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GENERAL STABILITY OF A RECIPROCAL TYPE FUNCTIONAL EQUATION IN THREE VARIABLES

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ABSTRACT. In this paper, we obtain the solution of a reciprocal type functional equation in three variables of the form

$$g\left(2(k-1)x_1+x_2+x_3\right)=\frac{g((k-1)x_1+x_2)g((k-1)x_1+x_3)}{g((k-1)x_1+x_2)+g((k-1)x_1+x_3)}$$

and investigate its generalized Hyers-Ulam stability where $k\geq 2$ is a positive integer, $g:X\to\mathbb{R}$ is a mapping with X as the space of non-zero real numbers and $2(k-1)x_1+x_2+x_3\neq 0$, $g((k-1)x_1+x_2)+g((k-1)x_1+x_2)\neq 0$, for all $x_1,x_2,x_3\in X$. We also provide counter-examples for non-stability.

1. INTRODUCTION

An inquisitive question that was given a serious thought by S.M. Ulam [42] concerning the stability of group homomorphisms gave rise to the stability problem of functional equations. The laborious intellectual strivings of D.H. Hyers [15] did not go in vain because he was the first to come out with a partial answer to solve the question posed by Ulam on Banach spaces. In course of time, the theorem formulated by Hyers was generalized by T. Aoki [4] for additive mappings and by Th.M. Rassias [40] for linear mappings by taking into consideration an unbounded Cauchy difference.

The findings of Th.M. Rassias have exercised a delectable influence on the development of what is addressed as the generalized Hyers-Ulam-Rassias stability of functional equations. A generalized and modified form of the theorem evolved by Th.M. Rassias was advocated by P. Gavruta [13] who replaced the unbounded Cauchy difference by driving into study a general control function within the viable approach designed by Th.M. Rassias. In 1982-1989, a generalization of the result of D.H. Hyers was proved by J.M. Rassias using weaker conditions controlled by a product of different powers of norms ([31], [32], [33]). The investigation of stability of functional equations involving with the mixed type product-sum of powers of norms is introduced by J.M. Rassias [34].

A further research materialized by F. Skof [41] found solution to Hyers-Ulam-Rassias stability problem for quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

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for a class of functions $f : A \to B$, where A is a normed space and B is a Banach space. The functional equation (1.1) is used to characterize inner product spaces ([1], [2], [24]). The stability problems of several functional equations have been extensively investigated by a number of mathematicians, posed with creative thinking and critical dissent who have arrived at interesting results (see [5], [8], [9], [11], [14], [16], [25], [35]).

Functional equations find a lot of application in information theory, information science, measure of information, coding theory, computer graphics, spatial filtering in image processing, behavioral and social sciences, astronomy, number theory, fuzzy system models, economics, stochastic processes, mechanics, cryptography and physics.

In 1998, S.M. Jung [18] investigated the Hyers-Ulam-Rassias stability for the Jensen functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y) \tag{1.2}$$

and applied the stability result to the study of an asymptotic behaviour of the additive mappings.
In 2008, W.G. Park and J.H. Bae [29] obtained the general solution and the stability of the functional equation

$$\begin{aligned} &f(x+y+z, u+v+w) + f(x+y-z, u+v+w) \\ &\quad + 2f(x, u, -w) + 2f(y, v, -w) \\ &\qquad = f(x+y, u+w) + f(x+y, v+w) + f(x+z, u+w) \\ &\qquad\quad + f(x-z, u+v-w) + f(y+z, v+w) + f(y-z, u+v-w). \end{aligned} \tag{1.3}$$

The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) = x^3 + ax + b - y^2$ having level curves as elliptic curves is a solution of (1.3). The stability result of the functional equation (1.3) is related with the canonical height function of the elliptic curves.

In the year 1996, Isac and Th.M. Rassias [17] were the first to provide applications of stability theorem of functional equations for the proof of new fixed point theorems with applications.

Usually, the stability problem for functional equations is solved by direct method in which the exact solution of the functional equation is explicitly constructed as a limit of a (Hyers) sequence, starting from the given approximate solution of f (see [3], [10], [16], [18], [19]). In the year 2003, Radu [30] proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative. This method was recently been used by many authors (see [20], [21], [22], [23], [27], [28]).

Cadariu and Radu ([6], [7]) applied a fixed point method to investigate the Jensen's and Cauchy additive functional equations.

In the year 2010, K. Ravi and B.V. Senthil Kumar [36] investigated the generalized Hyers-Ulam stability for the reciprocal functional equation

$$g(x+y) = \frac{g(x)g(y)}{g(x) + g(y)} \tag{1.4}$$

where $g : X \to Y$ is a mapping on the spaces of non-zero real numbers. The reciprocal function $g(x) = \frac{c}{x}$ is a solution of the functional equation (1.4). The functional equation (1.4) holds good for the **"Reciprocal formula"** of any electric circuit with two resistors connected in parallel.

S.M. Jung [22] applied fixed point method for proving Hyers-Ulam stability for the reciprocal functional equation (1.4).

K. Ravi, J.M. Rassias and B.V. Senthil Kumar [37] disucssed the generalized Hyers-Ulam stability for the reciprocal functional equation in several variables of the form

$$g\left(\sum_{i=1}^{m}\alpha_i x_i\right) = \frac{\prod_{i=1}^{m} g(x_i)}{\sum_{i=1}^{m}\left[\alpha_i\left(\prod_{j=1, j\neq i}^{m} g(x_j)\right)\right]} \tag{1.5}$$

for arbitrary but fixed real numbers $(\alpha_1, \alpha_2, \ldots, \alpha_m) \neq (0, 0, \ldots, 0)$, so that $0 < \alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_m = \sum_{i=1}^{m}\alpha_i \neq 1$ and $g : X \to Y$ with X and Y are the spaces of non-zero real numbers.

Later, J.M. Rassias and et.al., [38] introduced the Reciprocal Difference Functional equation

$$r\left(\frac{x+y}{2}\right) - r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)} \tag{1.6}$$

and the Reciprocal Adjoint Functional equation

$$r\left(\frac{x+y}{2}\right) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)} \tag{1.7}$$

and investigated the generalized Hyers-Ulam stability of the equations (1.6) and (1.7).

Soon after, J.M. Rassias and et.al., [39] applied fixed point method to investigate the generalized Hyers-Ulam stability of the equations (1.6) and (1.7).

G.L. Forti [12] obtained the stability of the functional equation $\Psi \circ f \circ a = f$, by proving the following theorem.

Theorem 1.1. *Assume that (Y, d) is a complete metric space, K is a nonempty set, $f : K \to Y$, $\Psi : Y \to Y$, $a : K \to K$, $h : K \to [0, \infty)$, $\lambda \in [0, \infty)$, $d(\Psi \circ f \circ a(x), f(x)) \leq h(x)$ for $x \in K$,*

$$d(\Psi(x), \Psi(y)) \leq \lambda d(x, y) \qquad \text{for } x, y \in Y,$$

$$\text{and} \qquad H(x) = \sum_{i=0}^{\infty}\lambda^i h\left(a^i(x)\right) < \infty \qquad \text{for } x \in K.$$

Then, for every $x \in K$, the limit $r(x) = \lim_{n\to\infty}\Psi^n \circ f \circ a^n(x)$ exists and $r : K \to Y$ is a unique function such that $\Psi \circ r \circ a = r$ and $d(f(x), r(x)) \leq H(x)$, for $x \in K$.

In this paper, we obtain the general solution and investigate the generalized Hyers-Ulam stability of the reciprocal type functional equation in three variables of the form

$$g\left(2(k-1)x_1 + x_2 + x_3\right) = \frac{g((k-1)x_1 + x_2)g((k-1)x_1 + x_3)}{g((k-1)x_1 + x_2) + g((k-1)x_1 + x_3)} \tag{1.8}$$

where $k \geq 2$ is a positive integer, $g : X \to \mathbb{R}$ is a mapping with X as the space of non-zero real numbers and $2(k-1)x_1 + x_2 + x_3 \neq 0$, $g((k-1)x_1 + x_2) + g((k-1)x_1 + x_2) \neq 0$, for all $x_1, x_2, x_3 \in X$ using direct method, fixed point method and Theorem 1.1. We also provide counter-examples for non-stability.

Throughout this paper, let X be the space of non-zero real numbers. We also assume that $2(k-1)x_1 + x_2 + x_3 \neq 0$, $g(x) \neq 0$, $g((k-1)x_1 + x_2) \neq 0$,

$g((k-1)x_1+x_3) \neq 0$ and $g((k-1)x_1+x_2)+g((k-1)x_1+x_3) \neq 0$ for all $x, x_1, x_2, x_3 \in X$ to prove our main results. For the sake of convenience, let us define

$$\begin{aligned} D_3 g(x_1,x_2,x_3) &= g\left(2(k-1)x_1+x_2+x_3\right) \\ &\quad - \frac{g((k-1)x_1+x_2)g((k-1)x_1+x_3)}{g((k-1)x_1+x_2)+g((k-1)x_1+x_3)} \end{aligned} \tag{1.9}$$

for all $x_1, x_2, x_3 \in X$.

The paper is organised as follows. In Section 2, we present preliminaries required for proving our main results. In Section 3, we obtain the general solution of the functional equation (1.8). In Section 4, we investigate the generalized Hyers-Ulam stability of the equation (1.8) using direct method. In Section 5, we apply fixed point method to gain the generalized Hyers-Ulam stability of the equation (1.8) and in Section 6, we find the generalized Hyers-Ulam stability of the equation (1.8) in the sense of G.L. Forti. In Section 7, we illustrate counter-examples for singular cases.

2. PRELIMINARIES

In this section, we present the definition of generalized metric and fundamental result of fixed point theory.

Let A be a set. A function $d : A \times A \to [0, \infty]$ is called a generalized metric on A if d satisfies the following conditions:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in A$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for al $x, y, z \in A$.

We note that the only one difference of the generalized metric from the usual metric is that the range of the former is permitted to include infinity.

The following theorem is very useful for proving our results in Section 5, which is due to Margolis and Diaz [26].

Theorem 2.1. *Let (A, d) be a complete generalized metric space and let $J : A \to A$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in A$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{2.1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

1. $d\left(J^n x, J^{n+1} x\right) < \infty$ *for all* $n \geq n_0$;
2. *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
3. y^* *is the unique fixed point of* J *in the set* $Y = \{y \in A | d(J^{n_0} x, y) < \infty\}$;
4. $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

3. GENERAL SOLUTION OF FUNCTIONAL EQUATION (1.8)

Theorem 3.1. *A mapping $g : X \to \mathbb{R}$ satisfies the functional equation (1.8) for all $x_1, x_2, x_3 \in X$ if and only if there exists a reciprocal mapping $g : X \to \mathbb{R}$ satisfying the reciprocal functional equation (1.4) for all $x, y \in X$. Hence, every solution of functional equation (1.8) is also a reciprocal function.*

Proof. Let the mapping $g : X \to \mathbb{R}$ satisfy the functional equation (1.8). Replacing (x_1, x_2, x_3) by $\left(\frac{x+y}{k-1}, -y, -x\right)$ in (1.8), we obtain the equation (1.4).
Conversely, let the mapping $g : X \to \mathbb{R}$ satisfy the functional equation (1.4). Replacing (x, y) by $((k-1)x_1 + x_2, (k-1)x_1 + x_3)$ in (1.4), we obtain the equation (1.8), which completes the proof of Theorem 3.1. □

4. GENERALIZED HYERS-ULAM STABILITY OF EQUATION (1.8) USING DIRECT METHOD

Theorem 4.1. *Let $g : X \to \mathbb{R}$ be a mapping satisfying (1.9) and*

$$|D_3 g(x_1, x_2, x_3)| \leq \phi(x_1, x_2, x_3) \tag{4.1}$$

for all $x_1, x_2, x_3 \in X$, where $\phi : X \times X \times X \to \mathbb{R}$ be a given function such that

$$\Phi(x) = \sum_{i=0}^{\infty} 2^{i+1} \phi\left(\frac{2^i x}{k}, \frac{2^i x}{k}, \frac{2^i x}{k}\right) \tag{4.2}$$

with the condition

$$\lim_{n\to\infty} 2^n \phi\left(\frac{2^n x}{k}, \frac{2^n x}{k}, \frac{2^n x}{k}\right) = 0 \tag{4.3}$$

holds for every $x \in X$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ which satisfies (1.8) and the inequality

$$|g(x) - r(x)| \leq \Phi(x) \tag{4.4}$$

for all $x \in X$.

Proof. Replacing (x_1, x_2, x_3) by $\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right)$ in (4.1) and multiplying by 2, we get

$$|2g(2x) - g(x)| \leq 2\phi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right) \tag{4.5}$$

for all $x \in X$. Again replacing x by $2x$ in (4.5), multiplying by 2 and summing the resulting inequality with (4.5), we get $\left|2^2 g(2^2 x) - g(x)\right| \leq \sum_{i=0}^{1} 2^{i+1} \phi\left(\frac{2^i x}{k}, \frac{2^i x}{k}, \frac{2^i x}{k}\right)$ for all $x \in X$. Proceeding further and using induction on a positive integer n, we obtain

$$\begin{aligned} |2^n g(2^n x) - g(x)| &\leq \sum_{i=0}^{n-1} 2^{i+1}\phi\left(\frac{2^i x}{k}, \frac{2^i x}{k}, \frac{2^i x}{k}\right) \\ &\leq \sum_{i=0}^{\infty} 2^{i+1}\phi\left(\frac{2^i x}{k}, \frac{2^i x}{k}, \frac{2^i x}{k}\right) \end{aligned} \tag{4.6}$$

for all $x \in X$. In order to prove the convergence of the sequence $\{2^n g(2^n x)\}$, replace x by $2^p x$ in (4.6) and multiply by 2^p, we find that for $n > p > 0$

$$\begin{aligned} \left|2^p g(2^p x) - 2^{n+p} g\left(2^{n+p} x\right)\right| &= 2^p \left|g(2^p x) - 2^n g\left(2^{n+p} x\right)\right| \\ &\leq \sum_{i=0}^{\infty} 2^{p+i+1}\phi\left(\frac{2^{p+i} x}{k}, \frac{2^{p+i} x}{k}, \frac{2^{p+i} x}{k}\right) \\ &\to 0 \text{ as } p \to \infty. \end{aligned}$$

Thus the sequence $\{2^n g(2^n x)\}$ is a Cauchy sequence. Allowing $n \to \infty$ in (4.6), we arrive (4.4) with $r(x) = \lim_{n\to\infty} 2^n g(2^n x)$. To show that r satisfies (1.8), replacing (x, y) by $(2^n x, 2^n y)$ in (4.1) and multiplying by 2^n, we obtain

$$2^n \left|D_3 g(2^n x_1, 2^n x_2, 2^n x_3)\right| \leq 2^n \phi(2^n x_1, 2^n x_2, 2^n x_3). \tag{4.7}$$

Allowing $n \to \infty$ in (4.7), we see that r satisfies (1.8) for all $x_1, x_2, x_3 \in X$. To prove r is a unique reciprocal mapping satisfying (1.8), let $r_1 : X \to \mathbb{R}$ be another reciprocal mapping which satisfies (1.8) and the inequality (4.4). Clearly $r_1(2^n x) = 2^{-n} r_1(x)$, $r(2^n x) = 2^{-n} r(x)$ and using (4.4), we arrive

$$\begin{aligned} |r_1(x) - r(x)| &\leq 2^n \left|r_1(2^n x) - r(2^n x)\right| \\ &\leq 2^n \left(\left|r_1(2^n x) - g(2^n x)\right| + \left|g(2^n x) - r(2^n x)\right|\right) \\ &\leq 2\sum_{i=0}^{\infty} 2^{n+i+1}\phi\left(\frac{2^{n+i} x}{k}, \frac{2^{n+i} x}{k}, \frac{2^{n+i} x}{k}\right) \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

which implies that r is unique. This completes the proof of Theorem 4.1. □

Theorem 4.2. *Let $g : X \to \mathbb{R}$ be a mapping satisfying (1.9) and (4.1), for all $x_1, x_2, x_3 \in X$, where $\phi : X \times X \times X \to \mathbb{R}$ be a given function such that*

$$\Phi(x) = \sum_{i=0}^{\infty} \frac{1}{2^i}\phi\left(\frac{x}{2^{i+1}k}, \frac{x}{2^{i+1}k}, \frac{x}{2^{i+1}k}\right) \tag{4.8}$$

with the condition

$$\lim_{n\to\infty} \frac{1}{2^n}\phi\left(\frac{x}{2^{n+1}k}, \frac{x}{2^{n+1}k}, \frac{x}{2^{n+1}k}\right) = 0 \tag{4.9}$$

holds for every $x \in X$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ which satisfies (1.8) and the inequality

$$|g(x) - r(x)| \leq \Phi(x) \tag{4.10}$$

for all $x \in X$.

Proof. The proof is obtained by replacing (x_1, x_2, x_3) by $\left(\frac{x}{2k}, \frac{x}{2k}, \frac{x}{2k}\right)$ in (4.1) and proceeding by similar arguments as in Theorem 4.1. □

Corollary 4.3. *Let $g : X \to \mathbb{R}$ be a mapping and let there exist real numbers $q \neq -1$ and $\theta_1 \geq 0$ such that*

$$|D_3 g(x_1, x_2, x_3)| \leq \theta_1 \left(\sum_{i=1}^{3} |x_i|^q \right) \tag{4.11}$$

for all $x_1, x_2, x_3 \in X$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ satisfying (1.8) and

$$|g(x) - r(x)| \leq \begin{cases} \frac{6\theta_1}{k^q(1-2^{q+1})}|x|^q, & \text{for } q < -1 \\ \frac{6\theta_1}{k^q(2^{q+1}-1)}|x|^q, & \text{for } q > -1 \end{cases} \tag{4.12}$$

for every $x \in X$.

Proof. The proof follows immediately by taking $\phi(x_1, x_2, x_3) = \theta_1 \left(\sum_{i=1}^{3} |x_i|^q \right)$, for all $x_1, x_2, x_3 \in X$ in Theorems 4.1 and 4.2 respectively. □

Corollary 4.4. *Let $g : X \to \mathbb{R}$ be a mapping and let there exist a real number $q \neq -1$. Let there exist $\theta_2 \geq 0$ such that*

$$|D_3 g(x_1, x_2, x_3)| \leq \theta_2 \left(\prod_{i=1}^{3} |x_i|^{\frac{q}{3}} \right) \tag{4.13}$$

for all $x_1, x_2, x_3 \in X$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ satisfying (1.8) and

$$|g(x) - r(x)| \leq \begin{cases} \frac{2\theta_2}{k^q(1-2^{q+1})}|x|^q, & \text{for } q < -1 \\ \frac{2\theta_2}{k^q(2^{q+1}-1)}|x|^q, & \text{for } q > -1 \end{cases} \tag{4.14}$$

for every $x \in X$.

Proof. The required results in Corollary 4.4 can be easily derived by considering $\phi(x_1, x_2, x_3) = \theta_2 \left(\prod_{i=1}^{3} |x_i|^{\frac{q}{3}} \right)$, for all $x_1, x_2, x_3 \in X$ in Theorems 4.1 and 4.2 respectively. □

Corollary 4.5. *Let $\theta_3 \geq 0$ and $q \neq -1$ be real numbers, and $g : X \to \mathbb{R}$ be a mapping satisfying the functional inequality*

$$|D_3 g(x_1, x_2, x_3)| \leq \theta_3 \left(\prod_{i=1}^{3} |x_i|^{\frac{q}{3}} + \left(\sum_{i=1}^{3} |x_i|^q \right) \right) \tag{4.15}$$

for all $x_1, x_2, x_3 \in X$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ satisfying (1.8) and

$$|g(x) - r(x)| \leq \begin{cases} \frac{8\theta_3}{k^q(1-2^{q+1})}|x|^q, & \text{for } q < -1 \\ \frac{8\theta_3}{k^q(2^{q+1}-1)}|x|^q, & \text{for } q > -1 \end{cases} \tag{4.16}$$

for every $x \in X$.

Proof. By choosing $\phi(x_1, x_2, x_3) = \theta_3 \left(\prod_{i=1}^{3} |x_i|^{\frac{q}{3}} + \left(\sum_{i=1}^{3} |x_i|^q \right) \right)$, for all $x_1, x_2, x_3 \in X$ in Theorems 4.1 and 4.2 respectively, the proof of Corollary 4.5 is complete. $\square$

5. GENERALIZED HYERS-ULAM STABILITY OF EQUATION (1.8) USING FIXED POINT METHOD

Theorem 5.1. *Suppose that the mapping $g : X \to \mathbb{R}$ satisfies the inequality*

$$|D_3 g(x_1, x_2, x_3)| \leq \phi(x_1, x_2, x_3) \tag{5.1}$$

for all $x_1, x_2, x_3 \in X$, where $\phi : X \times X \times X \to \mathbb{R}$ is a given function. If there exists $L < 1$ such that the mapping

$$x \to \Phi(x) = 2\phi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right)$$

has the property

$$\Phi(2x) \leq \frac{1}{2} L \Phi(x), \ \text{for all } x \in X$$

and the mapping ϕ has the property

$$\lim_{n\to\infty} 2^n \phi\left(2^n x_1, 2^n x_2, 2^n x_3\right) = 0 \tag{5.2}$$

for all $x_1, x_2, x_3 \in X$, then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ such that

$$|g(x) - r(x)| \leq \frac{1}{1-L} \Phi(x) \tag{5.3}$$

for all $x \in X$.

Proof. Define a set S by

$$S = \{h : X \to \mathbb{R} | h \text{ is a function}\}$$

and introduce the generalized metric d on S as follows:

$$d(g,h) = d_\Phi(g,h) = \inf\{C \in \mathbb{R}_+ : |g(x) - h(x)| \leq C\Phi(x),\ \text{for all } x \in X\}. \tag{5.4}$$

Now, we show that (S,d) is complete. Using the idea from [21], we prove the completeness of (S,d). Let $\{h_n\}$ be a Cauchy sequence in (S,d). Then for any $\epsilon > 0$, there exists an integer $N_\epsilon > 0$ such that $d(h_m, h_n) \leq \epsilon$, for all $m, n \geq N_\epsilon$. From (5.4), we arrive

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall m, n \geq N_\epsilon, \forall x \in X : |h_m(x) - h_n(x)| \leq \epsilon\Phi(x). \tag{5.5}$$

If x is a fixed number, (5.5) implies that $\{h_n(x)\}$ is a Cauchy sequence in $(\mathbb{R}, |.|)$. Since $(\mathbb{R}, |.|)$ is complete, $\{h_n(x)\}$ converges for all $x \in X$. Therefore, we can define a function $h : X \to \mathbb{R}$ by

$$h(x) = \lim_{n\to\infty} h_n(x)$$

and hence $h \in S$. Letting $m \to \infty$ in (5.5), we have

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall n \geq N_\epsilon, \forall x \in X : |h(x) - h_n(x)| \leq \epsilon\Phi(x).$$

By considering (5.4), we arrive

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall n \geq N_\epsilon : d(h, h_n) \leq \epsilon,$$

which implies that the Cauchy sequence $\{h_n\}$ converges to h in (S,d). Hence (S,d) is complete.

Define a mapping $\sigma : S \to S$ by

$$\sigma h(x) = 2h(2x) \qquad (x \in X) \tag{5.6}$$

for all $h \in S$. We claim that σ is strictly contractive on S. For any given $g, h \in S$, let $C_{gh} \in [0, \infty]$ be an arbitrary constant with $d(g,h) \leq C_{gh}$. Hence

$$\begin{aligned} d(g,h) < C_{gh} &\Rightarrow |g(x) - h(x)| \leq C_{gh}\Phi(x),\ \text{for all } x \in X \\ &\Rightarrow |2g(2x) - 2h(2x)| \leq 2C_{gh}\Phi(2x),\ \text{for all } x \in X \\ &\Rightarrow |2g(2x) - 2h(2x)| \leq LC_{gh}\Phi(x),\ \text{for all } x \in X \\ &\Rightarrow d(\sigma g, \sigma h) \leq LC_{gh}. \end{aligned}$$

Therefore, we see that

$$d(\sigma g, \sigma h) \leq Ld(g,h),\ \text{for all } g, h \in S.$$

That is, σ is strictly contractive mapping of S, with the Lipschitz constant L. Now, replacing (x_1, x_2, x_3) by $\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right)$ in (5.1) and multiplying by 2, we get

$$|2g(2x) - g(x)| \leq 2\phi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right) = \Phi(x)$$

for all $x \in X$. Hence (5.4) implies that $d(\sigma f, f) \leq 1$. Hence by applying the fixed point alternative Theorem 2.1, there exists a function $r : X \to \mathbb{R}$ satisfying the following:

(1) r is a fixed point of σ, that is

$$r(2x) = \frac{1}{2} r(x) \tag{5.7}$$

for all $x \in X$. The mapping r is the unique fixed point of σ in the set

$$\mu = \{f \in S : d(f, g) < \infty\}.$$

This implies that r is the unique mapping satisfying (5.7) such that there exists $C \in (0, \infty)$ satisfying

$$|r(x) - g(x)| \leq C\Phi(x), \ \forall x \in X.$$

(2) $d(\sigma^n g, r) \to 0$ as $n \to \infty$. Thus, we have

$$\lim_{n\to\infty} 2^n g\left(2^n x\right) = r(x) \tag{5.8}$$

for all $x \in X$.

(3) $d(g, r) \leq \frac{1}{1-L} d(\sigma g, r)$, which implies

$$d(g, r) \leq \frac{1}{1-L}.$$

Thus the inequality (5.3) holds. Hence from (5.1), (5.2) and (5.8), we have

$$\begin{aligned} |D_3 g(x_1, x_2, x_3)| &= \lim_{n\to\infty} 2^n |D_3 g(2^n x_1, 2^n x_2, 2^n x_3)| \\ &\leq \lim_{n\to\infty} 2^n \phi(2^n x_1, 2^n x_2, 2^n x_3) \\ &= 0 \end{aligned}$$

for all $x_1, x_2, x_3 \in X$. Hence r is a solution of the functional equation (1.8). By Theorem 3.1, $r : X \to \mathbb{R}$ is a reciprocal mapping.

Next, we show that r is the unique reciprocal mapping satisfying (1.8) and (5.3). Suppose, let $r_1 : X \to \mathbb{R}$ be another reciprocal mapping satisfying (1.8) and (5.3). Then from (1.8), we have that r_1 is a fixed point of σ. Since $d(g, r_1) < \infty$, we have

$$r_1 \in S^* = \{f \in S | d(f, g) < \infty\}.$$

From Theorem 2.1 (3) and since both r and r_1 are fixed points of σ, we have $r = r_1$. Therefore, r is unique. Hence, there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ satisfying (1.8) and (5.3), which completes the proof of Theorem 5.1. □

Theorem 5.2. *Suppose that the mapping $g : X \to \mathbb{R}$ satisfies the inequality (5.1) for all $x_1, x_2, x_3 \in X$, where $\phi : X \times X \times X \to \mathbb{R}$ is a given function. If there exists $L < 1$ such that the mapping*

$$x \to \Phi(x) = \phi\left(\frac{x}{2k}, \frac{x}{2k}, \frac{x}{2k}\right)$$

has the property

$$\Phi\left(\frac{x}{2}\right) \le 2L\Phi(x), \text{ for all } x \in X$$

and the mapping ϕ has the property

$$\lim_{n\to\infty} 2^{-n}\phi\left(2^{-n}x_1, 2^{-n}x_2, 2^{-n}x_3\right) = 0 \tag{5.9}$$

for all $x_1, x_2, x_3 \in X$, then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ such that

$$|g(x) - r(x)| \le \frac{1}{1-L}\Phi(x) \tag{5.10}$$

for all $x \in X$.

Proof. The proof of Theorem 5.2 goes through the same way as in Theorem 5.1. □

Corollary 5.3. *Let $g : X \to \mathbb{R}$ be a mapping and let there exist real numbers $q \neq -1$ and $\theta_1 \ge 0$ such that (4.11) holds for all $x_1, x_2, x_3 \in X$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ satisfying (1.8) and (4.12) for every $x \in X$.*

Proof. The proof is obtained by assuming $\phi(x_1, x_2, x_3) = \theta_1\left(\sum_{i=1}^{3}|x_i|^q\right)$, for all $x_1, x_2, x_3 \in X$ and $L = 2^{-q-1}, L = 2^{q+1}$ in Theorems 5.1 and 5.2 respectively. □

Corollary 5.4. *Let $g : X \to \mathbb{R}$ be a mapping and let there exist a real number $q \neq -1$. Let there exist $\theta_2 \ge 0$ such that (4.13) holds for all $x_1, x_2, x_3 \in X$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ satisfying (1.8) and (4.14) for every $x \in X$.*

Proof. It is easy to derive the required results in Corollary 5.4 by considering $\phi(x_1, x_2, x_3) = \theta_2\left(\prod_{i=1}^{3}|x_i|^{\frac{q}{3}}\right)$, for all $x_1, x_2, x_3 \in X$ and $L = 2^{-q-1}, L = 2^{q+1}$ in Theorems 5.1 and 5.2 respectively. □

Corollary 5.5. *Let $\theta_3 \ge 0$ and $q \neq -1$ be real numbers, and $g : X \to \mathbb{R}$ be a mapping satisfying the functional inequality (4.15) for all $x_1, x_2, x_3 \in X$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ satisfying (1.8) and (4.16) for every $x \in X$.*

Proof. The proof is complete by choosing $\phi(x_1,x_2,x_3) = \theta_3\left(\prod_{i=1}^{3}|x_i|^{\frac{q}{e}} + \left(\sum_{i=1}^{3}|x_i|^q\right)\right)$, for all $x_1,x_2,x_3 \in X$ and $L = 2^{-q-1}, L = 2^{q+1}$ in Theorems 5.1 and 5.2 respectively. □

6. GENERALIZED HYERS-ULAM STABILITY OF EQUATION (1.8) IN THE SENSE OF G.L. FORTI

Theorem 6.1. *Suppose that the mapping $g : X \to \mathbb{R}$ satisfies the inequality*

$$|D_3 g(x_1,x_2,x_3)| \leq \phi(x_1,x_2,x_3) \tag{6.1}$$

for all $x_1,x_2,x_3 \in X$, where $\phi : X \times X \times X \to \mathbb{R}$ is a given function. Suppose there exists $\beta \in (0,\infty)$ such that $2\beta < 1$,

$$\phi\left(\frac{2x_1}{k},\frac{2x_2}{k},\frac{2x_3}{k}\right) \leq \beta\phi\left(\frac{x_1}{k},\frac{x_2}{k},\frac{x_3}{k}\right) \tag{6.2}$$

for all $x_1,x_2,x_3 \in X$ and $k(>1) \in \mathbb{Z}^+$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ such that

$$|r(x) - g(x)| \leq \frac{2}{1-2\beta}\phi\left(\frac{x}{k},\frac{x}{k},\frac{x}{k}\right) \tag{6.3}$$

for all $x \in X$.

Proof. Replacing (x_1,x_2,x_3) by $\left(\frac{x}{k},\frac{x}{k},\frac{x}{k}\right)$ in (6.1), we get

$$|g(2x) - \frac{1}{2}g(x)| \leq \phi\left(\frac{x}{k},\frac{x}{k},\frac{x}{k}\right), \text{ for all } x \in X.$$

Hence, we obtain

$$\begin{aligned}|2g(2x) - g(x)| &= 2\left|g(2x) - \frac{1}{2}g(x)\right| \\ &\leq 2\phi\left(\frac{x}{k},\frac{x}{k},\frac{x}{k}\right), \text{ for } x \in X.\end{aligned}$$

Considering $f = \frac{1}{2}g$, $\Psi(z) = 2z$, $\lambda = 2$, $h(x) = 2\phi\left(\frac{x}{k},\frac{x}{k},\frac{x}{k}\right)$, $a(x) = 2x$ and $d(x,y) = |x-y|$, for all $x,y \in X$ in Theorem 1.1, we see that the limit $r(x)$ exists and $|r(x) - g(x)| \leq H(x)$, for all $x \in X$. Using (6.1), we obtain

$$2^n|D_3 g\left(2^n x_1, 2^n x_2, 2^n x_3\right)| \leq (2\beta)^n\phi(x_1,x_2,x_3) \tag{6.4}$$

for all $x_1,x_2,x_3 \in X$ and $n \in \mathbb{N}$. Allowing $n \to \infty$ in (6.4), we see that r satisfies (1.8). Next we show that r is the unique reciprocal mapping satisfying (1.8) and (6.3). Suppose, let $r_1 : X \to \mathbb{R}$ be another reciprocal mapping satisfying (1.8) and (6.3), and $|r_1(x) - g(x)| \leq H(x)$, for all $x \in X$. Then $\Psi \circ r_1 \circ a = r_1$ and hence by Theorem 1.1, $r = r_1$, which proves that r is unique. □

Theorem 6.2. *Suppose that the mapping $g : X \to \mathbb{R}$ satisfies (6.1), for all $x_1, x_2, x_3 \in X$, where ϕ is a function defined as in Theorem 6.1. Suppose there exists $\beta \in (0, \infty)$ such that $\frac{\beta}{2} < 1$,*

$$\phi\left(\frac{x_1}{2k}, \frac{x_2}{2k}, \frac{x_3}{2k}\right) \leq \beta\phi\left(\frac{x_1}{k}, \frac{x_2}{k}, \frac{x_3}{k}\right) \tag{6.5}$$

for all $x_1, x_2, x_3 \in X$ and $k(> 1) \in \mathbb{Z}^+$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ such that

$$|r(x) - g(x)| \leq \frac{2\beta}{2 - \beta}\phi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right) \tag{6.6}$$

for all $x \in X$.

Proof. Replacing (x_1, x_2, x_3) by $\left(\frac{x}{2k}, \frac{x}{2k}, \frac{x}{2k}\right)$ in (6.1), we get

$$\left|g(x) - \frac{1}{2}g\left(\frac{x}{2}\right)\right| \leq \phi\left(\frac{x}{2k}, \frac{x}{2k}, \frac{x}{2k}\right), \text{ for all } x \in X.$$

The rest of the proof is obtained by taking $f = g$, $\Psi(z) = \frac{1}{2}z$, $\lambda = \frac{1}{2}$, $h(x) = \phi(\frac{x}{2k}, \frac{x}{2k}, \frac{x}{2k})$, $a(x) = \frac{1}{2}x$ and $d(x, y) = |x - y|$, for all $x, y \in X$ in Theorem 1.1 and using similar arguments as in Theorem 6.1. □

Corollary 6.3. *Let $g : X \to \mathbb{R}$ be a mapping and let there exist real numbers $q \neq -1$ and $\theta_1 \geq 0$ such that (4.11) holds for all $x_1, x_2, x_3 \in X$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ satisfying (1.8) and (4.12) for every $x \in X$.*

Proof. The proof is obtained by assuming $\phi(x_1, x_2, x_3) = \theta_1 \left(\sum_{i=1}^{3} |x_i|^q\right)$, for all $x_1, x_2, x_3 \in X$ and $\beta = 2^q, \beta = \frac{1}{2^q}$ in Theorems 6.1 and 6.2 respectively. □

Corollary 6.4. *Let $g : X \to \mathbb{R}$ be a mapping and let there exist a real number $q \neq -1$. Let there exist $\theta_2 \geq 0$ such that (4.13) holds for all $x_1, x_2, x_3 \in X$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ satisfying (1.8) and (4.14) for every $x \in X$.*

Proof. It is easy to derive the required results in Corollary 6.4 by considering $\phi(x_1, x_2, x_3) = \theta_2 \left(\prod_{i=1}^{3} |x_i|^{\frac{q}{3}}\right)$, for all $x_1, x_2, x_3 \in X$ and $\beta = 2^q, \beta = \frac{1}{2^q}$ in Theorems 6.1 and 6.2 respectively. □

Corollary 6.5. *Let $\theta_3 \geq 0$ and $q \neq -1$ be real numbers, and $g : X \to \mathbb{R}$ be a mapping satisfying the functional inequality (4.15) for all $x_1, x_2, x_3 \in X$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ satisfying (1.8) and (4.16) for every $x \in X$.*

Proof. The proof is complete by choosing $\phi(x_1, x_2, x_3) = \theta_3 \left(\prod_{i=1}^{3} |x_i|^{\frac{q}{e}} + \left(\sum_{i=1}^{3} |x_i|^q \right) \right)$, for all $x_1, x_2, x_3 \in X$ and $\beta = 2^q, \beta = \frac{1}{2^q}$ in Theorems 6.1 and 6.2 respectively. $\square$

Remark 6.6. From Section 4, Section 5 and Section 6, we observe that the results obtained in Corollaries 4.3, 5.3 and 6.3 are the same. Similarly the upper bounds in the Corollaries 4.4, 5.4 and 6.4 are identical. Also, we find that the stability results in Corollaries 4.5, 5.5 and 6.5 are similar. Therefore, from the above results, we conclude that the method of G.L. Forti is the easiest method in comparison with the direct method and fixed point method.

7. COUNTER-EXAMPLES

The following example illustrates the fact that the functional equation (1.8) is not stable for $q = -1$ in Corollary 4.3.
Example 7.1. Let $\varphi : X \to \mathbb{R}$ be a mapping defined by

$$\varphi(x) = \begin{cases} \frac{c_1}{x} & \text{for } x \in (1, \infty) \\ c_1 & \text{otherwise} \end{cases}$$

where $c_1 > 0$ is a constant, and define a mapping $g : X \to X$ by

$$g(x) = \sum_{n=0}^{\infty} \frac{\varphi(2^{-n}x)}{2^n}, \text{for all } x \in X.$$

Then the mapping g satisfies the inequality

$$|D_3 g(x_1, x_2, x_3)| \leq 3c_1 \left(\sum_{i=1}^{3} |x_i|^{-1} \right) \tag{7.1}$$

for all $x_i \in X, i = 1, 2, 3$. Therefore there do not exist a reciprocal mapping $r : X \to \mathbb{R}$ and a constant $\delta > 0$ such that

$$|g(x) - r(x)| \leq \delta |x|^{-1} \tag{7.2}$$

for all $x \in X$.

Proof. $|g(x)| \leq \sum_{n=0}^{\infty} \frac{|\varphi(2^{-n}x)|}{|2^n|} \leq \sum_{n=0}^{\infty} \frac{c_1}{2^n} = 2c_1$. Hence g is bounded by $2c_1$. If $\left(\sum_{i=1}^{3} |x_i|^{-1} \right) \geq 1$, then the left hand side of (4.1) is less than $3c_1$. Now, suppose that $0 < \left(\sum_{i=1}^{3} |x_i|^{-1} \right) < 1$. Then there exists a positive integer m such that

$$\frac{1}{2^{m+1}} < \left(\sum_{i=1}^{3} |x_i|^{-1} \right) < \frac{1}{2^m}. \tag{7.3}$$

Hence $\left(\sum_{i=1}^{3} |x_i|^{-1} \right) < \frac{1}{2^m}$ implies

$$\begin{aligned} & 2^m |x_i|^{-1} < 1, \text{ for } i = 1, 2, 3 \\ \text{or} \quad & \frac{x_i}{2^m} > 1, \text{ for } i = 1, 2, 3 \\ \text{or} \quad & \frac{x_i}{2^{m-1}} > 2 > 1, \text{ for } i = 1, 2, 3 \end{aligned}$$

and consequently

$$\frac{1}{2^{m-1}}(2(k-1)x_1+x_2+x_3), \frac{1}{2^{m-1}}((k-1)x_1+x_2), \frac{1}{2^{m-1}}((k-1)x_1+x_3) > 1.$$

Therefore, for each value of $n = 0, 1, 2, \ldots, m-1$, we obtain

$$\frac{1}{2^n}(2(k-1)x_1+x_2+x_3), \frac{1}{2^n}((k-1)x_1+x_2), \frac{1}{2^n}((k-1)x_1+x_3) > 1$$

and $D_3\varphi\left(\frac{1}{2^n}x_1, \frac{1}{2^n}x_2, \frac{1}{2^n}x_3\right) = 0$, for $n = 0, 1, 2, \ldots, m-1$. Using (7.3) and the definition of g, we obtain

$$\begin{aligned}
&\frac{|D_3g(x_1,x_2,x_3)|}{(|x_1|^{-1}+|x_2|^{-1}+|x_3|^{-1})}\\
&\quad\le \sum_{n=m}^{\infty} \frac{\left|\varphi\left(2^{-n}(2(k-1)x_1+x_2+x_3)\right) - \frac{\varphi\left(2^{-n}((k-1)x_1+x_2)\right)\varphi\left(2^{-n}((k-1)x_1+x_3)\right)}{\varphi(2^{-n}((k-1)x_1+x_2))+\varphi(2^{-n}((k-1)x_1+x_3))}\right|}{2^n\left(|x_1|^{-1}+|x_2|^{-1}+|x_3|^{-1}\right)}\\
&\quad\le \sum_{k=0}^{\infty} \frac{\frac{3}{2}c_1}{2^k2^m\left(|x_1|^{-1}+|x_2|^{-1}+|x_3|^{-1}\right)}\\
&\quad\le \sum_{k=0}^{\infty} \frac{\frac{3}{2}c_1}{2^k} = \frac{3}{2}c_1\left(1-\frac{1}{2}\right)^{-1} = 3c_1, \text{ for all } x, y \in X.
\end{aligned}$$

That is, the inequality (7.1) holds true. Now, assume that there exists a reciprocal mapping $r : X \to \mathbb{R}$ satisfying (7.2). Therefore, we have

$$|g(x)| \le (\delta+1)|x|^{-1}. \tag{7.4}$$

However, we can choose a positive integer p with $pc_1 > \delta + 1$. If $x \in (1, 2^{p-1})$, then $2^{-n}x \in (1, \infty)$ for all $n = 0, 1, 2, \ldots, p-1$ and therefore

$$|g(x)| = \sum_{n=0}^{\infty} \frac{\varphi(2^{-n}x)}{2^n} \ge \sum_{n=0}^{m-1} \frac{\frac{c_1}{2^{-n}x}}{2^n} = \frac{pc_1}{x} > (\delta+1)x^{-1}$$

which contradicts (7.4). Therefore, the reciprocal type of functional equation (1.8) is not stable for $q = -1$ in Corollary 4.3. □

The following example illustrates the fact that the functional equation (1.8) is not stable for $q = -1$ in Corollary 4.5.

Example 7.2. Let $\psi : X \to \mathbb{R}$ be a mapping defined by

$$\psi(x) = \begin{cases} \frac{c_2}{x} & \text{for } x \in (1, \infty) \\ c_2 & \text{otherwise} \end{cases}$$

where $c_2 > 0$ is a constant, and define a mapping $g : X \to \mathbb{R}$ by

$$g(x) = \sum_{n=0}^{\infty} \frac{\psi(2^{-n}x)}{2^n}, \text{for all } x \in X.$$

Then the mapping g satisfies the inequality

$$|D_3g(x_1,x_2,x_3)| \le 3c_2\left(\prod_{i=1}^{3}|x_i|^{-\frac{1}{3}} + \left(\sum_{i=1}^{3}|x_i|^{-1}\right)\right)$$

for all $x_i \in X, i = 1, 2, 3$. Therefore there do not exist a reciprocal mapping $r : X \to \mathbb{R}$ and a constant $\delta > 0$ such that

$$|g(x) - r(x)| \leq \delta |x|^{-1}, \text{ for all } x \in X.$$

Proof. The proof is analogous to the proof of Example 7.1. □

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