



EQUIVALENCE OF STURM-LIOUVILLE PROBLEM WITH FINITELY MANY δ -INTERACTIONS AND MATRIX EIGENVALUE PROBLEMS

ABDULLAH KABLAN*, MEHMET AKİF ÇETİN

Faculty of Arts and Sciences, Department of Mathematics, Gaziantep University, Gaziantep, 27310, Turkey

**Corresponding author: kablan@gantep.edu.tr*

ABSTRACT. The purpose of this article is to show the matrix representations of Sturm-Liouville operators with finitely many δ -interactions. We show that a Sturm-Liouville problem with finitely many δ -interactions can be represented as a finite dimensional matrix eigenvalue problem which has the same eigenvalue with the former Sturm-Liouville operator. Moreover an example is also presented.

1. INTRODUCTION

According to classical spectral theory, a Sturm–Liouville problem (SLP) consisting of the equation

$$-(py')' + qy = \lambda wy, \text{ on } J = (a, b)$$

and boundary conditions has infinite spectrum under some assumptions. Atkinson in his book [1] suggested that if the coefficients of SLP satisfy some conditions, the problem may have finite eigenvalues. Then in [2], Kong, Wu and Zettl obtained the following result: For every positive integer n , we can construct a class of regular self-adjoint and nonself-adjoint SLP with exactly n eigenvalues by choosing p and w such that $1/p$ and w are alternatively zero on consecutive subintervals.

Recently, there has been much attention paid to the SLPs with finite spectrum. For a comprehensive treatment of the subject we refer the reader to the book by Zettl [3], and the papers by Kong, Wu and Zettl [2], Ao, Sun, and Zhang [4], [5] and Ao, Bo and Sun [6], [7]. In 2009, the equivalence of SLP with

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a matrix eigenvalue problem was first constructed by Volkmer and Zettl in [8]. By equivalence of matrix eigenvalue problems for the SLPs with finite spectrum we mean to construct a matrix eigenvalue problem with exactly the same eigenvalues as the corresponding SLP. Then, the matrix representations of SLPs with finite spectrum are extended to various problems. For the SLPs see [8]- [11] and for fourth order boundary value problems see [12]- [16].

The goal of this paper is to find the matrix representation of the following Sturm-Liouville problem with finitely many δ -interactions:

$$-(py')' + \sum_{n=1}^{\infty} \alpha_n \delta(x - x_n)y + qy = \lambda wy, \text{ on } J = (a, b), \tag{1.1}$$

where $J = (a, x_1) \cup (x_1, x_2) \cup \dots \cup (x_n, b)$, $x_1, \dots, x_n \in (a, b)$ with $-\infty < a < b < \infty$, α_j 's are real numbers, $\delta(x)$ is the Dirac delta function and $\lambda \in \mathbb{C}$ is a spectral parameter. Sturm-Liouville equations with Dirac delta function potentials often appear in quantum mechanics. For example, such an equations had been used for modelling of atomic and molecular systems including atomic lattices, quantum heterostructures, semiconductors, organic fluorescent materials, solar cells etc. (see [17], [18], [19] and citations of them). Recently, we generalize the finite spectrum result to the problem (1.1) in [20]. The equation (1.1) is equivalent to the many-point boundary value problem, (see [19]). So we can understand problem (1.1) as studying the equation

$$-(py')' + qy = \lambda wy, \text{ on } J, \tag{1.2}$$

and n transmission conditions

$$C_j Y(x_j-) = Y(x_j+), \quad Y = \begin{bmatrix} y \\ py' \end{bmatrix}, \quad j = 1, 2, \dots, n \tag{1.3}$$

where x_j 's are inner discontinuity points and

$$C_j = \begin{bmatrix} 1 & 0 \\ \alpha_j & 1 \end{bmatrix}.$$

Additionally, let us consider the boundary conditions of the form

$$AY(a) + BY(b) = 0, \quad A, B \in M_2(\mathbb{C}) \tag{1.4}$$

where $A = (a_{ij})_{2 \times 2}$, $B = (b_{ij})_{2 \times 2}$ are complex valued 2×2 matrices and $M_2(\mathbb{C})$ denotes the set of square matrices of order 2 over \mathbb{C} . Here, the coefficients fulfill the following minimal conditions:

$$r = \frac{1}{p}, \quad q, \quad w \in L(J, \mathbb{C}), \tag{1.5}$$

where $L(J, \mathbb{C})$ denotes the complex valued functions which are Lebesgue integrable on J .

The BC (1.3) is said to be self-adjoint if the following two conditions are satisfied:

$$\text{rank}(A, B) = 2, \quad AEA^* = BEB^* \quad \text{with } E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (1.6)$$

It is well known that under the condition (1.5), the BCs (1.3) fall into two disjoint classes: separated and coupled. The separated boundary conditions have the canonical representation:

$$\begin{aligned} \cos \alpha y(a) - \sin \alpha (py')(a) &= 0, \quad 0 \leq \alpha < \pi \\ \cos \beta y(b) - \sin \beta (py')(b) &= 0, \quad 0 < \beta \leq \pi. \end{aligned} \quad (1.7)$$

The real coupled boundary conditions have the canonical representation:

$$Y(b) = KY(a) \quad \text{with } K = (k_{s,t})_{2 \times 2}, \quad k_{s,t} \in \mathbb{R}, \quad \det(K) = 1. \quad (1.8)$$

Let $u = y$ and $v = (py')$. Then we have the system representation of equation (1.2)

$$u' = rv, \quad v' = (q - \lambda w)u, \quad \text{on } J. \quad (1.9)$$

2. MATRIX REPRESENTATIONS OF SLPs WITH FINITELY MANY δ -INTERACTIONS

Definition 2.1. A Sturm-Liouville equation with finitely many δ -interactions (1.1) or equivalently the equation (1.2) with transmission condition (1.3) is said to be of Atkinson type if, for some integers $m_j \geq 1$, $j = 0, 1, \dots, n$, there exists a partition of the interval J

$$\begin{aligned} a &= x_{00} < x_{01} < x_{02} < \dots < x_{0,2m_0+1} = x_1, \\ x_1 &= x_{10} < x_{11} < x_{12} < \dots < x_{1,2m_1+1} = x_2, \\ &\vdots \\ x_{n-1} &= x_{n-1,0} < x_{n-1,1} < x_{n-1,2} < \dots < x_{n-1,2m_{n-1}+1} = x_n, \\ x_n &= x_{n0} < x_{n1} < x_{n2} < \dots < x_{n,2m_n+1} = b \end{aligned} \quad (2.1)$$

such that for each $j \in \{0, 1, \dots, n\}$

$$\begin{aligned} r = \frac{1}{p} = 0 \quad \text{on } (x_{j,2k}; x_{j,2k+1}], \quad k = 0, 1, \dots, m_j - 1 \quad \text{and } [x_{j,2m_j}; x_{j,2m_j+1}), \\ \int_{x_{j,2k}}^{x_{j,2k+1}} w \neq 0, \quad \int_{x_{j,2k}}^{x_{j,2k+1}} q \neq 0, \quad k = 0, 1, \dots, m_j, \end{aligned} \quad (2.2)$$

and

$$q = w = 0 \quad \text{on } [x_{j,2k+1}; x_{j,2k+2}], \quad \int_{x_{j,2k+1}}^{x_{j,2k+2}} r \neq 0, \quad k = 0, 1, \dots, m_j - 1. \quad (2.3)$$

Our main aim in this section is to construct matrix eigenvalue problems in such a way that its eigenvalues are exactly the same as those of the corresponding SLPs with finitely many δ -Interactions of Atkinson type.

Definition 2.2. *A SLP with finitely many δ -Interactions of Atkinson type is said to be equivalent to a matrix eigenvalue problem if the former has exactly the same eigenvalues as the latter.*

We begin by stating some additional notation. For each $j \in \{0, 1, \dots, n\}$ given (2.1)-(2.3), let

$$\begin{aligned}
 p_{jk} &= \left(\int_{x_{j,2k-1}}^{x_{j,2k}} r \right)^{-1}, \quad k = 1, 2, \dots, m_j; \\
 q_{jk} &= \int_{x_{j,2k}}^{x_{j,2k+1}} q, \quad w_{jk} = \int_{x_{j,2k}}^{x_{j,2k+1}} w, \quad k = 0, 1, \dots, m_j.
 \end{aligned}
 \tag{2.4}$$

and let introduce the notation

$$m = \sum_{j=0}^n m_j.
 \tag{2.5}$$

We note from (2.2) and (2.3) that $p_{jk}, w_{jk} \in \mathbb{R} \setminus \{0\}$, and no sign restrictions are imposed on them.

From (2.2) and (2.3) we can make the following observation: For any solution u, v of (1.9), u is constant on the intervals where r is identically zero and v is constant on the intervals where both q and w are both identically zero. Let

$$\begin{aligned}
 u_{0k} &= u(x), \quad x \in [x_{0,2k}; x_{0,2k+1}], \quad k = 0, 1, \dots, m_0 - 1, \\
 u_{0m_0} &= u(x), \quad x \in [x_{0,2m_0}; x_{0,2m_0+1}), \\
 u_{j0} &= u(x), \quad x \in (x_{j0}; x_{j1}], \quad j = 1, 2, \dots, n \\
 u_{jk} &= u(x), \quad x \in [x_{j,2k}; x_{j,2k+1}], \quad k = 1, 2, \dots, m_j - 1, \quad j = 1, 2, \dots, n - 1 \\
 u_{jm_j} &= u(x), \quad x \in [x_{j,2m_j}; x_{j,2m_j+1}), \quad j = 0, 1, \dots, n - 1 \\
 u_{nk} &= u(x), \quad x \in [x_{n,2k}; x_{n,2k+1}], \quad k = 1, 2, \dots, m_n \\
 v_{jk} &= v(x), \quad x \in [x_{j,2k-1}; x_{j,2k}), \quad k = 1, 2, \dots, m_j, \quad j = 0, 1, \dots, n
 \end{aligned}
 \tag{2.6}$$

and set

$$v_{j0} = v(x_{j0+}), \quad v_{j,m_j+1} = v(x_{j,2m_j+1-}), \quad j = 0, 1, \dots, n.
 \tag{2.7}$$

Lemma 2.1. *Assume Eq. (1.2) is of Atkinson type. Then for each $j = 0, 1, \dots, n$ and for any solution u, v of Eq. (1.9), we have*

$$p_{jk}(u_{jk} - u_{j,k-1}) = v_{jk}, \quad k = 1, 2, \dots, m_j,
 \tag{2.8}$$

$$v_{j,k+1} - v_{jk} = u_{jk}(q_{jk} - \lambda w_{jk}), \quad k = 0, 1, \dots, m_j.
 \tag{2.9}$$

following system:

$$[-k_{11}k_{21} + k_{11}^2 (p_{j+1,m_{j+1}} + q_{j+1,m_{j+1}} - \lambda w_{j+1,m_{j+1}})] u_{j0} \tag{2.25}$$

$$= (\lambda w_{j0} - p_{j1} - q_{j0})u_{j0} + p_{j1}u_{j1} + k_{11}p_{j+1,m_{j+1}}u_{j+1,m_{j+1}-1}$$

$$p_{j,k+1} (u_{j,k+1} - u_{jk}) - p_{jk} (u_{jk} - u_{j,k-1}) = u_{jk} (q_{jk} - \lambda w_{jk}) \tag{2.26}$$

$$p_{j+1,1} (u_{j+1,1} - u_{j+1,0}) - v_{j+1,0} = u_{j+1,0}(q_{j+1,0} - \lambda w_{j+1,0}) \tag{2.27}$$

$$p_{j+1,m_{j+1}} (k_{11}u_{j0} - u_{j+1,m_{j+1}-1}) - p_{j+1,m_{j+1}-1}u_{j+1,m_{j+1}-1} \tag{2.28}$$

$$= p_{j+1,m_{j+1}-1}u_{j+1,m_{j+1}-2} + u_{j+1,m_{j+1}-1}(q_{j+1,m_{j+1}-1} - \lambda w_{j+1,m_{j+1}-1})$$

Then, by Lemma 2.1, any solution of system (1.9), hence of (1.2), is uniquely determined by a solution of system (2.25)-(2.28). □

Theorem 2.3. Consider the boundary condition (1.8) with $k_{12} \neq 0$. Define the $(m + 1) \times (m + 1)$ matrix which is tridiagonal except for the $(1, m + 1)$ and $(m + 1, 1)$ entries

$$P_2 = \begin{bmatrix} M_0 & & & & & & & & & \frac{1}{k_{12}} \\ N_1 & M_1 & & & & & & & & \\ & N_2 & M_2 & & & & & & & \\ & & & \ddots & \ddots & & & & & \\ & & & & & N_n & M_n & & & \\ \frac{1}{k_{12}} & & & & & & N_{n+1} & M_{n+1} & & \end{bmatrix}$$

and diagonal matrices

$$Q_2 = \text{diag} (q_{00}, q_{01}, \dots, q_{0,m_0-1}, q_{0m_0} + q_{10}, q_{11}, \dots, q_{n,m_n-1}, q_{nm_n})$$

$$W_2 = \text{diag} (w_{00}, w_{01}, \dots, w_{0,m_0-1}, w_{0m_0} + w_{10}, w_{11}, \dots, w_{n,m_n-1}, w_{nm_n})$$

Then SLP with transmission conditions (1.2), (1.3), (1.8) is equivalent to matrix eigenvalue problem

$$(P_2 + Q_2)U = \lambda W_2U \tag{2.29}$$

where

$$U = [u_{00}, u_{01}, \dots, u_{0m_0}, u_{11}, \dots, u_{1m_1}, \dots, u_{n1}, \dots, u_{nm_n}]^T,$$

and the elements of the matrix P_3 are defined as follows: For each $j = 0, 1, \dots, n$ the matrices N_{j+1} 's and for each $j = 0, 1, \dots, n - 1$ the matrices M_{j+1} 's are defined as in Theorem 2.2. On the other hand, the 1×2 matrix

$$M_0 = \begin{bmatrix} p_{01} - \frac{k_{11}}{k_{12}} & -p_{01} \end{bmatrix},$$

In this case, the matrices in (1.3) and (1.4) become

$$C_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

respectively. Now, let's take a partition of the interval J as follows:

$$\begin{aligned} a &= -3 < -2 < -1 < 0 = x_1 \\ x_1 &= 0 < 2 < 3 < 4 < 5 < 6 = x_2 = b. \end{aligned} \tag{3.5}$$

This yields that $m_0 = 1, m_1 = 2$ and define the piecewise constant functions p, q, w are as follows:

$$p(x) = \begin{cases} \infty, & (-3, -2) \\ 1, & (-2, -1) \\ \infty, & (-1, 0) \\ \infty, & (0, 2) \\ \frac{1}{2}, & (2, 3) \\ \infty, & (3, 4) \\ \frac{1}{4}, & (4, 5) \\ \infty, & (5, 6) \end{cases} \quad q(x) = \begin{cases} 0, & (-3, -2) \\ 0, & (-2, -1) \\ 1, & (-1, 0) \\ 2, & (0, 2) \\ 0, & (2, 3) \\ 3, & (3, 4) \\ 0, & (4, 5) \\ 4, & (5, 6) \end{cases} \quad w(x) = \begin{cases} 1, & (-3, -2) \\ 0, & (-2, -1) \\ 3, & (-1, 0) \\ 4, & (0, 2) \\ 0, & (2, 3) \\ 1, & (3, 4) \\ 0, & (4, 5) \\ 2, & (5, 6) \end{cases} \tag{3.6}$$

By using the similar method as given in [4], [5] or [20] we have the following two eigenvalues

$$\lambda_1 = 0.67442, \quad \lambda_2 = 3.75739. \tag{3.7}$$

On the other hand, if we find the values p_{jk}, q_{jk}, w_{jk} from (2.4), and use Theorem 2.1 we get the matrices

$$P_{0\pi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & \frac{5}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad Q_{0\pi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad W_{0\pi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.8}$$

and so the matrix eigenvalue problem

$$(P_{0\pi} + Q_{0\pi})U = \lambda W_{0\pi}U, \tag{3.9}$$

which is equivalence of SLP with finitely many δ -interactions in (3.1). Indeed, if we find the eigenvalues of the matrix eigenvalue problem (3.9) we obtain the eigenvalues in (3.7).

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