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TIME CONTROL CHARTS THROUGH NHPP BASED ON DAGUM DISTRIBUTION

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Abstract. Statistical process control is a method of monitoring product in its development process using statistical techniques with the presumption that the products produced under identical process condition shall not always be alike with respect to some quality characteristic(s). However, if the observed variations are within the tolerable limits statistical process control (SPC) methods would pass them for acceptance. This philosophy is adopted to decide the reliability and quality of a product by defining some quality measures and proposing a probability model for the quality measurements. The well known Dagum distribution(DD) is considered to propose a product reliability based on non-homogenous Poisson process (NHPP). Its mean value function is taken as a quality characteristic and SPC limits for it are developed. These control limits are exemplified to a live failure data to detect the out of control signals for the quality of the product based on the failure data and compared with Exponential distribution(ED).

1. Introduction

Life time data generally contain the failure times of sample products or interfailure times or number of failures experienced in a given time. Assuming a suitable probability model the reliability of the product

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is computed and the quality with respect to reliability would be assessed. From a different point of view if the specific life time data contain, times between failures, also called inter failure times, probability limits for such a data can be constructed in a parametric approach. Taking central line at the median of the distribution of the data, the probability limits as usual control limits we can think of a control chart for the data. Points above the upper control limit of such a data would be an encouraging characteristic of the product because they lead to a large gap between successive failures so that the uptime of the product is large. Hence the product is preferable. That is detection of out of control above the UCL is desirable and its causes are to preserved or encouraged. Similarly detection of out of control below the LCL results in shorter gaps between successive failures.The assignable causes for this detection are to be minimized or eliminated. Points within the control limits indicate a smooth failure phenomenon. Thus such a set of control limits would be helpful in assessing the quality of the product based on inter failure time data. Any manufactured product is prone to failures for known or unknown reasons. A failed product can be rectified to bring it back to functioning through a testing process. In this procedure the data of observed product failures would throw some light on the quality of the product. There are various methods of measuring the product quality and the most popular among them is product reliability. Non homogenous Poisson processes are suitable models to compute product reliability in the statistical science. The earliest works in this direction can be attributed to those of Chan *et al*(2000) [1], Xie *et al*(2002) [4], Pham and Zhang(2003) [3] and Kim(2013) [2]. All these attempts are focussed on the mathematical model of the type

$$\mathrm{P(N(t+s)-N(t)=y)=\frac{e^{-\lambda s}(\lambda s)^y}{y!}, y=0,1,2,...} \tag{1.1}$$

where *N(t)* indicates the random number of occurrences of an event in the interval *[0,t]*. This mathematical model indicates that the changes in *N(t)* from one time period to another time period say *[t,t+s]* depend only on the length of the interval s but not on the extremities *t,t+s* of the interval. λ is called the failure intensity. In the above equation $E[N(t)] = \lambda t, \forall t$. If we think of a Poisson process whose mean depends on the starting t and also the length of the interval s such a Poisson process can be explained by an equation as

$$\mathrm{P(N(t)=y)=\frac{e^{-m(t)}(m(t))^y}{y!}, y=0,1,2,...} \tag{1.2}$$

In this equation *m(t)* is a positive valued, non decreasing, continuous function and is called the mean value function. Equation (1.2) is called a Non Homogenous Poisson Process. If a product system when put to use fails with probability *F(t)* before time t, if 'θ' stands for the unknown eventual number of failures that it is likely to experience, then the average number of failures expected to be experienced before time t is $\theta F(t)$. Hence $\theta F(t)$ can be taken as the mean value function of an NHPP. In the theory of probability, *F(t)* is called the cumulative distribution function (CDF) of a continuous non negative valued random variable. Thus an NHPP designed to study the failure process of a product can be constructed as a Poisson process with mean value function based on the cumulative distribution function of a continuous positive valued random variable.

With this backdrop, we consider the well known Dagum distribution (DD) as F(t) to generate a growth model based Non Homogenous Poisson Process (NHPP). For such a model we developed the statistically admissible control limits for the mean value function and demonstrate the same how a graphical procedure called a statistical process control (SPC) chart based on the mean value function would help in detecting out of control signals for the product quality. The rest of the paper is organized as follows:
The basic distribution characteristics of Dagum distribution (DD)and its properties are presented in section 2. Control chart monitoring the time between failures based on the mean value function, statistically tolerable limits for the failure time random variable, comparison with Exponential model and findings are given in section 3. Monitoring the production process based on the mean value function using order statistics, comparison with Exponential model and the findings are given in section 4. Summary and Conclusions are given in section 5.

2. **Distribution and its properties**

In the present paper we consider the CDF of DD as the genesis of mean value function of our SQC. This model is an increasing failure rate (IFR). Such a distribution is proved to be having a number of important applications in survival analysis, a proxy concept to reliability theory.
The probability density function (pdf) of Dagum distribution is given by

$$f(x) = \frac{ap}{x}\left(\frac{(\frac{x}{b})^{ap}}{((\frac{x}{b})^{a}+1)^{p+1}}\right), x>0, a>0, b>0, p>0. \tag{2.1}$$

Its cumulative distribution function (cdf) is

$$F(x) = \left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}, x>0, a>0, b>0, p>0. \tag{2.2}$$

The Dagum distribution is a skewed, unimodal distribution on the positive real line. The mean, median and variance of Dagum distbution are respectively

$$Mean \ = \ -\frac{b}{a}\frac{\Gamma\left(-\frac{1}{a}\right)\Gamma\left(\frac{1}{a}+p\right)}{\Gamma\left(p\right)}, if a>1. \tag{2.3}$$

$$Median \ = \ b\left(-1+2^{\frac{1}{p}}\right)^{-\frac{1}{a}}. \tag{2.4}$$

$$Variance \ = \ -\frac{b^2}{a^2}\left(2a\frac{\Gamma\left(-\frac{2}{a}\right)\Gamma\left(\frac{2}{a}+p\right)}{\Gamma\left(p\right)}+\left(\frac{\Gamma\left(-\frac{1}{a}\right)\Gamma\left(\frac{1}{a}+p\right)}{\Gamma\left(p\right)}\right)^2\right), if a>2. \tag{2.5}$$

The NHPP with θF(x) as the mean value function for our present study is

$$m(x) = \theta\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}, \theta>0, a>0, b>0, p>0. \tag{2.6}$$

Thus our proposed model contains 3 parameters namely θ, a and b, where θ stands for the unknown number of faults present in the product

3. Control chart monitoring the time between failures based on mean value function

Let F(x) be the cumulative distribution function of a continuous positive valued random variable, f(x) be its probability density function. If the random variable is taken as representing inter failure time of a device, a control chart of such data would be based on 0.9973 probability limits of the times between failure random variable say t analogous to the Shewhart's theory of variable control charts. These limits and the central line are respectively the solutions of the following equations taking equi-tailed probabilities.

$$F(t) = 0.00135 \tag{3.1}$$

$$F(t) = 0.5 \tag{3.2}$$

$$F(t) = 0.99865 \tag{3.3}$$

Let t_U, t_C and t_L be respectively the solutions of equations (3.1), (3.2) and (3.3) in the standard form

$$t_L = F^{-1}(0.00135) \tag{3.4}$$

$$t_C = F^{-1}(0.5) \tag{3.5}$$

$$t_U = F^{-1}(0.99865) \tag{3.6}$$

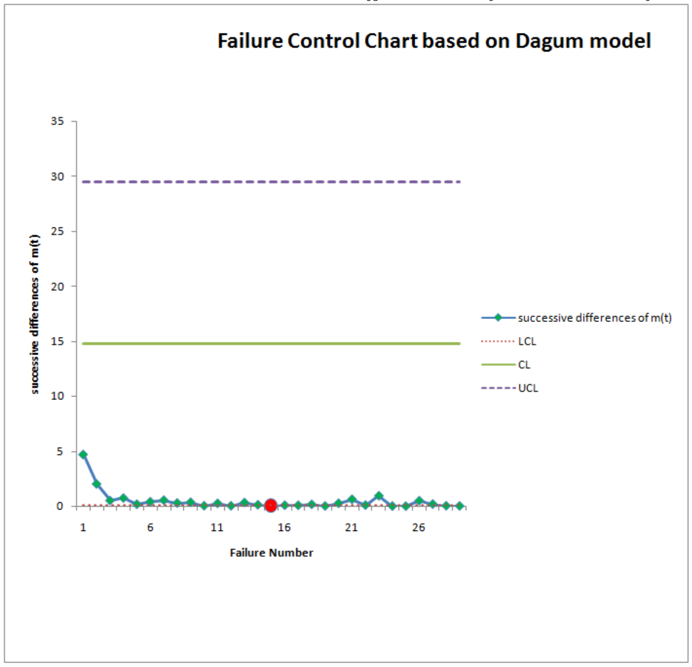
The NHPP with F(θ,x) as the mean value function for our present study is

$$m(x) = \theta\left(1 + \left(\frac{x}{b}\right)^{-a}\right)^{-p}, \theta > 0, a > 0, b > 0, p > 0. \tag{3.7}$$

The time control chart based on the mean value function corresponding to inter failure time together with three parallel lines to the horizontal axis at t_L, t_C and t_U for the data of Kim(2013) [2] is given below.

Estimated values of m(t) at the given failure times $t_1, t_2, ..t_n$ along with the successive differences of these estimates are given in Table 3. The successive differences would indicate the estimated number of failures between consecutive failure times. The graph through $[t_i, \triangle\hat{m}(t_i)]$ i=1,2,..,n-1 along with three parallel horizontal lines at $m(t_L), m(t_C), m(t_U)$ would be the required control chart and is given in Figure 1.

Fig 1. Control chart based on successive differences of mean value function of DD



3.1. **Comparative study.**

For comparison, we take Exponential model, the most frequently used model in reliability studies. The cumulative distribution function of Exponential distribution(ED)is

$$F(x) = 1 - e^{-bx}, \ x > 0, b > 0. \tag{3.8}$$

The NHPP with θF(x) as the mean value function for our present study is

$$m(x) = \theta[1 - e^{-bx}], \ x > 0, \theta > 0, b > 0. \tag{3.9}$$

Table 1. *Failure time data*

Failure number	Failure time (hours)	Failure number	Failure time (hours)	Failure number	Failure time (hours)
1	9	11	71	21	116
2	21	12	77	22	149
3	32	13	78	23	156
4	36	14	87	24	247
5	43	15	91	25	249
6	45	16	92	26	250
7	50	17	95	27	337
8	58	18	98	28	384
9	63	19	104	29	396
10	70	20	105	30	405

Table 2. *Parameter estimates of Dagum distribution and their control limits*

Dagum model						
a	b	p	$\hat{\theta}$	$m(t_L)$	$m(t_C)$	$m(t_U)$
0.6	0.5	5	29.47	0.0398	14.7349	29.4302

Table 3. *Successive differences based on the mean value function of DD*

Failure number	Failure time (hours)	m(t)	Successive differences of m(t) $\triangle\hat{m}(t)$	Failure number	Failure time (hours)	m(t)	Successive differences of m(t) $\triangle\hat{m}(t)$
1	9	13.07228	4.721	16	92	23.7886	0.095
2	21	17.7929	2.036	17	95	23.8839	0.091
3	32	19.829	0.524	18	98	23.9751	0.17
4	36	20.352	0.75	19	104	24.1455	0.027
5	43	21.103	0.184	20	105	24.1725	0.274
6	45	21.287	0.415	21	116	24.4469	0.635
7	50	21.702	0.557	22	149	25.0821	0.108
8	58	22.261	0.296	23	156	25.1904	0.959
9	63	22.556	0.363	24	247	26.1494	0.015
10	70	22.919	0.048	25	249	26.1643	0.007
11	71	22.967	0.267	26	250	26.1717	0.509
12	77	23.234	0.042	27	337	26.6811	0.199
13	78	23.276	0.343	28	384	26.8799	0.045
14	87	23.619	0.137	29	396	26.9248	0.032
15	91	23.7553	0.033	30	405	26.9571	

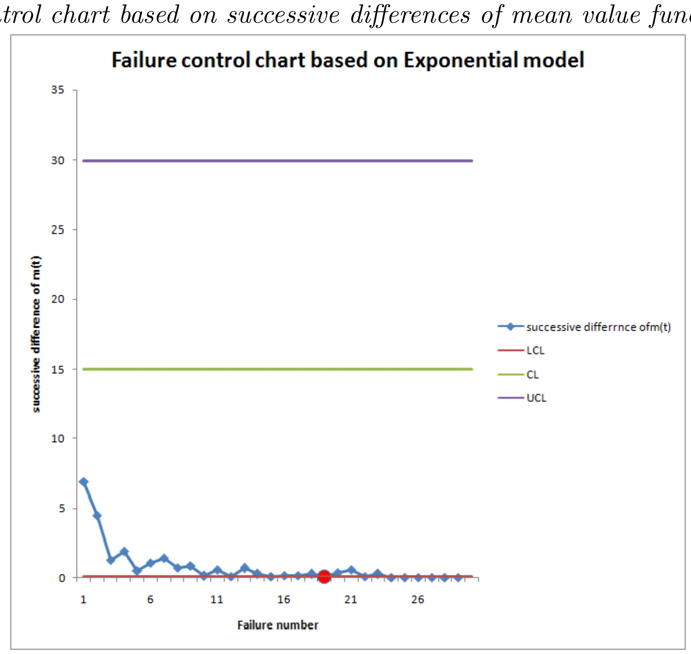
Table 4. *Parameter estimates of Exponential model and their control limits*

Exponential model				
b	$\hat{\theta}$	$m(t_L)$	$m(t_C)$	$m(t_U)$
0.03	30.00016	0.0405	15.0008	29.9596

Table 5. *Successive differences based on the mean value function of ED*

Failure number	Failure time (hours)	m(t)	Successive differences of m(t) $\triangle\hat{m}(t)$	Failure number	Failure time (hours)	m(t)	Successive differences of m(t) $\triangle\hat{m}(t)$
1	9	7.09865	6.924	16	92	28.1014	0.163
2	21	14.0223	4.491	17	95	28.2648	0.149
3	32	18.5133	1.299	18	98	28.4142	0.261
4	36	19.8122	1.93	19	104	28.6754	0.039
5	43	21.742	0.481	20	105	28.7146	0.361
6	45	22.2229	1.083	21	116	29.0759	0.581
7	50	23.7345	1.428	22	149	29.6567	0.065
8	58	24.7345	0.733	23	156	29.7218	0.26
9	63	25.468	0.858	24	247	29.982	0.001
10	70	26.3264	0.109	25	249	29.2831	0.0005
11	71	26.435	0.587	26	250	29.9836	0.0153
12	77	27.0223	0.088	27	337	29.9989	0.00009
13	78	27.1103	0.684	28	384	29.9999	0.000009
14	87	27.7941	0.249	29	396	30	0.000005
15	91	28.0436	0.058	30	405	30	

Fig 2. Control chart based on successive differences of mean value function of ED



4. Monitoring the production process based on mean value function using order statistics

Let $x_1, x_2, .., x_n$ be a random sample of size *n representing* n inter failure times of a product governed by the probability model of a continuous random variable X. Let F(x) be the cumulative distribution function of X. These inter failure times can be used for assessing the failure phenomenon with respect to two limits of reference called control limits with a pre specified coverage probability. Thus the time control chart plotted for inter failure times would indicate alarms, advantages and stable failure process. If r is a natural number (<n), the summations $\sum_{i=1}^{r} X_i$, $\sum_{i=r+1}^{2r} X_i$, $\sum_{i=2r+1}^{3r} X_i$ etc represent the lapse of time consecutively between every r^{th} failure. A control chart for times between every r^{th} failure would throw light on the out of control signals than that of inter failure times. Xie *et al*(2002) [4] named such a control chart as t_r-control chart and developed control limits using the sampling distribution of $\sum_{i=1}^{r} X_i$. They have taken the example of exponential distribution and used the theory that the sum of exponential variates is a gamma variate to get the percentiles of t_r-control chart with the help of cumulative summations. If the inter failure times are not exponentials, the control limits of t_r-chart of Xie *et al*(2002) [4] can not be used.

Overcoming this drawback we suggest the following alternative approach to get control limits of t_r-chart for Dagum distribution. If $(X_1, X_2, .., X_r)$; $(X_{r+1}, X_{r+2}, .., X_{2r})$; $(X_{2r+1}, X_{2r+2}, .., X_{3r})$; etc are regarded as independent samples of size r each, i.i.d random variables having F(x) as their common model. $Y_1 = X_1$, $Y_2 = \sum_{i=1}^{2} X_i$, $Y_3 = \sum_{i=1}^{3} X_i$, ..., $Y_r = \sum_{i=1}^{r} X_i$ becomes an ordered sample of size r representing the time to first failure, time to second failure, time to third failure,....., time to r^{th} failure respectively. Thus, the t_r-chart is the control chart with Y_r as the points on it representing the time to every r^{th} failure. Therefore, when r is fixed, the percentiles of highest order statistics in a sample of size r would serve the purpose of control limits for the t_r-chart.

Let F(x) be the cumulative distribution function of a continuous positive valued random variable. If the random variable is taken as representing inter failure time of a device, a control chart of such data with order statistics would be based on 0.9973 probability limits of the times between failure random variable say t. These limits and the central line are respectively the solutions of the following equations taking equi-tailed probabilities.

$$[F(t)]^r = 0.00135 \tag{4.1}$$

$$[F(t)]^r = 0.5 \tag{4.2}$$

$$[F(t)]^r = 0.99865 \tag{4.3}$$

Let t_U, t_C and t_L be respectively the solutions of equations (3.1), (3.2) and (3.3) in the standard form

$$t_L = F^{-1}(0.00135^{\frac{1}{r}}) \tag{4.4}$$

$$t_C = F^{-1}(0.5^{\frac{1}{r}}) \tag{4.5}$$

$$t_U = F^{-1}(0.99865^{\frac{1}{r}}) \tag{4.6}$$

The NHPP with θ.F(x) as the mean value function for our present study is

$$m(x) = \theta\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}, \theta > 0, a > 0, b > 0, p > 0. \tag{4.7}$$

The above model is illustrated for the example of 60 failure times considered by Xie *et al*(2002) [4]. For a ready reference the data is produced in table 6.

Table 6. *Failure time data of the components*

Failure number	Time	Failure number	Time	Failure number	Time	Failure number	Time
1	1065.55	16	2932.96	31	35.85	46	239.66
2	535.8	17	987.67	32	362.8	47	93.78
3	540.53	18	1816.18	33	357.85	48	680.45
4	716.2	19	117.21	34	334.48	49	4.83
5	2525.43	20	190.65	35	80.13	50	102.91
6	1264.18	21	943.99	36	1939.0	51	479.05
7	479.44	22	1084.48	37	77.88	52	156.67
8	1783.22	23	2306.54	38	4.03	53	1286.24
9	473.67	24	6.56	39	98.67	54	443.97
10	2265.42	25	3111.51	40	17.19	55	360.03
11	2191.75	26	283.86	41	289.79	56	414.66
12	1097.26	27	659.39	42	63.99	57	128.9
13	597.59	28	683.48	43	2.46	58	36.1
14	971.16	29	36.14	44	697.68	59	197.31
15	3157.29	30	754.16	45	1167.33	60	418.12

Table 7. *Accumulation failure time for every three failures*

Observation	Accumulation of 3 failures	Observation	Accumulation of 3 failures
1	2141.88	11	756.5
2	4505.81	12	2353.61
3	2736.33	13	180.58
4	5554.43	14	370.97
5	4726.04	15	1867.47
6	5736.81	16	1013.89
7	1251.85	17	586.79
8	3397.58	18	1886.88
9	4054.76	19	903.59
10	1473.78	20	651.53

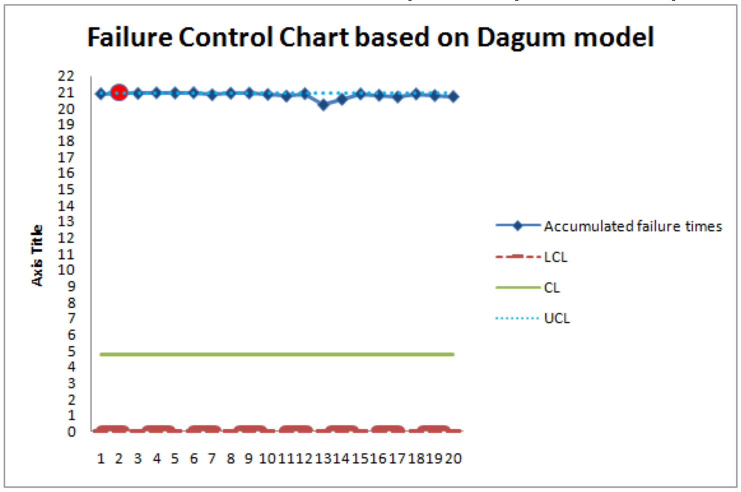
Table 8. *Parameter estimates of Dagum distribution and their control limits*

Dagum model						
a	b	p	$\hat{\theta}$	$m(t_L)$	$m(t_C)$	$m(t_U)$
0.8	0.1	0.5	20.9884	0.000015	4.7555	20.9278

Table 9. *Mean value function for accumulated failure times of DD*

Observation	Accumulation of 3 failures	m(t)	Observation	Accumulation of 3 failures	m(t)
1	2141.88	20.881	11	756.5	20.741
2	4505.81	20.929	12	2353.61	20.888
3	2736.33	20.899	13	180.58	20.222
4	5554.43	20.938	14	370.97	20.554
5	4726.04	20.931	15	1867.47	20.868
6	5736.81	20.939	16	1013.89	20.793
7	1251.85	20.823	17	586.79	20.686
8	3397.58	20.913	18	1886.88	20.869
9	4054.76	20.923	19	903.59	20.774
10	1473.78	20.843	20	651.53	20.711

Fig 3. Control chart based on accumulated failures of mean value function of DD



4.1. **Comparative study based on accumulated failure times.**

We compare our model under study with the exponential model using the data set given in table 6 and the results are as follows:

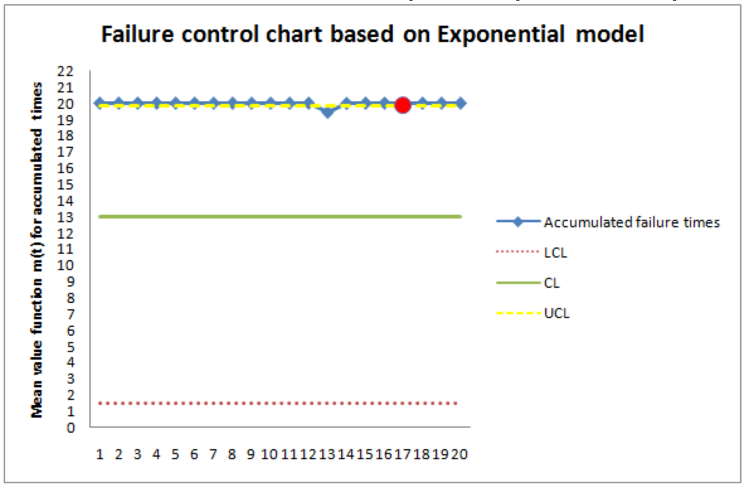
TABLE 10. *Parameter estimates of Exponential model and their control limits*

Exponential model				
b	$\hat{\theta}$	$m(t_L)$	$m(t_C)$	$m(t_U)$
0.02	20.00004	1.5023	13.0130	19.8830

TABLE 11. *Mean value function for accumulated failure times of ED*

Observation	Accumulation of 3 failures	m(t)	Observation	Accumulation of 3 failures	m(t)
1	2141.88	20.00004	11	756.5	20.00004
2	4505.81	20.00004	12	2353.61	20.00004
3	2736.33	20.00004	13	180.58	19.45987
4	5554.43	20.00004	14	370.97	19.98805
5	4726.04	20.00004	15	1867.47	20.00004
6	5736.81	20.00004	16	1013.89	20.00004
7	1251.85	20.00004	17	586.79	19.99988
8	3397.58	20.00004	18	1886.88	20.00004
9	4054.76	20.00004	19	903.59	20.00004
10	1473.78	20.00004	20	651.53	20

Fig 4. Control chart based on accumulated failures of mean value function of ED



5. **Summary & Conclusions**

In Figure 1, the first out of control situation is noticed at the 15^{th} failure with the corresponding successive difference of $m(t)$ falling below LCL and hence a preferable out-of-control signal for the product. Where as in Figure 2, it is noticed at 19^{th} failure. The earlier the failure, one can alert the process and assignable cause for this is to be investigated and can be promoted. There are many charts which use statistical techniques. It is important to use the best chart for the given data, situation and need. In the first part of the paper, the control chart for estimated number of failures in successive failure time intervals against the serial order of the failure interval is developed with the associated control lines and central line at same serial point on that of Kim(2013) [2].

Similarly for the control limits based on the accumulated failure times also shows that Dagum distribution is better model when compared with that of the exponential model used by Xie *et al*(2002) [4]. From the figure 3 and 4 we can observe that 2^{nd} and 19^{th} accumulated failure time is out of the limits respectively. The earlier the failure group, one need not to wait till the last group failure occurs.

References

[1] Chan, L.Y., Xie, M. and Goh, T.N. Cumulative quality control charts for monitoring production process, Int. J. Product. Res., 38(2)(2000), 397–408.

[2] Kim, H.C. Assessing software reliability based on NHPP using SPC, Int. J. Softw. Eng. Appl., 7(6)(2013), 61–70.

[3] Pham, H. and Zhang, X. Non homogeneous poisson process software reliability and cost models with testing coverage, Eur. J. Oper. Res., (145)(2003), 443–454.

[4] Xie, M., Goh, T.N. and Ranjan, P. Some effective control chart procedurs for reliability monitoring, Reliab. Eng. Syst. Safety, 77(2002), 143–150.