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FRACTIONAL OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WHOSE FIRST DERIVATIVES ARE s-PREINVEX IN THE SECOND SENSE

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ABSTRACT. In this paper, we establish an fractional identity. Using this new identity we derives some fractional Ostrowski's inequalities for functions whose first derivatives are s-preinvex in the second sense.

1. INTRODUCTION

In 1938, A.M. Ostrowski proved an interesting integral inequality, given by the following theorem

Theorem 1.1. *[10] Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping in the interior I° of I, and $a, b \in I^\circ$, with $a < b$. If $|f'| \leq M$ for all $x \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall x \in [a,b]. \tag{1.1}$$

In the last decades, the inequality (1.1) has attracted much interest by many researchers, and considerable papers have been published concerning the generalizations, variants, and extensions of the inequality (1.1), for more detail we refer readers to [4, 7–9, 13, 16, 17] and references cited therein.

Recently, lot of efforts have been made by many mathematicians to generalize the classical convexity. Hanson [3], introduced a new class of generalized convex functions, called invex functions. In [1], the authors gave the concept of preinvex functions which is special case of invexity, and many authors have study their basic properties, and their role in optimization, variational inequalities and equilibrium problems, we refer readers to [11, 12, 15, 20, 21].

İşcan [5] established the following Ostrowski inequalities for functions whose derivatives are preinvex

Theorem 1.2. *[5, Theorem 2.2] Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function and $|f'|$ is preinvex function on A. If f' is integrable on $[a, a + \eta(b, a)]$, then the following inequality*

$$\left| f(x) - \tfrac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right| \leq \tfrac{\eta(b,a)}{6}$$
$$\times \left\{ \left(3\left(\tfrac{x-a}{\eta(b,a)}\right)^2 - 2\left(\tfrac{x-a}{\eta(b,a)}\right)^3 + 2\left(\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\right)^3 \right) |f'(a)| \right.$$
$$\left. + \left(1 - 3\left(\tfrac{x-a}{\eta(b,a)}\right)^2 + 4\left(\tfrac{x-a}{\eta(b,a)}\right)^3\right) |f'(b)| \right\}$$

holds for each $x \in [a, a + \eta(b, a)]$.

Theorem 1.3. *[5, Theorem 2.8] Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function and $|f'|^q$ is*

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preinvex function on $[a, a+\eta(b,a)]$ for some fixed $q \geq 1$. If f' is integrable on $[a, a+\eta(b,a)]$, then the following inequality

$$\begin{aligned}
&\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right| \leq \eta(b,a) \left(\tfrac{1}{2}\right)^{1-\frac{1}{q}} \\
&\times \left\{ \left(\tfrac{x-a}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)} \left(\tfrac{(x-a)^2(3\eta(b,a)-2x+2a)}{6\eta^3(b,a)} |f'(a)|^q + \frac{1}{3}\left(\tfrac{x-a}{\eta(b,a)}\right)^3 |f'(b)|^q \right)^{\frac{1}{q}} \right. \\
&+ \left(\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)} \left(\frac{1}{3}\left(\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\right)^3 |f'(a)|^q \right. \\
&\left. \left. + \left(\frac{1}{6} + \tfrac{(x-a)^2(2x-3\eta(b,a)-2a)}{6\eta^3(b,a)}\right) |f'(b)|^q \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

holds for each $x \in [a, a+\eta(b,a)]$.

Kirmaci [7] established the following midpoint inequalities for differentiable convex functions

Theorem 1.4. *[7, Theorem 2.2] Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ (I° is the interior of I) with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\tfrac{a+b}{2}\right) \right| \leq \tfrac{b-a}{8} \left(|f'(a)| + |f'(b)|\right).$$

Theorem 1.5. *[7, Theorem 2.3] Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ (I° is the interior of I) with $a < b$ and let $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, then we have*

$$\begin{aligned}
\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\tfrac{a+b}{2}\right) \right| \leq\ & \tfrac{b-a}{16} \left(\tfrac{4}{p+1}\right)^{\frac{1}{p}} \left(\left(3|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \right. \\
& \left. + \left(|f'(a)|^{\frac{p}{p-1}} + 3|f'(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \right).
\end{aligned}$$

Wang et al. [18] established the following midpoint inequalities for functions whose the power of the absolute value of the first derivatives are preinvex

Theorem 1.6. *[18, Theorem 3.1] Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a differentiable function. If $|f'|^q$ is preinvex on A for $q \geq 1$, then for every $a, b \in A$ with $\eta(b,a) \neq 0$ we have*

$$\begin{aligned}
&\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du - f\left(\tfrac{2a+\eta(b,a)}{2}\right) \right| \\
\leq\ & \tfrac{|\eta(b,a)|}{8} \left(\left(\tfrac{|f'(a)|^q + 2|f'(b)|^q}{3}\right)^{\frac{1}{q}} + \left(\tfrac{2|f'(a)|^q + |f'(b)|^q}{3}\right)^{\frac{1}{q}} \right).
\end{aligned}$$

Theorem 1.7. *[18, Corollary 3.2] Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a differentiable function. If $|f'|$ is preinvex on A, then for every $a, b \in A$ with $\eta(b,a) \neq 0$ we have*

$$\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du - f\left(\tfrac{2a+\eta(b,a)}{2}\right) \right| \leq \tfrac{|\eta(b,a)|}{8} \left(|f'(a)| + |f'(b)|\right).$$

Motivated by these results, in this paper we establish an fractional identity, and then using this equality we derive some Ostrowski's inequalities for functions whose first derivatives in absolute value are s-preinvex in the second sense.

2. Preliminaries

In this section we recall some concepts of convexity that are well known in the literature. Throughout this section I is an interval of $\mathbb{R}$.

Definition 2.1. *[14] A function $f : I \to \mathbb{R}$ is said to be convex, if*

$$f\left(tx+\left(1-t\right)y\right)\leq tf\left(x\right)+\left(1-t\right)f(y)$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Definition 2.2. *[2] A nonnegative function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0, 1]$, if the following inequality*

$$f(tx+(1-t)y)\ \leq t^{s}f(x)+(1-t)^{s}f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Let K be a subset in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions.

Definition 2.3. *[20] A set K is said to be invex at x with respect to η, if*

$$x+t\eta\left(y,x\right)\in K$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

K is said to be an invex set with respect to η if K is invex at each $x \in K$.

Definition 2.4. *[20] A function f on the invex set K is said to be preinvex with respect to η, if*

$$f\left(x+t\eta\left(y,x\right)\right)\leq\left(1-t\right)f\left(x\right)+tf(y)$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Definition 2.5. *[19] A nonnegative function f on the invex set $K \subseteq [0, \infty)$ is said to be s-preinvex in the second sense with respect to η, for some fixed $s \in (0, 1]$, if*

$$f\left(x+t\eta\left(y,x\right)\right)\leq\left(1-t\right)^{s}f\left(x\right)+t^{s}f(y)$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Definition 2.6. *[6] Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$\begin{aligned} J_{a^{+}}^{\alpha}f(x) &= \frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{x}\left(x-t\right)^{\alpha-1}f(t)dt, \qquad x>a \\ J_{b^{-}}^{\alpha}f(x) &= \frac{1}{\Gamma\left(\alpha\right)}\int_{x}^{b}\left(t-x\right)^{\alpha-1}f(t)dt, \qquad b>x \end{aligned}$$

respectively, where $\Gamma(\alpha) = \int\limits_{0}^{\infty} e^{-t}t^{\alpha-1}dt$, is the Gamma function and $J^{0}_{a^+}f(x) = J^{0}_{b^-}f(x) = f(x)$.

3. Main Results

In what follows $\eta : K \times K \to \mathbb{R}$, and $K \subset \mathbb{R}$ an invex subset with respect to η, and $a, b \in K^{\circ}$ the interior of K such that $[a, a + \eta(b, a)] \subset K$. At first, we prove the following lemma.

Lemma 3.1. *Let $f : [a, a + \eta(b, a)] \to \mathbb{R}$ be a differentiable function with $a < a + \eta(b, a)$. If $f' \in L([a, a + \eta(b, a)])$, then the following equality for fractional integrals*

$$\begin{aligned} &\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}\right)f(x)-\tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J_{x^{-}}^{\alpha}f(a)+J_{x^{+}}^{\alpha}f(a+\eta\left(b,a\right))\right) \\ =\ &\eta\left(b,a\right)\left(\int_{0}^{\frac{x-a}{\eta(b,a)}}t^{\alpha}f'(a+t\eta\left(b,a\right))dt-\int_{\frac{x-a}{\eta(b,a)}}^{1}\left(1-t\right)^{\alpha}f'(a+t\eta\left(b,a\right))dt\right) \qquad (3.1) \end{aligned}$$

holds for all $x \in [a, a + \eta(b, a)]$.

Proof. Integrating by parts right hand side of (3.1), we get

$$\begin{aligned}
&\eta(b,a)\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^\alpha f'(a+t\eta(b,a))dt - \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^\alpha f'(a+t\eta(b,a))dt\right) \\
= &\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^\alpha f(x) - \alpha\int_0^{\frac{x-a}{\eta(b,a)}} t^{\alpha-1} f(a+t\eta(b,a))dt\right. \\
&\left.+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^\alpha f(x) - \alpha\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{\alpha-1} f(a+t\eta(b,a))dt\right) \\
= &\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^\alpha + \left(1-\tfrac{x-a}{\eta(b,a)}\right)^\alpha\right) f(x) - \alpha\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{\alpha-1} f(a+t\eta(b,a))dt\right. \\
&\left.+\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{\alpha-1} f(a+t\eta(b,a))dt\right). \qquad (3.2)
\end{aligned}$$

Using the change of variable $u = a + t\eta(b,a)$, (3.2) becomes

$$\begin{aligned}
&\eta(b,a)\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^\alpha f'(a+t\eta(b,a))dt - \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^\alpha f'(a+t\eta(b,a))dt\right) \\
= &\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^\alpha + \left(1-\tfrac{x-a}{\eta(b,a)}\right)^\alpha\right) f(x) \\
&-\tfrac{\alpha}{(\eta(b,a))^\alpha}\left(\int_a^x (u-a)^{\alpha-1} f(u)du + \int_x^{a+\eta(b,a)} (\eta(b,a)+a-u)^{\alpha-1} f(u)du\right) \\
= &\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^\alpha + \left(1-\tfrac{x-a}{\eta(b,a)}\right)^\alpha\right) f(x) - \tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^\alpha}\left(J_{x^-}^\alpha f(a) + J_{x^+}^\alpha f(a+\eta(b,a))\right),
\end{aligned}$$

which is the desired result. □

Theorem 3.1. *Let $f : [a, a+\eta(b,a)] \to \mathbb{R}$ be a differentiable function such that $\eta(b,a) > 0$ and $f' \in L([a, a+\eta(b,a)])$. If $|f'|$ is s-preinvex in the second sense for some fixed $s \in (0,1]$, then the following inequality for fractional integrals*

$$\begin{aligned}
&\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^\alpha + \left(1-\tfrac{x-a}{\eta(b,a)}\right)^\alpha\right) f(x) - \tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^\alpha}\left(J_{x^-}^\alpha f(a) + J_{x^+}^\alpha f(a+\eta(b,a))\right)\right| \\
\leq\ &\eta(b,a)\left(\left(B_{\frac{x-a}{\eta(b,a)}}(\alpha+1, s+1) + \tfrac{1}{\alpha+s+1}\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+s+1}\right)|f'(a)|\right. \\
&\left.+\left(\tfrac{1}{\alpha+s+1}\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+s+1} + B(s+1,\alpha+1) - B_{\frac{x-a}{\eta(b,a)}}(s+1,\alpha+1)\right)|f'(b)|\right)
\end{aligned} \qquad (3.3)$$

holds for all $x \in [a, a+\eta(b,a)]$.

Proof. From Lemma 3.1, and properties of modulus, we have

$$\begin{aligned}&\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}\right)f(x)-\tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(a+\eta(b,a))\right)\right|\\ \leq\ &\eta(b,a)\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha}\left|f'(a+t\eta(b,a))\right|dt+\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha}\left|f'(a+t\eta(b,a))\right|dt\right).\end{aligned}$$

Using the s-preinvexity of $|f'|$, we obtain

$$\begin{aligned}&\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}\right)f(x)-\tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(a+\eta(b,a))\right)\right|\\ \leq\ &\eta(b,a)\left(\int_0^{\frac{x-a}{\eta(b,a)}}\left(t^{\alpha}(1-t)^{s}|f'(a)|+t^{\alpha+s}|f'(b)|\right)dt\right.\\ &\left.+\int_{\frac{x-a}{\eta(b,a)}}^{1}\left((1-t)^{\alpha+s}|f'(a)|+(1-t)^{\alpha}t^{s}|f'(b)|\right)dt\right)\\ =\ &\eta(b,a)\left(\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha}(1-t)^{s}dt+\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha+s}dt\right)|f'(a)|\right.\\ &\left.+\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha+s}dt+\int_{\frac{x-a}{\eta(b,a)}}^{1}t^{s}(1-t)^{\alpha}dt\right)|f'(b)|\right)\\ =\ &\eta(b,a)\left(\left(B_{\frac{x-a}{\eta(b,a)}}(\alpha+1,s+1)+\tfrac{1}{\alpha+s+1}\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+s+1}\right)|f'(a)|\right.\\ &\left.+\left(\tfrac{1}{\alpha+s+1}\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+s+1}+B(s+1,\alpha+1)-B_{\frac{x-a}{\eta(b,a)}}(s+1,\alpha+1)\right)|f'(b)|\right),\end{aligned}\tag{3.4}$$

where we use the facts that

$$\begin{aligned}\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha}(1-t)^{s}dt&=B_{\frac{x-a}{\eta(b,a)}}(\alpha+1,s+1)\\ \int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha+s}dt&=\tfrac{1}{\alpha+s+1}\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+s+1}\\ \int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha+s}dt&=\tfrac{1}{\alpha+s+1}\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+s+1}\\ \int_{\frac{x-a}{\eta(b,a)}}^{1}t^{s}(1-t)^{\alpha}dt&=B(s+1,\alpha+1)-B_{\frac{x-a}{\eta(b,a)}}(s+1,\alpha+1).\end{aligned}\tag{3.5}$$

The proof is completed. □

Remark 3.1. *In Theorem 3.1, if we put $\alpha=s=1$, we obtain Theorem 2.2 from [5].*

Corollary 3.1. *In Theorem 3.1, if we choose* $x = \frac{2a+\eta(b,a)}{2}$*, then the following midpoint inequality holds for fractional integrals*

$$\begin{aligned}
&\left| f\left(\tfrac{2a+\eta(b,a)}{2}\right) - \tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}} \left(J^{\alpha}_{\frac{2a+\eta(b,a)}{2}^{-}} f(a) + J^{\alpha}_{\frac{2a+\eta(b,a)}{2}^{+}} f(a+\eta(b,a)) \right) \right| \\
\leq\ & \eta(b,a) \left(\left(B_{\frac{1}{2}}(\alpha+1, s+1) + \tfrac{1}{(\alpha+s+1)2^{\alpha+s+1}} \right) |f'(a)| \right. \\
&\left. + \left(\tfrac{1}{(\alpha+s+1)2^{\alpha+s+1}} + B(s+1,\alpha+1) - B_{\frac{1}{2}}(s+1,\alpha+1) \right) |f'(b)| \right)
\end{aligned}$$

Remark 3.2. *In Corollary 3.1, if we put* $\alpha = s = 1$*, we obtain Corollary 3.2 from [18]. Moreover if we take* $\eta(b,a) = b-a$*, we obtain Theorem 2.2 from [7].*

Theorem 3.2. *Let* $f : [a, a+\eta(b,a)] \to \mathbb{R}$ *be a differentiable function such that* $\eta(b,a) > 0$ *and* $f' \in L([a, a+\eta(b,a)])$ *and let* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. If* $|f'|^q$ *is s-preinvex in the second sense for some fixed* $s \in (0,1]$*, then the following inequality for fractional integrals*

$$\begin{aligned}
&\left| \left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha} + \left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha} \right) f(x) - \tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}} \left(J^{\alpha}_{x^-} f(a) + J^{\alpha}_{x^+} f(a+\eta(b,a)) \right) \right| \\
\leq\ & \tfrac{\eta(b,a)}{(s+1)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}} \left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}} \left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{s+1} |f'(b)|^q \right. \right. \\
&\left. + \left(1 - \left(1-\tfrac{x-a}{\eta(b,a)}\right)^{s+1}\right) |f'(a)|^q \right)^{\frac{1}{q}} + \left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}} \\
&\left. \times \left(\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{s+1} |f'(a)|^q + \left(1-\left(\tfrac{x-a}{\eta(b,a)}\right)^{s+1}\right) |f'(b)|^q \right)^{\frac{1}{q}} \right) \qquad (3.6)
\end{aligned}$$

holds for all $x \in [a, a+\eta(b,a)]$.

Proof. From Lemma 3.1, properties of modulus, and Hölder's inequality, we have

$$\begin{aligned}
&\left| \left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha} + \left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha} \right) f(x) - \tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}} \left(J^{\alpha}_{x^-} f(a) + J^{\alpha}_{x^+} f(a+\eta(b,a)) \right) \right| \\
&\leq \eta(b,a) \left(\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{\alpha p} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} |f'(a+t\eta(b,a))|^q dt \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{\alpha p} dt \right)^{\frac{1}{p}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 |f'(a+t\eta(b,a))|^q dt \right)^{\frac{1}{q}} \right) \\
=\ & \tfrac{\eta(b,a)}{(\alpha p+1)^{\frac{1}{p}}} \left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} |f'(a+t\eta(b,a))|^q dt \right)^{\frac{1}{q}} \right. \\
&\left. + \left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 |f'(a+t\eta(b,a))|^q dt \right)^{\frac{1}{q}} \right). \qquad (3.7)
\end{aligned}$$

Since $|f'|^q$ is s-preinvex, we deduce

$$
\begin{aligned}
&\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha}+\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha}\right)f(x)-\frac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{x^-}f(a)+J^{\alpha}_{x^+}f(a+\eta(b,a))\right)\right| \\
\leq\;& \frac{\eta(b,a)}{(\alpha p+1)^{\frac{1}{p}}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}}\left(\int_0^{\frac{x-a}{\eta(b,a)}}(1-t)^s|f'(a)|^q+t^s|f'(b)|^q\,dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}}\left(\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^s|f'(a)|^q+t^s|f'(b)|^q\,dt\right)^{\frac{1}{q}}\right) \\
=\;& \frac{\eta(b,a)}{(s+1)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{s+1}|f'(b)|^q\right.\right. \\
&\left.+\left(1-\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1}\right)|f'(a)|^q\right)^{\frac{1}{q}}+\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}} \\
&\left.\times\left(\left(1-\frac{x-a}{\eta(b,a)}\right)^{s+1}|f'(a)|^q+\left(1-\left(\frac{x-a}{\eta(b,a)}\right)^{s+1}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right),
\end{aligned}
$$

which completes the proof. □

Corollary 3.2. *In Theorem 3.2, if we choose $x=\frac{2a+\eta(b,a)}{2}$, then the following midpoint inequality holds for fractional integrals*

$$
\begin{aligned}
&\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{\frac{2a+\eta(b,a)}{2}^-}f(a)+J^{\alpha}_{\frac{2a+\eta(b,a)}{2}^+}f(a+\eta(b,a))\right)\right| \\
\leq\;& \frac{\eta(b,a)}{2^{\alpha+\frac{1}{p}}(s+1)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\left(\frac{|f'(b)|^q+(2^{s+1}-1)|f'(a)|^q}{2^{s+1}}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{2^{s+1}}\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

Remark 3.3. *In Corollary 3.2, if we choose $\alpha=s=1$, and $\eta(b,a)=b-a$, we obtain Theorem 2.3 from [7].*

Theorem 3.3. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a differentiable function such that $\eta(b,a)>0$ and $f'\in L([a,a+\eta(b,a)])$ and let $q>1$. If $|f'|^q$ s-preinvex in the second sense for some fixed $s\in(0,1]$, then the following inequality for fractional integrals*

$$
\begin{aligned}
&\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha}+\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha}\right)f(x)-\frac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{x^-}f(a)+J^{\alpha}_{x^+}f(a+\eta(b,a))\right)\right| \\
\leq\;& \frac{\eta(b,a)}{(\alpha+1)^{1-\frac{1}{q}}}\left(\left(\left(\frac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)}\right)\left(B_{\frac{x-a}{\eta(b,a)}}(\alpha+1,s+1)|f'(a)|^q\right.\right. \\
&\left.+\frac{1}{\alpha+s+1}\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+s+1}|f'(b)|^q\right)^{\frac{1}{q}}+\left(1-\frac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)} \\
&\times\left(\frac{1}{\alpha+s+1}\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+s+1}|f'(a)|^q\ ?\right. \\
&\left.\left.+\left(B(s+1,\alpha+1)-B_{\frac{x-a}{\eta(b,a)}}(s+1,\alpha+1)\right)|f'(b)|^q\right)^{\frac{1}{q}}\right) \qquad (3.8)
\end{aligned}
$$

holds for all $x\in[a,a+\eta(b,a)]$.

Proof. From Lemma 3.1, properties of modulus, and power mean inequality, we have

$$\begin{aligned}
&\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}\right)f(x)-\tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(a+\eta(b,a))\right)\right|\\
\leq\ &\eta(b,a)\left(\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha}\left|f'(a+t\eta(b,a))\right|^{q}dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha}\left|f'(a+t\eta(b,a))\right|^{q}dt\right)^{\frac{1}{q}}\right)\\
=\ &\frac{\eta(b,a)}{(\alpha+1)^{1-\frac{1}{q}}}\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)}\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha}\left|f'(a+t\eta(b,a))\right|^{q}dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)}\left(\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha}\left|f'(a+t\eta(b,a))\right|^{q}dt\right)^{\frac{1}{q}}\right).
\end{aligned} \tag{3.9}$$

Since $|f'|^q$ is s-preinvex, we deduce

$$\begin{aligned}
&\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}\right)f(x)-\tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(a+\eta(b,a))\right)\right|\\
&\leq\frac{\eta(b,a)}{(\alpha+1)^{1-\frac{1}{q}}}\left(\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)}\right)\right.\\
&\times\left(|f'(a)|^{q}\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha}(1-t)^{s}dt+|f'(b)|^{q}\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha+s}dt\right)^{\frac{1}{q}}+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)}\\
&\left.\times\left(|f'(a)|^{q}\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha+s}dt+|f'(b)|^{q}\int_{\frac{x-a}{\eta(b,a)}}^{1}t^{s}(1-t)^{\alpha}dt\right)^{\frac{1}{q}}\right).
\end{aligned} \tag{3.10}$$

Substituting (3.5) into (3.10), we obtain the desired result. □

Remark 3.4. *In Theorem 3.3, If we take $\alpha=s=1$, we obtain Theorem 2.8 from [5].*

Corollary 3.3. *In Theorem 3.3, if we choose $x=\frac{2a+\eta(b,a)}{2}$, then the following midpoint inequality holds for fractional integrals*

$$\begin{aligned}
&\left|\frac{1}{2^{\alpha-1}}f\left(\tfrac{2a+\eta(b,a)}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J_{\frac{2a+\eta(b,a)}{2}^-}^{\alpha}f(a)+J_{\frac{2a+\eta(b,a)}{2}^+}^{\alpha}f(a+\eta(b,a))\right)\right|\\
\leq\ &\frac{\eta(b,a)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}(\alpha+1)^{1-\frac{1}{q}}}\left(\left(B_{\frac{1}{2}}(\alpha+1,s+1)|f'(a)|^{q}+\tfrac{1}{(\alpha+s+1)2^{\alpha+s+1}}|f'(b)|^{q}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\tfrac{1}{(\alpha+s+1)2^{\alpha+s+1}}|f'(a)|^{q}+\left(B(s+1,\alpha+1)-B_{\frac{1}{2}}(s+1,\alpha+1)\right)|f'(b)|^{q}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Remark 3.5. *In Corollary 3.3, if we put $\alpha=s=1$, we obtain Theorem 3.1 from [18].*

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