International Journal of Analysis and Applications
ISSN 2291-8639
Volume 15, Number 2 (2017), 125-137
DOI: 10.28924/2291-8639-15-2017-125

SOME INTEGRAL INEQUALITIES USING QUANTUM CALCULUS APPROACH

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Abstract. The aim of this paper is to introduce a new class of preinvex functions which is called as generalized beta preinvex functions. We show that this class includes some other new classes of preinvex functions. We derive some new integral inequalities using the approach of quantum calculus. These integral inequalities involve generalized preinvex functions and q-Euler-Beta functions. Our results can be viewed as new quantum estimates for trapezoidal like inequalities. Some new special cases are also discussed which can be deduced from the main results of the paper.

1. Introduction and Preliminaries

The property of convexity of a function has attracted several researchers over the years and consequently this property has been generalized in different dimensions according to need, for some useful details, interested readers are referred to [3,4,6,9,14,15,17,18,24,25,28,34] and the references therein. Varošanec [33] introduced the notion of h-convex functions which not only generalizes the class of classical convex function but also generalizes several other classes of convex functions, such as Breckner type of s-convex functions, Godunova-Levin-Dragomir type of s-convex functions, Godunova-Levin functions and P-functions. So naturally the class of h-convex functions is quite unifying one. In recent years several authors have investigated the class of h-convex functions with respect to integral inequalities. Recently Tunç et al. [32] introduced the notion of so-called tgs-convex functions as:

Definition 1.1 ([32]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that f is tgs-convex function on I, if*

$$f((1-t)u+tv) \leq t(1-t)[f(u)+f(v)], \quad \forall u,v \in I, t \in]0,1[. \tag{1.1}$$

Note that f is tgs-concave function if $-f$ is tgs-convex function. Also if $t = 0, 1$, then, according to the hypothesis the function is equal to zero.

Remark 1.1. *It has been noticed that the class of tgs-convex functions is also contained in the class of h-convex functions by taking suitable choice of the function $h(\cdot)$.*

Hanson [9] introduced the notion of invex functions while studying mathematical programming. Ben Israel and Mond introduced the class of the invex sets and then using invex sets as domain they defined the notion of preinvex functions. They have shown that the differentiable preinvex functions imply invex functions. Under certain suitable conditions, one can show that these two classes of are equivalent. Noor [17] had shown that the minimum of the differentiable preinvex functions on the invex sets can be characterized by a class of variational inequalities, which is known as variational-like inequalities. Recently Noor et al. [25] extended the classes of preinvex functions and h-convex functions and introduced the notion of h-preinvex functions. This class not only contains the classes of preinvex functions and h-convex functions but also other classes of convex functions.

Inequalities play pivotal role in mathematical analysis. Convexity property of functions has also a close relationship with theory of inequalities. This relationship has attracted several authors and resultantly numerous new inequalities have been obtained via convex functions and also for its other generalizations. For some more information, see [2,5,6,8,10,14,16,20–22,24,27].

Received 24th June, 2017; accepted 12th August, 2017; published 1st November, 2017.
2010 *Mathematics Subject Classification.* 26A51; 26D15.
Key words and phrases. convex; generalized preinvex; functions; quantum; q-differentiable; integral inequalities.

In this paper, we define a new class of preinvex functions which is called as generalized preinvex functions. We obtain some new quantum bounds which involve primarily the property of generalized preinvexity. Some special cases are also discussed which can be deduced from the main results of the paper. It is expected that interested readers may find some novel applications of generalized preinvex functions and its related inequalities in other fields of pure and applied sciences. This is the main motivation of this paper.

2. Preliminaries

Let $\mathscr{K}_\eta$ be a nonempty closed set in $\mathbb{R}^n$. Let $f : \mathscr{K}_\eta \to \mathbb{R}$ be a continuous function and let $\eta(.,.) : \mathscr{K}_\eta \times \mathscr{K}_\eta \to \mathbb{R}^n$ be a continuous bifunction. First of all, we recall some known results and concepts.

Definition 2.1 ([1]). *A set $\mathscr{K}_\eta$ is said to be invex set with respect to $\eta(.,.)$, if*

$$u + t\eta(v,u) \in \mathscr{K}_\eta, \quad \forall u, v \in \mathscr{K}_\eta, t \in [0,1]. \tag{2.1}$$

The invex set $\mathscr{K}_\eta$ is also called η-connected set. If $\eta(v,u) = v - u$, then we have classical convex set $(1-t)u + tv \in \mathscr{K}$.

Definition 2.2 ([34]). *A function $f : \mathscr{K}_\eta \to \mathbb{R}$ is said to be preinvex with respect to arbitrary bifunction $\eta(.,.)$, if*

$$f(u + t\eta(v,u)) \leq (1-t)f(u) + tf(v), \quad \forall u, v \in \mathscr{K}_\eta, t \in [0,1]. \tag{2.2}$$

For $\eta(v,u) = v - u$ in (2.2), the preinvex functions reduces to classical convex functions.

Definition 2.3. *A function $f : \mathscr{K} \to \mathbb{R}$ is said to be convex in the classical sense, if*

$$f(u + t(v-u)) \leq (1-t)f(u) + tf(v), \quad \forall u, v \in \mathscr{K}_\eta, t \in [0,1].$$

Noor et al. [25] have defined the class of h-preinvex functions as:

Definition 2.4 ([25]). *Let $h : J \to \mathbb{R}$ where $(0,1) \subseteq J$ be an interval in $\mathbb{R}$, and let $\mathscr{K}_\eta$ be an invex set with respect to $\eta(.,.)$. A function $f : \mathscr{K}_\eta \to \mathbb{R}$ is called h-preinvex with respect to $\eta(.,.)$, if*

$$f(u + t\eta(v,u)) \leq h(1-t)f(u) + h(t)f(v), \quad u, v \in \mathscr{K}_\eta, t \in (0,1).$$

If above inequality is reversed, then f is said to be h-preincave with respect to bifunction $\eta(.,.)$. Noor et al. [25] have shown that the class of h-preinvex functions includes several other classes of preinvex functions as special cases.

Remark 2.1. *In this paper function $\eta(.,.) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is supposed to have the following property:*

$$\eta(v + t_1\eta(u,v), v + t_2\eta(u,v)) = (t_1 - t_2)\eta(u,v), \quad \forall t_1, t_2 \in [0,1], t_1 \leq t_2. \tag{2.3}$$

In this case the following consequences hold:

1. *If $t_1 = t_2 = 0$ then (2.3) implies that $\eta(v,v) = 0$ for all $v \in \mathbb{R}$.*
2. *If $t_1 = 0$ and $t_2 = t > 0$ then $\eta(v, v + t\eta(u,v)) = -t\eta(u,v)$ for all $u, v \in \mathbb{R}$. This is the first requirement of Condition C introduced in [13].*
3. *If $\eta(u,v) > 0$ for some $(u,v) \in \mathbb{R}$ then $\eta(v, v + t\eta(u,v)) \leq 0$ for all $t \in [0,1]$. It means that property (2.3) implies that function η has not constant sign on $\mathbb{R} \times \mathbb{R}$.*

Now we define the class of so-called generalized beta preinvex functions.

Definition 2.5. *Let $\mathscr{K}_\eta$ be an invex set with respect to bifunction $\eta(.,.)$ and $g, h : (0,1) \to \mathbb{R}$ be real functions. A function $f : \mathscr{K}_\eta \to \mathbb{R}$ is said to be generalized beta preinvex with respect to bifunction $\eta(.,.)$, if*

$$f(u + t\eta(v,u)) \leq g(t)h(1-t)f(u) + g(1-t)h(t)f(v), \quad \forall u, v \in \mathscr{K}_\eta, t \in]0,1[.$$

We now discuss some special cases of Definition 2.5.

I. If we take $g(t) = t^\alpha$ and $h(t) = t^\gamma$, where $\alpha, \gamma \in [0,1]$, then we have following definition of (α,γ)-preinvex functions.

Definition 2.6. *Let $\mathscr{K}_\eta$ be an invex set with respect to bifunction $\eta(.,.)$. A function $f : \mathscr{K}_\eta \to \mathbb{R}$ is said to be (α,γ)-preinvex with respect to bifunction $\eta(.,.)$, if*

$$f(u+t\eta(v,u)) \leq t^\alpha(1-t)^\gamma f(u) + (1-t)^\alpha t^\gamma f(v), \quad \forall u,v \in \mathscr{K}_\eta, t \in]0,1[, \alpha,\gamma \in [0,1].$$

II. If we take $g(t) = t^\alpha$ and $h(t) = t^\alpha$, where $\alpha \in [0,1]$, then we have following definition of α-preinvex functions.

Definition 2.7. *Let $\mathscr{K}_\eta$ be an invex set with respect to bifunction $\eta(.,.)$. A function $f : \mathscr{K}_\eta \to \mathbb{R}$ is said to be α-preinvex with respect to bifunction $\eta(.,.)$, if*

$$f(u+t\eta(v,u)) \leq [t^\alpha(1-t)^\alpha][f(u)+f(v)], \quad \forall u,v \in \mathscr{K}_\eta, t \in]0,1[, \alpha \in [0,1].$$

III. If we take $g(t) \equiv 1$, then Definition 2.5 reduces to the definition of h-preinvex functions [25].

We now define another new class of generalized preinvex functions. This class is not included in the class of generalized beta preinvex functions.

Definition 2.8. *Let $\mathscr{K}_\eta$ be an invex set with respect to bifunction $\eta(.,.)$. A function $f : \mathscr{K}_\eta \to \mathbb{R}$ is said to be generalized preinvex with respect to bifunction $\eta(.,.)$, if*

$$f(u+t\eta(v,u)) \leq [t^\alpha(1-t)^\gamma][f(u)+f(v)], \quad \forall u,v \in \mathscr{K}_\eta, t \in]0,1[, \alpha,\gamma \in [0,1].$$

Note that the class of generalized preinvex functions is included in the class of h-preinvex functions. Also if $\eta(v,u) = v-u$ and $\alpha = 1 = \gamma$, then we have classical tgs-convex functions.

We now recall some previously known concepts and results of quantum calculus. These results will be helpful in the development of our main results. These results are mainly due to Tariboon et al. [30,31].

Let $J = [a,b] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant. The q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a,b]$ is defined as follows.

Definition 2.9. *Let $f : J \to \mathbb{R}$ be a continuous function and let $x \in J$. Then q-derivative of f on J at x is defined as*

$$\mathscr{D}_q f(x) = \frac{f(x) - f(qx+(1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \tag{2.4}$$

A function f is q-differentiable on J if $\mathscr{D}_q f(x)$ exists for all $x \in J$.

Definition 2.10. *Let $f : J \to \mathbb{R}$ is a continuous function. A second-order q-derivative on J, which is denoted as $\mathscr{D}_q^2 f$, provided $\mathscr{D}_q f$ is q-differentiable on J is defined as $\mathscr{D}_q^2 f = \mathscr{D}_q(\mathscr{D}_q f) : J \to \mathbb{R}$. Similarly higher order q-derivative on J is defined by $\mathscr{D}_q^n f := J \to \mathbb{R}$.*

Lemma 2.1. *Let $\alpha \in \mathbb{R}$, then*

$$\mathscr{D}_q (x-a)^\alpha = \Big(\frac{1-q^\alpha}{1-q}\Big)(x-a)^{\alpha-1}.$$

Tariboon et al. [30,31] defined the q-integral as:

Definition 2.11. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on I is defined as*

$$\int_a^x f(t)\mathrm{d}_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \tag{2.5}$$

for $x \in J$.

These integrals can be viewed as Riemann-type q-integral. Moreover, if $c \in (a, x)$, then the definite q-integral on J is defined by

$$\int_c^x f(t)\mathrm{d}_q t = \int_a^x f(t)\mathrm{d}_q t - \int_a^c f(t)\mathrm{d}_q t$$
$$= (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a)$$
$$- (1-q)(c-a)\sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a).$$

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a continuous function, then*

(1) $\mathscr{D}_q \int\limits_a^x f(t)\,\mathrm{d}_q t = f(x)$

(2) $\int\limits_c^x \mathscr{D}_q f(t)\mathrm{d}_q t = f(x) - f(c)$ *for* $x \in (c, x)$.

Theorem 2.2. *Let $f, g : I \to \mathbb{R}$ be a continuous functions, $\alpha \in \mathbb{R}$, then $x \in J$*

(1) $\int\limits_a^x [f(t) + g(t)]\,\mathrm{d}_q t = \int\limits_a^x f(t)\,\mathrm{d}_q t + \int\limits_a^x g(t)\,\mathrm{d}_q t$

(2) $\int\limits_a^x (\alpha f(t))(t)\,\mathrm{d}_q t = \alpha \int\limits_a^x f(t)\,\mathrm{d}_q t$

(3) $\int\limits_a^x f(t)\,{}_a\mathscr{D}_q g(t)\,\mathrm{d}_q t$

$= (fg)|_c^x - \int\limits_c^x g(qt + (1-q)a)\mathscr{D}_q f(t)\mathrm{d}_q t$ *for* $c \in (a, x)$.

Lemma 2.2. *Let $\alpha \in \mathbb{R} \setminus \{-1\}$, then*

$$\int_a^x (t-a)^{\alpha}\mathrm{d}_q t = \Big(\frac{1-q}{1-q^{\alpha+1}}\Big)(x-a)^{\alpha+1}.$$

We now recall the definitions of q-gamma ($\Gamma_q(.)$) and q-beta ($\mathbb{B}_q(.,.)$) functions.

Definition 2.12 ([12])**.** *For $\alpha > 0$, the $\Gamma_q(.)$ function is defined as:*

$$\Gamma_q(\alpha) = \int_0^{\frac{1}{1-q}} t^{\alpha-1}E_q^{-qt}\mathrm{d}_q t,$$

where E_q^x is one of the following q-analogues of the exponential function:

$$E_q^t := \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}\frac{t^n}{[n]!} = (1+(1-q)t)_q^{\infty} = \prod_{j=0}^{\infty}(1+q^j(1-q)t);$$

$$e_q^t := \sum_{n=0}^{\infty}\frac{t^n}{[n]!} = \frac{1}{(1-(1-q)t)_q^{\infty}} = \frac{1}{\prod\limits_{j=0}^{\infty}(1-q^j(1-q)t)}.$$

Definition 2.13 ([12])**.** *For $\alpha > 0, \gamma > 0$, the $\mathbb{B}_q(.,.)$ function is defined as:*

$$\mathbb{B}_q(\alpha,\gamma) = \int_0^1 t^{\alpha-1}(1-qt)_q^{\gamma-1}\mathrm{d}_q t,$$

where

$$(1-qt)_q^{\gamma-1} = \frac{(1-qt)_q^{\infty}}{(1-q^{\gamma}t)_q^{\infty}}.$$

q-Gamma and q-beta functions are related by the following relation:

$$\mathbb{B}_q(\alpha,\gamma)=\frac{\Gamma_q(\alpha)\Gamma_q(\gamma)}{\Gamma_q(\alpha+\gamma)}.$$

Also we have

$$(1-t)^{\gamma}\leq(1-qt)^{\gamma}\leq(1-qt)_q^{\gamma},$$

for $0\leq t\leq 1$, $\gamma>0$ and $0<q<1$. For more details one can consult [12].

Noor et al. [23] established a new quantum integral identity for first order q-differentiable functions, which reads as follows:

Lemma 2.3. *Let $f:I_\eta\to\mathbb{R}$ be a continuous function and $0<q<1$. If ${}_a\mathrm{D}_q f$ is an integrable function on I_η^0, then*

$$\begin{aligned}&R_f^{'}(a,a+\eta(b,a);q;\eta)\\&=\frac{q\eta(b,a)}{1+q}\int_0^1(1-(1+q)t)\,{}_a\mathrm{D}_q f(a+t\eta(b,a))\,{}_0\mathrm{d}_q t,\end{aligned}$$

where

$$R_f^{'}(a,a+\eta(b,a);q;\eta)=\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,{}_a\mathrm{d}_q x-\frac{qf(a)+f(a+\eta(b,a))}{1+q}$$

Noor et al. [19] also established the following integral identity for twice q-differentiable functions.

Lemma 2.4. *Let $f:I_\eta\to\mathbb{R}$ be a twice q-differentiable function on I_η° such that $\mathscr{D}_q^2 f$ be continuous and integrable on I_η where $0<q<1$, then*

$$R_f^{''}(a,a+\eta(b,a);q;\eta)=\frac{q^2\eta^2(b,a)}{1+q}\int_0^1 t(1-qt)\mathscr{D}_q^2 f(a+t\eta(b,a))\mathrm{d}_q t,$$

where

$$R_f^{''}(a,a+\eta(b,a);q;\eta)=\frac{qf(a)+f(a+\eta(b,a))}{1+q}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}_q x.$$

For some useful information on quantum calculus interested readers are referred to [7,11,12,29]. From now onwards $I=[a,a+\eta(b,a)]$ will be the interval unless otherwise specified.

3. Some q-Hermite-Hadamard type inequalities via generalized preinvex functions

In this section, we discuss our main results.

Theorem 3.1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be generalized preinvex function, where $\eta(b,a)>0$ and moreover $\eta(.,.)$ satisfies Condition C, then*

$$2^{\alpha+\gamma-1}f\left(\frac{a+\eta(b,a)}{2}\right)\leq\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}_q x\leq\mathbb{B}_q(\alpha+1,\gamma+1)[f(a)+f(b)].$$

Proof. Since it is given that f is a generalized preinvex function and $\eta(.,.)$ satisfies Condition C, then

$$f\left(\frac{2a+\eta(b,a)}{2}\right)\leq\frac{1}{2^{\alpha+\gamma}}[f(a+(1-t)\eta(b,a))+f(a+t\eta(b,a))].$$

q-integrating above inequality with respect to t on $[0,1]$, we have

$$2^{\alpha+\gamma-1}f\left(\frac{2a+\eta(b,a)}{2}\right)\leq\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}_q x.\tag{3.1}$$

Also

$$f(a+t\eta(b,a)) \le t^{\alpha}(1-t)^{\gamma}[f(a)+f(b)].$$

q-integrating above inequality with respect to t on $[0,1]$, we have

$$\frac{1}{\eta(b,a)}\int_{a}^{b} f(x)\mathrm{d}_q x \le \mathbb{B}_q(\alpha+1,\gamma+1)[f(a)+f(b)]. \tag{3.2}$$

On summation of inequalities (3.1) and (3.2), we get the required result. □

If $q \to 1$, $\alpha=\gamma=1$ and $\eta(b,a)=b-a$ in Theorem 3.1, we get Theorem 2.1 [32].
If $q \to 1$ and $\alpha=\gamma=1$ in Theorem 3.1, we have following new result for generalized preinvex functions

Corollary 3.1. *Under the assumptions of Theorem 3.1, if $q \to 1$ and $\alpha=\gamma=1$, then we have*

$$2f\left(\frac{a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{6}.$$

If $\alpha=\gamma=1$ in Theorem 3.1, we have following new result for tgs-preinvex functions

Corollary 3.2. *Under the assumptions of Theorem 3.1, if $\alpha=\gamma=1$, then we have*

$$2f\left(\frac{a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)\mathrm{d}_q x \le \left(\frac{1}{1+q}-\frac{1}{1+q+q^2}\right)[f(a)+f(b)].$$

Our next result is q-Hermite-Hadamard's inequality via product of two generalized preinvex functions.

Theorem 3.2. *Let $f,g : I \subset \mathbb{R} \to \mathbb{R}$ be two generalized preinvex functions. Moreover $\eta(.,.)$ satisfies Condition C and $\eta(b,a)>0$, then*
I. *The left side of the inequality reads as:*

$$\begin{aligned}&2^{2(\alpha+\gamma)-1} f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right) - \mathbb{B}_q(\alpha+\gamma+1,\alpha+\gamma+1)[M(a,b)+N(a,b)]\\ &\le \frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)g(x)\mathrm{d}_q x,\end{aligned}$$

II. *The right side of the inequality reads as:*

$$\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)g(x)\mathrm{d}_q x \le [M(a,b)+N(a,b)]\mathbb{B}_q(\alpha+\gamma+1,\alpha+\gamma+1),$$

where

$$M(a,b) = f(a)g(a)+f(b)g(b); \tag{3.3}$$
$$N(a,b) = f(a)g(b)+f(b)g(a). \tag{3.4}$$

Proof. **I.** Since it is given that f and g are generalized preinvex functions, then

$$\begin{aligned}&f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right)\\ &\le \frac{1}{2^{2(\alpha+\gamma)}}[f(a+(1-t)\eta(b,a))+f(a+t\eta(b,a))][g(a+(1-t)\eta(b,a))+g(a+t\eta(b,a))]\\ &= \frac{1}{2^{2(\alpha+\gamma)}}\{f(a+(1-t)\eta(b,a))g(a+(1-t)\eta(b,a))+f(a+t\eta(b,a))g(a+t\eta(b,a))\\ &\quad +f(a+t\eta(b,a))g(a+(1-t)\eta(b,a))+f(a+(1-t)\eta(b,a))g(a+t\eta(b,a))\}.\end{aligned}$$

q-integrating both sides of above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}
& f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right) \\
& \leq \frac{1}{2^{2(\alpha+\gamma)}}\left\{\int_0^1 f(a+(1-t)\eta(b,a))g(a+(1-t)\eta(b,a))\mathrm{d}_q t\right. \\
& \quad + \int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))\mathrm{d}_q t \\
& \quad \left. + \int_0^1 f(a+t\eta(b,a))g(a+(1-t)\eta(b,a))\mathrm{d}_q t + \int_0^1 f(a+(1-t)\eta(b,a))g(a+t\eta(b,a))\mathrm{d}_q t\right\} \\
& = \frac{1}{2^{2(\alpha+\gamma)}}\left\{\frac{2}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}_q x + 2\int_0^1 t^{\alpha+\gamma}(1-t)^{\alpha+\gamma}[f(a)+f(b)][g(a)+g(b)]\mathrm{d}_q t\right\} \\
& = \frac{1}{2^{2(\alpha+\gamma)-1}}\left\{\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}_q x + \mathbb{B}_q(\alpha+\gamma+1,\alpha+\gamma+1)\right. \\
& \qquad \left. \times [f(a)+f(b)][g(a)+g(b)]\right\} \\
& = \frac{1}{2^{2(\alpha+\gamma)-1}}\left\{\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}_q x + \mathbb{B}_q(\alpha+\gamma+1,\alpha+\gamma+1)[M(a,b)+N(a,b)]\right\}.
\end{aligned}$$

This completes the proof of first part.

II. Now we prove second part of the theorem. Using the hypothesis of the theorem that f and g are generalized preinvex functions, we have

$$f(a+t\eta(b,a))g(a+t\eta(b,a)) \leq t^{\alpha+\gamma}(1-t)^{\alpha+\gamma}[f(a)+f(b)][g(a)+g(b)].$$

q-integrating both sides of above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}
\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}_q x &= \int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))\mathrm{d}_q t \\
&\leq [f(a)+f(b)][g(a)+g(b)]\int_0^1 t^{\alpha+\gamma}(1-t)^{\alpha+\gamma}\mathrm{d}_q t \\
&= [M(a,b)+N(a,b)]\mathbb{B}_q(\alpha+\gamma+1,\alpha+\gamma+1).
\end{aligned}$$

This completes the proof of second part. □

Note that when $q \to 1$, $\alpha = \gamma = 1$ and $\eta(b,a) = b-a$ Theorem 3.2 reduces to previously known results, see [32].
When $q \to 1$ and $\alpha = \gamma = 1$ Theorem 3.2 reduces to new result for tgs-preinvexity.

Corollary 3.3. *Under the assumptions of Theorem 3.2, if $q \to 1$ and $\alpha = \gamma = 1$, then, we have*

I. $8f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{30}[M(a,b)+N(a,b)] \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}x,$

II. $\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}x \leq \frac{1}{30}[M(a,b)+N(a,b)],$

where $M(a,b)$ and $N(a,b)$ are given by (3.3) and (3.4) respectively.

When $\alpha=\gamma=1$ Theorem 3.2 reduces to new result for tgs-preinvexity.

Corollary 3.4. *Under the assumptions of Theorem 3.2, if $\alpha=\gamma=1$, then, we have*

I. $8f\left(\frac{2a+\eta(b,a)}{2}\right)g\left(\frac{2a+\eta(b,a)}{2}\right)-\psi_1(q)[M(a,b)+N(a,b)]\leq\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)}f(x)g(x)\mathrm{d}_qx,$

II. $\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)}f(x)g(x)\mathrm{d}_qx\leq\psi_1(q)[M(a,b)+N(a,b)],$

where $M(a,b)$ and $N(a,b)$ are given by (3.3), (3.4) and

$$\psi_1(q):=\frac{1}{1+q+q^2}+\frac{1}{1+q+q^2+q^3+q^4}-\frac{2}{1+q+q^2+q^3}, \tag{3.5}$$

respectively.

Theorem 3.3. *Let $f,g:I\subset\mathbb{R}\to\mathbb{R}$ be two generalized preinvex functions, then*

$$\begin{aligned}&\frac{1}{2\eta^2(b,a)\mathbb{B}_q(\alpha+\gamma+1,\alpha+\gamma+1)}\int\limits_a^{a+\eta(b,a)}\int\limits_a^{a+\eta(b,a)}\int\limits_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))\mathrm{d}_qt\mathrm{d}_qy\mathrm{d}_qx\\&\leq\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)}f(x)g(x)\mathrm{d}_qx+\mathbb{B}_q^2(\alpha+1,\gamma+1)[M(a,b)+N(a,b)],\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (3.3) and (3.4) respectively.

Proof. Since it is given that f and g are generalized preinvex functions, then

$$f(x+t\eta(y,x))g(x+t\eta(y,x))\leq t^{\alpha+\gamma}(1-t)^{\alpha+\gamma}[f(x)+f(y)][g(x)+g(y)].$$

q-integrating both sides of above inequality with respect to t on the interval $[0,1]$, we have

$$\begin{aligned}&\int\limits_0^1 f(x+t\eta(y,x))g(x+t\eta(y,x))\mathrm{d}_qt\\&\leq\int\limits_0^1 t^{\alpha+\gamma}(1-t)^{\alpha+\gamma}[f(x)+f(y)][g(x)+g(y)]\mathrm{d}_qt\\&=\mathbb{B}_q(\alpha+\gamma+1,\alpha+\gamma+1)[f(x)+f(y)][g(x)+g(y)].\end{aligned}$$

Now again q-integrating both sides of above inequality on $[a, a+\eta(b,a)] \times [a, a+\eta(b,a)]$, we have

$$
\begin{aligned}
&\int_a^{a+\eta(b,a)} \int_a^{a+\eta(b,a)} \int_0^1 f(x+t\eta(y,x))g(x+t\eta(y,x))\mathrm{d}_q t \mathrm{d}_q y \mathrm{d}_q x \\
&\leq \mathbb{B}_q(\alpha+\gamma+1, \alpha+\gamma+1) \int_a^{a+\eta(b,a)} \int_a^{a+\eta(b,a)} [f(x)+f(y)][g(x)+g(y)]\mathrm{d}_q y \mathrm{d}_q x \\
&= \mathbb{B}_q(\alpha+\gamma+1, \alpha+\gamma+1) \\
&\quad \times \int_a^{a+\eta(b,a)} \int_a^{a+\eta(b,a)} [f(x)g(x)+f(y)g(y)+f(x)g(y)+f(y)g(x)]\mathrm{d}_q y \mathrm{d}_q x \\
&= \mathbb{B}_q(\alpha+\gamma+1, \alpha+\gamma+1) \\
&\quad \times \left[\int_a^{a+\eta(b,a)} \int_a^{a+\eta(b,a)} [f(x)g(x)+f(y)g(y)]\mathrm{d}_q y \mathrm{d}_q x \right. \\
&\quad \left. + \int_a^{a+\eta(b,a)} f(x)\mathrm{d}_q x \int_a^{a+\eta(b,a)} g(y)\mathrm{d}_q y + \int_a^{a+\eta(b,a)} f(y)\mathrm{d}_q y \int_a^{a+\eta(b,a)} g(x)\mathrm{d}_q x \right].
\end{aligned}
$$

Using Theorem 3.1, we have

$$
\begin{aligned}
&\int_a^{a+\eta(b,a)} \int_a^{a+\eta(b,a)} \int_0^1 f(x+t\eta(y,x))g(x+t\eta(y,x))\mathrm{d}_q t \mathrm{d}_q y \mathrm{d}_q x \\
&\leq \mathbb{B}_q(\alpha+\gamma+1, \alpha+\gamma+1) \\
&\times \left[2\eta(b,a) \int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}_q x + 2\eta^2(b,a)\mathbb{B}_q^2(\alpha+1, \gamma+1)[M(a,b)+N(a,b)] \right] \\
&= 2\mathbb{B}_q(\alpha+\gamma+1, \alpha+\gamma+1) \\
&\quad \times \left[\eta(b,a) \int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}_q x + \eta^2(b,a)\mathbb{B}_q^2(\alpha+1, \gamma+1)[M(a,b)+N(a,b)] \right].
\end{aligned}
$$

Multiplying both sides of of above inequality by $\frac{1}{\eta^2(b,a)}$ completes the proof. □

Note that when $q \to 1$, $\alpha = \gamma = 1$ and $\eta(b,a) = b-a$ in Theorem 3.3, we get Theorem 2.4 [32]. If we take $q \to 1$ and $\alpha = \gamma = 1$ in Theorem 3.3, we get a new result for tgs-preinvexity.

Corollary 3.5. *Under the assumptions of Theorem 3.3, if $q \to 1$ and $\alpha = \gamma = 1$, then, we have*

$$
\begin{aligned}
&\frac{15}{\eta^2(b,a)} \int_a^{a+\eta(b,a)} \int_a^{a+\eta(b,a)} \int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))\mathrm{d}t\mathrm{d}y\mathrm{d}x \\
&\leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}x + \frac{1}{36}[M(a,b)+N(a,b)],
\end{aligned}
$$

where $M(a,b)$, $N(a,b)$ are given by (3.3) and (3.4), respectively.

If we take $\alpha = \gamma = 1$ in Theorem 3.3, we get a new result for tgs-preinvexity.

Corollary 3.6. *Under the assumptions of Theorem 3.3, if $\alpha=\gamma=1$, then, we have*

$$\frac{\psi_1^{-1}(q)}{2\eta^2(b,a)}\int_a^{a+\eta(b,a)}\int_a^{a+\eta(b,a)}\int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))\mathrm{d}_qt\mathrm{d}_qy\mathrm{d}_qx$$
$$\leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}_qx+\left(\frac{1}{1+q}-\frac{1}{1+q+q^2}\right)[M(a,b)+N(a,b)],$$

where $M(a,b)$, $N(a,b)$ and $\psi_1(q)$ are given by (3.3), (3.4) and (3.5), respectively.

Now we prove some q-Hermite-Hadamard type inequalities via q-differentiable generalized preinvex functions.

Theorem 3.4. *Let $f:I=[a,a+\eta(b,a)]\subset\mathbb{R}\to\mathbb{R}$ be a q-differentiable function on I° (the interior of I) with $\mathcal{D}_q$ be continuous and integrable on I where $0<q<1$. If $|\mathcal{D}_q|^r$, $r\geq 1$ is generalized preinvex function, then*

$$\left|R_f'(a,a+\eta(b,a);q;\eta)\right|\leq\frac{q\eta(b,a)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}\left(\psi_2(q)\{|\mathcal{D}_qf(a)|^r+|\mathcal{D}_qf(b)|^r\}\right)^{\frac{1}{r}},$$

where

$$\psi_2(q)=\mathbb{B}_q(\alpha+1,\gamma+1)-(1+q)\mathbb{B}_q(\alpha+2,\gamma+1). \tag{3.6}$$

Proof. Utilizing Lemma 2.3, property of modulus, power mean inequality and the hypothesis of the theorem, we have

$$\begin{aligned}
&\left|R_f'(a,a+\eta(b,a);q;\eta\right|\\
&=\left|\frac{q\eta(b,a)}{1+q}\int_0^1(1-(1+q)t)\mathcal{D}_qf(a+t\eta(b,a))\,\mathrm{d}_qt\right|\\
&\leq\frac{q\eta(b,a)}{1+q}\left(\int_0^1|1-(1+q)t|\mathrm{d}_q\right)^{1-\frac{1}{r}}\left(\int_0^1(1-(1+q)t)|\mathcal{D}_qf(a+\eta(b,a))|^r\mathrm{d}_qt\right)^{\frac{1}{r}}\\
&\leq\frac{q\eta(b,a)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}\left(\int_0^1(1-(1+q)t)[t^\alpha(1-t)^\gamma\{|\mathcal{D}_qf(a)|^r+|\mathcal{D}_qf(b)|^r\}]\mathrm{d}_qt\right)^{\frac{1}{r}}\\
&=\frac{q\eta(b,a)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}\{[\mathbb{B}_q(\alpha+1,\gamma+1)-(1+q)\mathbb{B}_q(\alpha+2,\gamma+1)]\{|\mathcal{D}_qf(a)|^r+|\mathcal{D}_qf(b)|^r\}\}^{\frac{1}{r}}\\
&=\frac{q\eta(b,a)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}\left(\psi_2(q)\{|\mathcal{D}_qf(a)|^r+|\mathcal{D}_qf(b)|^r\}\right)^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

Now using Lemma 2.4, we drive some more quantum estimates for Hermite-Hadamard type inequalities via twice q-differentiable generalized preinvex functions.

Theorem 3.5. *Let $f:I_\eta\to\mathbb{R}$ be a twice q-differentiable function on I_η° such that $\mathscr{D}_q^2f$ be continuous and integrable on I_η where $0<q<1$. If $|\mathscr{D}_q^2f(x)|$ is generalized preinvex function, then*

$$\left|R_f''(a,a+\eta(b,a);q;\eta)\right|\leq\frac{q^2\eta^2(b,a)}{1+q}\Omega_q(\alpha,\gamma)\{|\mathscr{D}_q^2f(a)|+|\mathscr{D}_q^2f(b)|\},$$

where

$$\Omega_q(\alpha,\gamma):=\mathbb{B}_q(\alpha+2,\gamma+1)-q\mathbb{B}_q(\alpha+3,\gamma+1). \tag{3.7}$$

Proof. Using Lemma 2.4, property of modulus and the fact that $|\mathscr{D}_q^2 f(x)|$ is generalized preinvex function, we have

$$\begin{aligned}
&\left|R_f''(a, a+\eta(b,a); q; \eta)\right| \\
&= \left|\frac{q^2\eta^2(b,a)}{1+q}\int_0^1 t(1-qt)\mathscr{D}_q^2 f(a+t\eta(b,a))\mathrm{d}_q t\right| \\
&\leq \frac{q^2\eta^2(b,a)}{1+q}\int_0^1 t(1-qt)|\mathscr{D}_q^2 f(a+t\eta(b,a))|\mathrm{d}_q t \\
&\leq \frac{q^2\eta^2(b,a)}{1+q}\int_0^1 t(1-qt)[t^\alpha(1-t)^\gamma\{|\mathscr{D}_q^2 f(a)|+|\mathscr{D}_q^2 f(b)|\}]\mathrm{d}_q t \\
&= \frac{q^2\eta^2(b,a)}{1+q}\left(\mathbb{B}_q(\alpha+2,\gamma+1)-q\mathbb{B}_q(\alpha+3,\gamma+1)\right)\{|\mathscr{D}_q^2 f(a)|+|\mathscr{D}_q^2 f(b)|\}.
\end{aligned}$$

This completes the proof. □

Theorem 3.6. *Let $f : I_\eta \to \mathbb{R}$ be a twice q-differentiable function on I_η° such that $\mathscr{D}_q^2 f$ be continuous and integrable on I_η where $0 < q < 1$. If $|\mathscr{D}_q^2 f(x)|^r$ is generalized preinvex function, then, for $\frac{1}{p}+\frac{1}{r}=1$, $r > 1$, we have*

$$\left|R_f''(a, a+\eta(b,a); q; \eta)\right| \leq \frac{q^2\eta^2(b,a)}{1+q}\Phi^{\frac{1}{p}}(\alpha,q)\left[\mathbb{B}_q(\alpha+1,\gamma+1)\{|\mathscr{D}_q^2 f(a)|^r+|\mathscr{D}_q^2 f(b)|^r\}\right]^{\frac{1}{r}},$$

where

$$\Phi(\alpha,q) = (1-q)\sum_{n=0}^{\infty}(q^n)^{p+1}(1-q^{n+1})^p. \tag{3.8}$$

Proof. Using Lemma 2.4, Holder's inequality and the fact that $|\mathscr{D}_q^2 f(x)|^r$ is generalized preinvex function, we have

$$\begin{aligned}
&\left|R_f''(a, a+\eta(b,a); q; \eta)\right| \\
&= \left|\frac{q^2\eta^2(b,a)}{1+q}\int_0^1 t(1-qt)\mathscr{D}_q^2 f(a+t\eta(b,a))\mathrm{d}_q t\right| \\
&\leq \frac{q^2\eta^2(b,a)}{1+q}\int_0^1 t(1-qt)|\mathscr{D}_q^2 f(a+t\eta(b,a))|\mathrm{d}_q t \\
&\leq \frac{q^2\eta^2(b,a)}{1+q}\left(\int_0^1 (t-qt^2)^p\mathrm{d}_q t\right)^{\frac{1}{p}}\left(\{|\mathscr{D}_q^2 f(a)|^r+|\mathscr{D}_q^2 f(b)|^r\}\int_0^1 t^\alpha(1-t)^\gamma\mathrm{d}_q t\right)^{\frac{1}{r}} \\
&= \frac{q^2\eta^2(b,a)}{1+q}\left((1-q)\sum_{n=0}^{\infty}(q^n)^{p+1}(1-q^{n+1})^p\right)^{\frac{1}{p}}\left[\mathbb{B}_q(\alpha+1,\gamma+1)\{|\mathscr{D}_q^2 f(a)|^r+|\mathscr{D}_q^2 f(b)|^r\}\right]^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

Theorem 3.7. *Let $f : I_\eta \to \mathbb{R}$ be a twice q-differentiable function on I_η° such that $\mathscr{D}_q^2 f$ be continuous and integrable on I_η where $0 < q < 1$. If $|\mathscr{D}_q^2 f(x)|^r$ is generalized preinvex function, then, for $r \geq 1$, we have*

$$\left|R_f''(a, a+\eta(b,a); q; \eta)\right| \leq \frac{q^2\eta^2(b,a)}{1+q}\Theta_q^{1-\frac{1}{r}}\Omega_q^{\frac{1}{r}}(\alpha,\gamma)\left[|\mathscr{D}_q^2 f(a)|^r+|\mathscr{D}_q^2 f(b)|^r\right]^{\frac{1}{r}},$$

where $\Omega_q(\alpha,\gamma)$ *is given by (3.7) and*

$$\Theta_q=\frac{1}{(1+q)(1+q+q^2)}. \tag{3.9}$$

respectively.

Proof. Using Lemma 2.4, power means inequality, and the fact that $|\mathscr{D}_q^2 f(x)|^r$ is generalized preinvex function, we have

$$\begin{aligned}
&\left|R_f''(a,a+\eta(b,a);q;\eta)\right|\\
&=\left|\frac{q^2\eta^2(b,a)}{1+q}\int_0^1 t(1-qt)\mathscr{D}_q^2 f(a+t\eta(b,a))\mathrm{d}_q t\right|\\
&\leq \frac{q^2\eta^2(b,a)}{1+q}\left(\int_0^1 t(1-qt)\mathrm{d}_q t\right)^{1-\frac{1}{r}}\left(\int_0^1 t(1-qt)|\mathscr{D}_q^2 f(a+t\eta(b,a))|^r\mathrm{d}_q t\right)^{\frac{1}{r}}\\
&\leq \frac{q^2\eta^2(b,a)}{1+q}\left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{r}}\\
&\qquad\times\left(\int_0^1 t^{\alpha+1}(1-qt)(1-t)^{\gamma}[|\mathscr{D}_q^2 f(a)|^r+|\mathscr{D}_q^2 f(b)|^r]\mathrm{d}_q t\right)^{\frac{1}{r}}\\
&=\frac{q^2\eta^2(b,a)}{1+q}\left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{r}}\left[\Omega_q(\alpha,\gamma)\{|\mathscr{D}_q^2 f(a)|^r+|\mathscr{D}_q^2 f(b)|^r\}\right]^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. ∎

Acknowledgements

The authors are thankful to anonymous referees for their helpful comments and suggestions. Authors are pleased to acknowledge the support of "the support of Distinguished Scientist Fellowship Program(DSFP), King Saud University, Riyadh, Saudi Arabia".

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