International Journal of Analysis and Applications
ISSN 2291-8639
Volume 4, Number 1 (2014), 58-67
http://www.etamaths.com

APPROXIMATING DERIVATIVES BY A CLASS OF POSITIVE LINEAR OPERATORS

BRAMHA DUTTA PANDEY[1,*] AND B. KUNWAR[2]

ABSTRACT. Some Direct Theorems for the linear combinations of a new class of positive linear operators have been obtained for both, pointwise and uniform simultaneous approximations. a number of well known positive linear operators such as Gamma Operators of Muller, Post-Widder and Modified Post-Widder Operators are special cases of this class of operators.

1. INTRODUCTION

During past few decades a number of sequences of positive linear operators (henceforth written as operator) both, of the summation and those defined by integrals have been introduced and studied by a number of authors. Some of wellknown operators of latter type are the Gamma operators of Müller [7], Post-Widder and Modified Post-Widder operators [6], Kunwar [4], Sikkema and Rathore [11].

Now we define our linear operator L_n [4] as

$$L_n(f;x) = D(m,n,\alpha)x^{mn+\alpha-1}\int_0^\infty u^{-mn-\alpha}e^{-n(\frac{x}{u})^m}f(u)du \tag{1}$$

where $D(m,n,\alpha) = \frac{|m|n^{n+\frac{\alpha-1}{m}}}{\Gamma n+\frac{\alpha-1}{m}}, m \in IR-\{0\}, n > 0, \alpha \in IR.$

The equation (1) defines a linear positive approximation methods, which contains as particular cases, a number of well known linear positive operators; e.g. Post-Widder and Modified Post-Widder operators [6], and the Gamma-operators of Muller [7] .

In the present paper we study the following problems:

(i) Is it possible to approximate the derivatives of f by the derivatives of $L_n(f)$?

(ii) Can we use certain linear combinations of L_n to obtain a better order of approximation?

We introduce notations and definitions used in this paper.

Throughout the paper IR^+ denotes the interval $(0,\infty), < a,b >$ open interval containing $[a,b] \subseteq IR^+, \chi_{\delta,x}(\chi^c_{\delta,x})$ the characteristic function of the interval $(x-\delta, x+\delta)$ $\{IR^+ - (x-\delta, x+\delta)\}$. The spaces $M(IR^+), M_b(IR^+), Loc(IR^+), L^1(IR^+)$ respectively denote the sets of complex valued measurable, bounded and measurable, locally integrable and Lebesgue integrable functions on IR^+.

Let $\Omega(>1)$ be a continuous function defined on IR^+.We call Ω a bounding function if for each $K \subseteq IR^+$ there exist positive numbers n_K and M_K such that

$$L_{n_K}(\Omega;x) < M_K, \ \ x \in K.$$

2000 *Mathematics Subject Classification.* Primary 41A35, 41A38; Secondary 41A25, 41A60.
Key words and phrases. Positive linear operators, Simultaneous approximation, Linear combinations.

here $\Omega(u) = u^{-a} + e^{bu^m} + u^c$, where $a, b, c > 0$.

For this bounding function

$D_\Omega = \{f : f$ is locally integrable on IR^+ and is such that $\limsup_{u\to 0} \frac{f(u)}{\Omega(u)}$ and $\limsup_{u\to\infty} \frac{f(u)}{\Omega(u)}$ exist$\}$

$D_\Omega^{(k)} = \{f : f \in D_\Omega$ and f is k- times cotinuously differentiable on IR^+ and $f^{(i)} \in D_\Omega, i = 1, 2, ..., k\}$

$C_b^m(IR^+) = \{f : f$ is m-times continuously differentiable and is such that $f^{(k)}$, $k = 0, 1, 2, ...m$ are bounded on $IR^+\}$.

2. SIMULTANEOUS APPROXIMATION FOR CONTINUOUS DERIVATIVES

We consider the elementary case of simultaneous approximation by the operators L_n wherein the derivatives of f are assumed to be continuous. We have termed this case elementary, for it is possible here to deduce the results on the simultaneous approximation: $(L_n f)^{(k)} \to f^{(k)} (k \in IN)$ from the corresponding results on the ordinary approximation: $L_n f \to f$.

Theorem 1. *: If $f \in D_\Omega^{(k)}$,then $L_n^{(k)}(f; x)$ for $x \in< a, b >$ exists for all sufficiently large n and*

(2) $\lim_{n\to\infty} L_n^{(k)}(f; x) = f^{(k)}(x)$, *uniformly for $x \in [a, b]$.*

Proof. We have

$$L_n(f; x) = D(m, n, \alpha)x^{mn+\alpha-1} \int_0^\infty u^{-mn-\alpha} e^{-n(\frac{x}{u})^m} f(u)du$$

A formal k-times differentiation within the integral sign and replacing α by $\alpha-k$, let the new operator be denoted by L_n^* and the corresponding $D(m, n, \alpha)$ be denoted by $D^*(m, n, \alpha)$.Then

$$L_n^{(k)}(f; x) = \frac{D(m,n,\alpha)}{D^*(m,n,\alpha)} L_n^*(f^{(k)}(x)) \tag{3}$$

Applying the known approximation $L_n f \to f$ to (3), we find that

$$L_n^{(k)}(f; x) = \frac{D(m,n,\alpha)}{D^*(m,n,\alpha)} L_n^*(f^{(k)}(x)) \to f^{(k)}(x) \text{ as } n \to \infty.$$

This completes the proof of the theorem. □

Theorem 2. *:If $f \in D_\Omega^{(k)}$. then at each $x \in IR^+$ where $f^{(k+2)}$ exists*

$$L_n^{(k)}(f; x) - f^{(k)}(x) = \frac{1}{2nm^2}[(m + k - 2\alpha + 2)kf^{(k)}(x)+ \tag{4}$$
$$+(m+2k-2\alpha+3)xf^{(k+1)}(x)+x^2 f^{(k+2)}(x)]+o(\tfrac{1}{n}), n \to \infty.$$

Further if $f^{(k+2)}$ exists and is continuous on $< a, b >$, then (4) holds uniformly in $x \in [a, b]$.

Proof. Using Voronovskaya formula [1], [6], [10],[11], [12] for L_n^* and (3), the result follows. □

In a similer manner one can prove the following results:

Theorem 3. *: If f is such that $f^{(k)}$ exists and is continuous on IR^+, then*

$$\left|L_n^{(k)}(f; x) - f^{(k)}(x)\right| \le \omega_{f^{(k)}}(n^{-\frac{1}{2}})[1+ \tag{5}$$
$$+\min\{x^2(\tfrac{1}{m^2} + o(1)), x(\tfrac{1}{m^2} + o(1))^{\frac{1}{2}}] + o(\tfrac{1}{n}),$$
$$(n \to \infty, x \in IR^+).$$

where $\omega_{f^{(k)}}$ is the modulus of continuity of $f^{(k)}$ [13] [2] [3].

Theorem 4. *: Let f be such that $f^{(k+1)}$ exists on IR^+. Then for $x \in IR^+$*

$$(6)\quad \left|L_n^{(k)}(f;x) - f^{(k)}(x)\right|$$
$$\leq \frac{k\left|f^{(k)}(x)\right|}{2nm^2}\{|m+k-2\alpha+2|\} + x\frac{\left|f^{(k+1)}(x)\right|}{2nm^2}\{|m+k-2\alpha+3|\}+$$
$$+\omega_{f^{(k+1)}}(n^{-\frac{1}{2}})[xn^{-\frac{1}{2}}\{\frac{1}{m^2(m-3)}+o(1)\} + \frac{x^2}{2n^{\frac{1}{2}}}\{\frac{1}{m^2(m-3)}+o(1)\}],$$
$$(n\to\infty, x\in IR^+).$$

3. Pointwise Simultaneous Approximation

In the present section we consider the "non-elementary" case of simultaneous approximation wherein assuming only that $f^{(k)}(x)$ exist at some point x, we solve the problem of pointwise approximation. Before proving this result we establish:

Lemma 1. *:Let $n > p \in IN$ (set of natural numbers). Then*

$$(7)\quad \frac{\partial^p}{\partial x^p}\{x^{\alpha+mn-1}u^{-mn}e^{-n(\frac{x}{u})^m}\} = x^{mn+\alpha-p-1}u^{p-mn}e^{-(n-p)(\frac{x}{u})^m}$$
$$\times\sum_{k=0}^{p}\sum_{\nu=0}^{[\frac{p-k}{2}]}(\tfrac{m}{u})^k n^{\nu+k}(\tfrac{x}{u})^{k(m-1)}e^{-k(\frac{x}{u})^m}[1-(\tfrac{x}{u})^m]^k g_{\nu,k,p}(x,u)$$

where $[x]$ denotes the integral part of $x \in IR^+$ and the function $g_{\nu,k,p}(x,u)$ are certain linear combinations of products of the powers of u^{-1}, x^{-1} and $\frac{\partial^k}{\partial x^k}\{(\frac{x}{u})^m e^{-(\frac{x}{u})^m}\}, k = 0,1,2,...,p$ and are independent of n.

Proof. We proceed by induction on p. We note that

$$(8)\quad \frac{\partial}{\partial x}\{x^{mn+\alpha-1}u^{-mn}e^{-n(\frac{x}{u})^m}\}$$
$$= x^{(\alpha-1)}(\tfrac{x}{u})^{m(n-1)}e^{-(n-1)(\frac{x}{u})^m}[\tfrac{(mn+\alpha-1)}{x}(\tfrac{x}{u})^m e^{-(\frac{x}{u})^m} - \tfrac{mn}{u}(\tfrac{x}{u})^{2m-1}e^{-(\frac{x}{u})^m}]$$

Putting $g_{0,0,1}(x,u) = \frac{(\alpha-1)}{x}(\frac{x}{u})^m e^{-(\frac{x}{u})^m}$

$g_{0,1,1}(x,u) = u^{-1}$

We observe that (8) is of the form (7). Hence the result is true for $p = 1$.

Next, let us assume that the lemma holds for a certain p. Then by the induction hypothesis,

$$(9)\quad \frac{\partial^{p+1}}{\partial x^{p+1}}\{x^{\alpha+mn-1}u^{-mn}e^{-n(\frac{x}{u})^m}\}$$
$$= x^{\alpha-1}(\tfrac{x}{u})^{m(n-p-1)}e^{-(n-p-1)(\frac{x}{u})^m}$$
$$\times\sum_{k=0}^{p+1}\sum_{\nu=0}^{[(\frac{p-k+1}{2})]}(\tfrac{m}{u})^k n^{\nu+k}\{(\tfrac{x}{u})^{m-1}e^{-(\frac{x}{u})^m} - (\tfrac{x}{u})^{2m-1}e^{-(\frac{x}{u})^m}\}^k g_{\nu,k,p+1}(x,u)$$

Wherewith $g_{\nu,k,p} \equiv 0$ for $k > p$ or $k < 0, \nu < 0$ or $\nu > [\frac{p-k}{2}]$, we have put

$$g_{\nu,k,p+1}(x,u) = \tfrac{mn+\alpha-1}{x}g_{\nu,k,p}(x,u)(\tfrac{x}{u})^m e^{-(\frac{x}{u})^m}$$
$$-\tfrac{mn}{u}\{\tfrac{m}{u}(\tfrac{x}{u})^{m-1}e^{-(\frac{x}{u})^m} - \tfrac{m}{u}(\tfrac{x}{u})^{2m-1}e^{-(\frac{x}{u})^m}\}g_{\nu,k,p}(x,u)+$$
$$+\tfrac{\partial}{\partial x}g_{\nu,k,p}(x,u) + \tfrac{1}{u}g_{\nu,k-1,p}(x,u)+$$
$$+(\tfrac{k+1}{u})\{\tfrac{m(m-1)}{u^2}(\tfrac{x}{u})^{m-2}e^{-(\frac{x}{u})^m} - (\tfrac{m}{u})^2(\tfrac{x}{u})^{2(m-1)}e^{-(\frac{x}{u})^m} -$$
$$-\tfrac{m(2m-1)}{u^2}(\tfrac{x}{u})^{2(m-1)}e^{-(\frac{x}{u})^m} + (\tfrac{m}{u})^2(\tfrac{x}{u})^{3m-2}e^{-(\frac{x}{u})^m}\}g_{\nu-1,k+1,p}(x,u).$$

For $k = 0,1,2,...,p+1$ and $\nu = 0,1,2,....,[\frac{p+1-k}{2}]$

It is clear that $g_{\nu,k,p+1}(x,u)$ satisfies the other required properties and hence the result is true for $p+1$. Hence it follows that (8) holds for all $p = 1,2,....$ This completes the proof. □

Theorem 5. *: Let $m \in IN$ and $f \in D_\Omega$, then*

$$(10)\quad \lim_{n\to\infty} L_n^{(k)}(f;x) = f^{(k)}(x).$$

whenever $x \in IR^+$ is such that $f^{(k)}(x)$ exists. Moreover if $f^{(k)}$ exists and is continuous on $< a,b >$, (10) holds uniformly in $x \in [a,b]$.

Proof. If $f^{(k)}(x)$ exists at some $x \in IR^+$, given an arbitrary $\epsilon > 0$ we can find a δ satisfying $x > \delta > 0$ s.t.

$$f(u) = \sum_{p=0}^{k} \frac{f^{(p)}(x)}{p!}(u-x)^p + h_x(u)(u-x)^k; \quad |u-x| \le \delta,$$

where $h_x(u)$ is certain measurable function on $[x-\delta, x+\delta]$ satisfying the inequality $|h_x(u)| \le \epsilon, |u-x| \le \delta$. Hence

(11) $$L_n^{(k)}(f;x) = \sum_{p=0}^{k} \frac{f^{(p)}(x)}{p!} \sum_{j=0}^{p} \binom{p}{j}(-1)^j L_n^{(k)}(u^{p-j};x) + \\ + L_n^{(k)}(h_x(u)(u-x)^k \chi_{\delta,x}(u);x) + L_n^{(k)}(f\chi^c_{\delta,x};x) \\ = \sum_1 + \sum_2 + \sum_3, \quad \text{(say)}.$$

Using the fact that L_n maps polynomials to polynomials and the basic convergence $Theorem 3$, we obtain

(12) $$\sum_1 = f^{(k)}(x) L_n(u^k;1) \to f^{(k)}(x), n \to \infty.$$

It follows from $Lemma 1$ that

$$L_n^{(k)}(h_x(u)(u-x)^k \chi_{\delta,x}(u);x) = x^{mn+\alpha-1} D(m,n,\alpha) \sum_{p=0}^{k} \sum_{\nu=0}^{[\frac{k-p}{2}]} n^{\nu+p} \\ \times \int_{x-\delta}^{x+\delta} u^{-mn-\alpha} h_x(u)(u-x)^m \{(\tfrac{x}{u})^m e^{-(\frac{x}{u})^m}\}^{(n-k)} \\ \times [\tfrac{\partial}{\partial x}\{(\tfrac{x}{u})^m e^{-(\frac{x}{u})^m}\}]^k g_{\nu,p,k}(x,u) du$$

The δ above can be chosen so small that

$$\left|\frac{\partial}{\partial x}\{(\tfrac{x}{u})^m e^{-(\frac{x}{u})^m}\}\right| \le A\,|u-x|\,, |u-x| < \delta,$$

where A is some constant. Since the functions $g_{\nu,p,k}(x,u)$ are bounded on $[x-\delta, x+\delta]$, it is clear that there exists a constant M_1 independent of n, ϵ and δ s.t. for all n sufficiently large,

$$\left| L_n^{(k)}(h_x(u)(u-x)^k \chi_{\delta,x}(u);x)\right| \le \epsilon M_1 \sum_{p=0}^{k} \sum_{\nu=0}^{[\frac{k-p}{2}]} n^{\nu+p-\frac{k+p}{2}}$$

by (3) where M_2 is another constant not depending on n,ϵ and δ. Since $\nu \le [\frac{k-p}{2}], \nu + p - \frac{p+k}{2} - [\frac{k-p}{2}] - \frac{k-p}{2} \le 0$ there exists a constant M independent of n,ϵ and δ s.t.

(13) $|\sum_2| \le M$ for all sufficiently large n. To estimate $\sum_3$, first of all we notice that there exist a positive integer p and a positive constant P such that

$$\left|[\{(\tfrac{m}{u})e^{-(\frac{x}{u})^m}(\tfrac{x}{u})^{m-1}\}\{1-(\tfrac{x}{u})^{m-1}\}]^k g_{\nu,p,k}(x,u)\right| \le P(1+u^{-m}), u \in IR^+$$

and $0 \le p \le k, 0 \le \nu \le [\frac{k-p}{2}]$. Hence by $Lemma 1$, we have

$$|\sum_3| \le P \sum_{p=0}^{k} \sum_{\nu=0}^{[\frac{k-p}{2}]} n^{\nu+p} x^{mn+\alpha-1} D(m,n,\alpha) \\ \times \int_0^\infty u^{-mn-\alpha}(1+u^{-m})(\tfrac{x}{u})^{mn-k} e^{-(n-k)(\frac{x}{u})^m} f(u)\chi^c_{\delta,x}(u) du \\ = P \sum_{p=0}^{k} \sum_{\nu=0}^{[\frac{k-p}{2}]} n^{\nu+p} \frac{D(m,n,\alpha)}{D(m,n-k,\alpha)} L_{n-k}(f\chi_{\delta,x};x) + \frac{D(m,n,\alpha)}{D^{**}(m,n-k,\alpha)} L^{**}_{n-k}(f\chi^c_{\delta,x};x)$$

where L_n^{**} corresponds to to the operator (1) with α replaced by $\alpha + m$ and $D^{**}(m,n,\alpha)$ refers to $D(m,n,\alpha)$ for L_n^{**}. We observe that

$$\lim_{n\to\infty} \frac{D(m,n,\alpha)}{D(m,n-k,\alpha)} = \lim_{n\to\infty} \frac{D(m,n,\alpha)}{D^{**}(m,n-k,\alpha)}$$

Also, by the definition of the operator L_n, we have

$$\lim_{n\to\infty} n^{\nu+p} L_{n-k}(f\chi^c_{\delta,x};x) = \lim_{n\to\infty} n^{\nu+p} L^{**}_{n-k}(f\chi^c_{\delta,x};x) \\ = 0$$

It follows that $\sum_3 \to 0$ as $n \to \infty$. In view of this fact and (11) − −(13), it follows that there exists an n_0 s.t.

$$\left|L_n^{(k)}(f;x) - f^{(k)}(x)\right| < (2+M)\epsilon, n > n_0.$$

Since M does not depend on ϵ we have (10).

The uniformity part is easy to derive from the above proof by noting that, to begin with, δ can be chosen independent of $x \in [a,b]$ so that $|h_x(u)| \leq \epsilon$ for $x \in [a,b]$ whenever $|u-x| \leq \delta$. Then, it is clear that the various constants occuring in the above proof can be chosen independent of $x \in [a,b]$. This completes the proof of the theorem. □

Finally, we show that the asymptotic formula of $Theorem 2$ remains valid in the pointwise simultaneous approximation as well. We observe that the difference between $Theorem 2$ and the following one lies in the assumptions of f. We have

Theorem 6. *: If $f \in D_\Omega$, then*

(14) $$L_n^{(k)}(f;x) - f^{(k)}(x) = -\frac{1}{2nm^2}[f^{(k)}(x)k\{(2\alpha-k-5)\}+ +xf^{(k+1)}(x)\{2(\alpha-k-3)+(3-k)\}+x^2 f^{(k+2)}(x)]+ +o(\tfrac{1}{n}), n \to \infty.$$

whenever $x \in IR^+$ is s.t. $f^{(k+2)}(x)$ exists. Also if $f^{(k+2)}(x)$ exists and is continuous on $< a,b >$, (14) holds uniformly in $x \in [a,b]$.

Proof. If $f^{(k+2)}$ exists, we have

$$f(u) = \sum_{p=0}^{k+2} \frac{f^{(p)}(x)}{p!}(u-x)^p + h(u,x),$$

where $h(u,x) \in D_\Omega$ and for any $\epsilon > 0$, there exist a $\delta > 0$ s.t. $|h(u,x)| \leq \epsilon\,|u-x|^{k+2}$ for all sufficiently $|u-x| \leq \delta$. Thus,

(15) $$L_n^{(k)}(f;x) = L_n^{(k)}(Q;x) + L_n^{(k)}(h(u,x);x),$$

where $Q = \sum_{p=0}^{k+2} \frac{f^{(p)}(x)}{p!}(u-x)^p$ is a polynomial in u.Clearly $Q \in D_\Omega^{(k)}$. Also,

$$Q^{(p)}(x) = f^{(p)}(x), \text{ for } p = k, k+1, k+2.$$

Hence, applying $Theorem 2$, we have

(16) $$L_n^{(k)}(Q;x) - f^{(k)}(x) = -\frac{1}{2nm^2}[k(2\alpha-k-m-2)f^{(k)}(x)+ +(2\alpha-2k-m-3)xf^{(k+1)}(x) + x^2 f^{(k+2)}(x)] + o(\tfrac{1}{n}),$$

$n \to \infty$.

To establish (14), it remains to show that

(17) $$\left|L_n^{(k)}(h(u,x);x)\right| \leq D(m,n,\alpha)x^{\alpha-1}\sum_{p=0}^{k}\sum_{\nu=0}^{[\frac{k-p}{2}]} mn^{\nu+p}\int_0^\infty x^{mn}u^{-mn-\alpha-1}e^{-n(\frac{x}{u})^m} \times \left|(\tfrac{x}{u})^{m-1}e^{-(\frac{x}{u})^m}\{1-(\tfrac{x}{u})\}^{m-1}\right| g_{\nu,p,k}(x,u)\{h(u,x)\chi^c_{\delta,x}(u)+ \epsilon\,|u-x|^{k+2}\}du$$

Proceeding as in the proof of $Theorem 5$, we find that the term corresponding to ϵ in the above is bounded by $\frac{\epsilon M}{n}$ for some M independent of ϵ and n and $\chi^c_{\delta,x}-$ term contributes only a $o(\frac{1}{n})$ quantity (in fact $o(\frac{1}{n^s})$ for an arbitrary $s > 0$). Then in view of arbitraryness of $\epsilon > 0$, (17) follows.

The uniformity part follows as a remark similar to that made for the proof of the uniformity part of $Theorem 5$. This completes the proof of the theorem. □

In the rest of the paper, we study the second problem.

4. Some Direct Theorems for Linear Combinations

In this section we give some direct theorems for the the linear combinations of the operators L_n. First, we give some definitions. The k^{th}-moment $\mu_{n,k}(x), k \in IN^0$ (set of non-negative integers) of the operators L_n [5] is defined by

(18) $$\mu_{n,k}(x) = L_n((u-x)^k; x) = x^k \tau_{n,k} \quad \text{(say)}.$$

Clearly, $\tau_{n,k}$ is independent of x.Now we first prove the lemma on the moments $\mu_{n,k}$.

Lemma 2. *: If $k \in IN^0$.Then there exist constants $\gamma_{k,\nu}, \nu \geq [\frac{k+1}{2}]$ s.t. the following asymptotic expansion is valid:*

(19) $$\tau_{\nu,k} = \sum_{\nu=[\frac{k+1}{2}]}^{\infty} \gamma_{k,\frac{\nu}{n}} \nu, \qquad n \to \infty.$$

Proof. Let $\frac{1}{3} < \gamma < \frac{1}{2}$. Then

$$\tau_{n,k} = \int_{1-n^{-\gamma}}^{1+n^{-\gamma}} s^{\alpha-k-2}(1-s)^k \exp[n \log\{e^{-1} - m^2 \tfrac{(s-1)^2}{2!} e^{-1} + \dots + \tfrac{(s-1)^{2p}}{2p!} (\tfrac{d^{2p}}{dx^{2p}} \{(\tfrac{x}{u})^m e^{-(\frac{x}{u})^m}\})_{\frac{x}{u}=1} + o((s-1)^{2p})\}]ds,$$

$(p \geq 2)$

$$= e^{-n} \int_{1-n^{-\gamma}}^{1+n^{-\gamma}} s^{\alpha-k-2}(1-s)^k \exp[-nm^2 \tfrac{(s-1)^2}{2}] \times \exp[\{C_3(s-1)^3 + C_4(s-1)^4 + \dots + C_{2p}(s-1)^{2p} + o((s-1)^{2p})\}]ds$$

$C_i's$ being constants.

$$= e^{-n} \int_{1-n^{-\gamma}}^{1+n^{-\gamma}} s^{\alpha-k-2}(1-s)^k \exp[-nm^2 \tfrac{(s-1)^2}{2}]\{1 + \sum_{3 \leq 3i \leq j \leq [2p+\frac{i-1}{\gamma}]} b_{ij} n^i (s-1)^j + o(n^{1-2p\gamma})\}ds$$

$b'_{ij}s$ depending on C'_is.

$$= e^{-n} \int_{1-n^{-\gamma}}^{1+n^{-\gamma}} \exp[-nm^2 \tfrac{(s-1)^2}{2}][\{\sum_{l=0}^{2p-\frac{1}{\gamma}} a_l (s-1)^{k+l}\} \times \{1 + \sum_{3 \leq 3i \leq j \leq [2p+\frac{i-1}{\gamma}]} b_{ij} n^i (s-1)^j\} + o(n^{1-(2p+k)\gamma})]ds$$

$$= e^{-n} \int_{1-n^{-\gamma}}^{1+n^{-\gamma}} \exp[-nm^2 \tfrac{(s-1)^2}{2}][\sum_{\substack{3 \leq 3i \leq j \leq [2p+\frac{i-1}{\gamma}] \\ 0 \leq l \leq [2p-\frac{1}{\gamma}]}} d_{ijl} n^i (s-1)^{j+k+l} + o(n^{1-(2p+k)\gamma})]ds$$

where $d'_{ijl}s$ are certain constants depending on a'_ls and $b'_{ij}s$ and vanish if $j+k+l$ is odd.

Using substitutions we get

$$= 2^{\frac{1}{2}} \frac{e^{-n}}{mn^{\frac{1}{2}}} \int_0^{-n} \frac{t^{[\frac{j+k+l+1}{2}]-\frac{1}{2}}}{e^t} [1 + \gamma m^2 \sum_{(0 \leq 3i \leq j \leq [2p+\frac{i-1}{\gamma}])} d^*_{ijl} n^{i-[\frac{j+k+l-1}{2}]} + o(n^{1-(2p+k)\gamma+1-2\gamma})]dt$$

where $d^*_{ijl} = d_{ijl}\{\frac{2}{m^2}\}^{[\frac{j+k+l-1}{2}]}$.

$$= 2^{\frac{1}{2}} \frac{e^{-n}}{mn^{\frac{1}{2}}} [\sum_{\substack{0 \leq 3i \leq j \leq [2p+\frac{i-1}{\gamma}] \\ 0 \leq l \leq [2p-\frac{1}{\gamma}]}} d^{**}_{ijl} n^{i-[\frac{j+k+l-1}{2}]} + o(n^{2-(2p+2+k)\gamma})]$$

where $d^{**}_{ijl} = d^*_{ijl}\Gamma(([\frac{j+k+l-1}{2}])^\gamma + \frac{1}{2})$ and we have made use of the fact that by enlarging the integral in the above from 0 to ∞, we are only adding the terms in n which decay exponentially and therefore can be absorbed in the o-term.

Next, we analyse the expression

$$\int_{(0,\infty)-(1-n^{-\gamma},1+n^{-\gamma})} s^{mn+\alpha-k-2}(1-s)^k e^{-ns^m} ds = E(n) \quad \text{(say)}.$$

We have for any positive integer q,

$$|E(n)| \leq n^{\gamma q} D^{**}(m,n,\alpha) L_n^{**}(|u-1|^{k+q}; 1),$$

where $D^{**}(m,n,\alpha)$ and L_n^{**} are the same as considered in the proof of $Theorem 5$.

By making use of an estimate for the operators L_n^{**}, we have

$$|E(n)| \leq An^{\gamma q - \frac{k+q}{2}} D^{**}(m,n,\alpha),$$

where A is certain constant not depending upon n. Again making use of the same estimate as above for $D^{**}(m,n,\alpha)$, we have

$e^n |E(n)| = o(n^{\gamma q - \frac{k+q+1}{2}})$.

Thus, choosing q s.t.

$p \geq \frac{2(2p+2+k)}{1-2\gamma}$, we have

$$\int_0^\infty s^{mn+\alpha-k-2}(1-s)^k e^{-ns^m} ds$$
$$= 2^{\frac{1}{2}} \frac{e^{-n}}{mn^{\frac{1}{2}}} [\sum_{\substack{(0\leq 3i\leq j\leq [2p+\frac{i-1}{\gamma}]) \\ 0\leq l\leq [2p-\frac{1}{\gamma}]}} d^{**}_{ijl} n^{i-[\frac{j+k+l-1}{2}]} + o(n^{2-(2p+k+2)\gamma})].$$

Now, for all indices under consideration we have

$$[\tfrac{j+k+l+1}{2}] - i = [\tfrac{j-2i+k+l+1}{2}] \geq [\tfrac{k+1}{2}],$$

and since p could be chosen arbitrarily large, there exist constants $C_{k,\nu}, \nu \geq [\frac{k+1}{2}]$ s.t. we have the following asymptotic expansion

$$\int_0^\infty s^{mn+\alpha-k-2}(1-s)^k e^{-ns^m} ds$$
$$= 2^{\frac{1}{2}} \frac{e^{-n}}{mn^{\frac{1}{2}}} \sum_{\nu=[\frac{k+1}{2}]}^\infty \frac{C_{k,\nu}}{n^\nu}$$

Noting that $C_{0,0} = 1$, it follows that there exist constants $\gamma_{k,\nu}, \nu \geq [\frac{k+1}{2}]$ s.t. (19) holds. This completes the proof of *Lemma*2. □

For any fixed set of positive constants $\alpha_i, i = 0,1,2,...,k$ following [9] the linear combination $L_{n,k}$ of the operators $L_{\alpha_i,n}, i = 0,1,2,...k$ is defined by

$$(20) \qquad L_{n,k}(f;x) = \frac{1}{\triangle} \begin{vmatrix} L_{\alpha_0 n}(f;x) & \alpha_0^{-1} & \alpha_0^{-2} & ... & ... & \alpha_0^{-k} \\ L_{\alpha_1 n}(f;x) & \alpha_1^{-1} & \alpha_1^{-2} & ... & ... & \alpha_1^{-k} \\ .... & ... & ... & ... & ... & ... \\ .... & ... & ... & ... & ... & ... \\ L_{\alpha_k n}(f;x) & \alpha_k^{-1} & \alpha_k^{-2} & ... & ... & \alpha_k^{-k} \end{vmatrix}$$

where $\triangle$ is the determinant obtained by replacing the operator column by the entries $'1'$. Clearly

$$(21) \qquad L_{n,k} = \sum_{j=0}^k C(j,k) L_{\alpha_j n},$$

for constants $C(j,k), j = 0,1,2,...,k$ which satisfy $\sum_{j=1}^k C(j,k) = 1$.

$L_{n,k}$ is called a linear combination of order k. $L_{n,0}$ denotes the operator L_n itself.

Theorem 7. *: If $f \in D_\Omega$. If at a point $x \in IR^+, f^{(2k+2)}$ exists, then*

$$(22) \qquad |L_{n,k}(f;x) - f(x)| = O(n^{-(k+1)}),$$
$$(23) \qquad |L_{n,k+1}(f;x) - f(x)| = o(n^{-(k+1)}),$$

where $k = 0,1,2,...$. Also, if $f^{(2k+2)}$ exists and is continuous on $< a,b > \subset IR^+$, (22) − −(23) hold uniformly on $[a,b]$.

Proof. First we show that

$$(24) \quad L_n(f;x) - f(x) = \sum_{j=1}^{2k+2} \frac{x^j f^{(j)}(x)}{j!} \tau_{n,j} + o(n^{-(k+1)}),$$

if $x \in IR^+$ is such that $f^{(2k+2)}$ exists and $f \in D_\Omega$.To prove (24) with the assumption on f, we have

$$f(u) - f(x) = \sum_{j=1}^{2k+2} \frac{f^{(j)}(x)}{j!}(u-x)^j + R_x(u); \quad u \to x,$$

where $R_x(u) = o((u-x)^{2k+2}), u \to x$. It is clear from the definition of $\tau_{n,j}$ that we only have to show that

(25) $\quad L_n(R_x(u); x) = o(n^{-(k+1)})$.

Obviously, $R_x(u) \in D_\Omega$. Now, given an arbitrary $\epsilon > 0$, we can choose a $\delta > 0$ s.t.

$$|R_x(u)| \leq \epsilon(u-x)^{2k+2}, |u-x| \leq \delta.$$

Hence, by using the basic properties of $L_n[1]$, we note that the result follows.In this case the uniformity part is obvious. Now, using $Lemma 2$ and (24) we get

(26) $\quad L_n(f;x) - f(x) = \sum_{j=1}^{2k+2} \frac{x^j f^{(j)}(x)}{j!} \sum_{\nu=[\frac{j+1}{2}]}^{k+1} \frac{\gamma_{j,\nu}}{n^\nu} + o(n^{-(k+1)}),$

which, in the uniformity case holds uniformly in $x \in [a,b]$.Since the coefficients $C(j,k)$ in (21) obviously satisfy the relation

(27) $\quad \sum_{j=0}^{k} C(j,k)\alpha_j^{-p} = 0, p = 1,2,3,...,k.$

In view of $(26), (22) - -(23)$ are immediate and so is the uniformity part. This completes the proof of $Theorem 7$. □

In the same spirit we have,

Theorem 8. *:.Let* $f \in D_\Omega$. *If* $0 \leq p \leq 2k+2$ *and* $f^{(p)}$ *exists and is continuous on* $< a,b > \subset IR^+$, *for each* $x \in [a,b]$ *and sufficiently large* n *then*

(28) $\quad |L_{n,k}(f;x) - f(x)| \leq \max[Cn^{-\frac{p}{2}}\omega(f^{(p)}; n^{-\frac{1}{2}}), C'n^{-(k+1)}]$

where $C = C(k)$ *and* $C' = C'(k,f)$ *are constants and* $\omega(f^{(p)};\delta)$ *denotes the local modulus of continuity of* $f^{(p)}$ *on* $< a,b >$.

Proof. :There exists a $\delta > 0$ s.t. $[a-\delta, b+\delta] \subset < a,b >$. It is clear that if $u \in < a,b >$, there exists an η lying between $x \in [a,b]$ and u s.t.

(29) $\quad \left| f(u) - f(x) - \sum_{j=1}^{p} \frac{f^{(j)}(x)}{j!}(u-x)^j \right| \leq \frac{|u-x|^p}{p!}(1+|u-x|\, n^{\frac{1}{2}})\omega(f^{(p)}; n^{-\frac{1}{2}}),$

using a well known result on modulus of continuity [13]. If the expression occuring within the modulus sign on $L.H.S.$ of the above inequality is denoted by $F_x(u)$, by a well known property of L_n, it follows that

$$L_{\alpha_j n}(F_x(u)\chi^c_{\delta,x}(u); x) = o(n^{-(k+1)}),$$

uniformly in $x \in [a,b]$. By (29), we have

(30) $\quad \left| L_{\alpha_j n}(F_x(u)\chi^c_{\delta,x}(u); x) \right| \leq \frac{b^p}{p!}(A_p + A_{p-1})(\alpha_j n)^{-\frac{p}{2}}\omega(f^{(p)}; n^{-\frac{1}{2}})$

for all n sufficiently large and $x \in [a,b]$. Here A_p, A_{p-1} are constants depending on p. Hence, for a constant C_p independent of f such that for all $x \in [a,b]$,

(31) $\quad \left| L_{n,k}(F_x(u)\chi^c_{\delta,x}(u); x) \right| \leq C_p n^{-\frac{p}{2}}\omega(f^{(p)}; n^{-\frac{1}{2}}).$

Applying the result (22) for the functions $1, u, u^2, u^3, ..., u^p$, we find that there exists a constant C'' depending on $\max\{|f'(x)|, ..., |f^{(p)}(x)|; x \in [a,b]\}$ and p such that for all $x \in [a,b]$,

(32) $\quad \left| L_{n,k}(\sum_{j=1}^{p} \frac{f^{(j)}(x)}{j!}(u-x)^j; x) \right| \leq C'' n^{-(k+1)}$

Now, (28) is clear from $(30) - -(32)$. This completes the proof of the Theorem. □

Theorem 9. *:Let* $f \in D_\Omega$. *If at a point* $x \in IR^+, f^{(2k+p+2)}$ *exists then*

(33) $\quad \left| L^{(p)}_{n,k}(f;x) - f^{(p)}(x) \right| = O(n^{-(k+1)}), and$

(34) $\quad \left| L^{(p)}_{n,k+1}(f;x) - f^{(p)}(x) \right| = o(n^{-(k+1)}),$

where $k = 0,1,2,...$. *Also, if* $f^{(2k+p+2)}$ *exists and is continuous on* $< a,b > \subset IR^+$, $(33) - - - (34)$ *hold uniformly in* $x \in [a,b]$.

Proof. If $f^{(2k+p+2)}$ exists , we can find a neighbourhood (a', b') of x s.t. $f^{(p)}$ exists and is continuous on (a', b'). Let $g(u)$ be an infinitely differentiable function with *supp* $g \subseteq (a', b')$ s.t. $g(u) = 1$ for $u \in [x-\delta, x+\delta]$ for some $\delta > 0$. Then an application of $Lemma 1$ shows that

(35) $L_{n,k}^{(p)}(f(u) - f(u)g(u); x) = o(n^{-(k+1)})$.

In the uniformity case, we consider a $g(u)$ with *supp* $g \subset < a, b >$ with $g(u) = 1$,

for $u \in [a-\delta, b+\delta] \subseteq < a, b >$ and then (34) holds uniformly in $x \in [a, b]$. since $f(u)g(u) \in C_b^{(p)} IR^+$ we have

(36) $L_n^{(p)}(fg; x) = x^{-p} L_n(u^p\{f(u)g(u)\}^{(p)}; x)$.

Now, since $u^p\{f(u)g(u)\}^{(p)}$ is $(2k+2)-$times differentiable at x (and continuously differentiable on $(a-\delta, b+\delta)$ in the uniformity case), applying $Theorem 7$ we have

(37) $\left|L_{n,k}^{(p)}(fg; x) - f^{(p)}(x)\right| = O(n^{-(k+1)}), and$

(38) $\left|L_{n,k+1}^{(p)}(fg; x) - f^{(p)}(x)\right| = o(n^{-(k+1)})$,

where, in the uniformity case these holds in $x \in [a, b]$. Thus, combining $(35) - - - (38)$, we get $(33) - - - (34)$. This completes the proof . □

Theorem 10. *: Let $m \in IN$, and $f \in D_\Omega$. If $0 \leq q \leq 2k+2$ and $f^{(p+q)}$ exists and is continuous on $< a, b > \subseteq IR^+$ for each $x \in [a, b]$, then for all sufficiently large n,*

(39) $\left|L_{n,k}^{(p)}(f; x) - f^{(p)}(x)\right| \leq \max\{C_p n^{-(\frac{k}{2})} \omega(f^{(p+q)}; n^{-\frac{1}{2}}), C_p' n^{-(k+1)}\}$

where $C_p = C_p(k), C_p' = C_p'(k, f)$ are constants and $\omega(f^{(p+q)}; \delta)$ denotes the local modulus of continuity of $f^{(p+q)}$ on $< a, b >$.

Proof. : The proof of this Theorem follows from $Lemma 1$ and $Theorem 5 - - 9$. □

References

[1] BRAMHA DUTTA PANDEY AND B. KUNWAR, On a class of positive linear operators(communicated).
[2] R. A. DEVORE AND G. G. LORENTZ, Constructive Approximation. Springer-Verlag Berlin Heidelberg, New York, (1993).
[3] P.P. KOROVKIN, Linear Operators and Approximation Theory (1960), Delhi. (Translated from Russian edition of 1959).
[4] B. Kunwar, Approximation of analytic functions by a class of linear positive operators, J. Approx. Theory 44. 173-182 (1985).
[5] G. G. LORENTZ, Bernstein Polynomials, Toronto (1953).
[6] C. P. MAY, Saturation and inverse theorems for combination of a class of exponential type operators. Canad. J. Math. 28,(1976), 1224-50.
[7] M. W.MULLER, Die Folge der Gamma operatoren.Thesis,Technische Hoschule,Stuttgort.(1967).
[8] Paula Anamaria Piţul, Evaluation of the approximation order by positive linear operators, Ph. D. Thesis, Universitatea Babeş Bolyai(Romania) 2007.
[9] R. K. S. RATHORE, Linear combinations of linear positive operators and generating relations in special functions, Dissertation, IIT Delhi (India) (1973).
[10] R. K. S. RATHORE, Approximation of unbounded functions with Linear Positive Operators, Doctoral Dissertation, Technische Hogeschool Delft (1974).
[11] P. C. SIKKEMA AND R. K. S. RATHORE,Convolution with powers of bell shaped functions. Report, Dept. of Math. Technische Hogeschool (1976).

[12] P. C. SIKKEMA, Approximation formulae of Voronovskaya- type for certain convolution operators, J. Approx. Theory 26(1979), 26-45.

[13] A. F. TIMAN, Theory of Approximation of Functions of a Real variable, Peargamon Press (1963).

[1]Department of Applied Sciences and Humanities, Institute of Engineering and Technology, Lucknow-21 (India)

[2]Department of Applied Sciences and Humanities, Institute of Engineering and Technology, Lucknow-21 (India)

[*]Corresponding author

The author is thankful to Councel of Scientific and Industrial Research, INDIA for providing financial assistance for this research work under grant no.09/827(0004).