{n+1}, Sx_{n+1}) &\leq \alpha G(Sx_{n-1}, Sx_{n-1}, Sx_n). \end{aligned}$$

Continuing in the same way, we have

$$G(Sx_n, Sx_n, Sx_{n+1}) \leq \alpha^n G(Sx_0, Sx_0, Sx_1).$$

Then for all $n, m \in N, n < m$, we have by (VGM-2) we prove the remaining part of the proof.

□

4. Acknowledgement

The authors thank the referees for their careful reading of the manuscript and for their suggestions.

References

1. C.D. Aliprantis, K.C. Border, "Infinite Dimensional Analysis", Springer-Verlag, Derlin, 1999.
2. C.D. Aliprantis and R. Tourky, *Cones and Duality, in: Graduate studies in Mathematics*, Amer. Math. Soc. **84** (2007). 215–240.

3. I. Beg, M. Abbas and T. Nazir, *Generalized cone metric spaces*, J. Nonlinear Sci. Appl. **3** (2010), no. 1, 23–31.
4. L. G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **322** (2007), 1468–1476.
5. A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28** (1969), 326–329.
6. Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear and Convex Anal. **7** (2006), no. 2, 289–297.
7. Z. Mustafa, W. Shatanawi and M. Bataineh, *Existence of fixed point result in G-metric spaces,* , Int. J. Math. Math. Sci. 2009(2009), page 10, Article ID 283028.
8. Z. Mustafa and B. Sims, *Fixed point theorems for contractive mappings in complete G-metric space*, Fixed Point Theory Appl. 2009(2009), page 10, Article ID 917175.
9. Z. Mustafa, H. Obiedat and F. Awawdeh, Some of fixed point theorem for mapping on complete G-metric spaces, *Fixed Point Theory Appl.,* 2008(2008), Article ID 189870,page 12.
10. W. Shatanawi: *Some fixed point theorems in ordered G-metric spaces and applications,* Abstr. Appl. Anal.,(2011), Article ID 126205, 11 p.
11. W. Shatanawi, Fixed point theory for contractive mappings satisfying - maps in G-metric spaces, *Fixed Point Theory Appl.* 2010(2010), Article ID 181650, 9 pages.
12. M. Abbas, T. Nazir and P. Vetro: Common fixed point results for three maps in G-metric spaces, Filomat, 25(2011), 1-17.
13. H. Aydi: *A fixed point result involving a generalized weakly contractive condition in G-metric spaces*, Bull. Math. Anal. Appl., 3(2011), No. 4, 180-188.

[1]Department of Mathematics, Govt. Science & Commerce College, Benazir Bhopal-India

[2]Department of Mathematics, Govt. Science P.G. College, Reewa -India

*Corresponding author