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INEQUALITIES OF FEJÉR TYPE RELATED TO GENERALIZED CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. This paper deals with some Fejér type inequalities related to (η_1, η_2)-convex functions. In fact the difference between the right and middle part of Fejér inequality is estimated without using Hölder's inequality when the absolute value of the derivative of considered function is (η_1, η_2)-convex. Furthermore we give two estimation results when the derivative of considered function is bounded and satisfies a Lipschitz condition.

1. Introduction and Preliminaries

The Fejér integral inequality for convex functions has been proved in [5]:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then*

$$f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{1.1}$$

where $g : [a, b] \to [0, +\infty)$ is integrable and symmetric to $x = \frac{a+b}{2}$ $\Big(g(x) = g(a + b - x), \forall x \in [a, b]\Big)$.

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The estimation for difference of the right and middle part in (1.1) is an interesting problem. The following theorem has been proved in [12], that estimates the difference between the right and middle part in (1.1) using Hölder's inequality when the absolute value of the derivative of considered function is convex.

Theorem 1.2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $w : [a, b] \to [0, +\infty)$ be a differentiable mapping and symmetric to $\frac{a+b}{2}$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx - \frac{1}{b-a}\int_a^b f(x)w(x)dx\right| \leq \frac{1}{2}\left[\int_0^1 \big(g(x)\big)^p dt\right]^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}, \tag{1.2}$$

where

$$g(x) = \left|\int_{a+(b-a)t}^{b-(b-a)t} w(x)dx\right|,$$

for $t \in [0, 1]$.

The preinvex functions as a generalization of convex functions was considered by Ben-Israel and Mond in [1] and Hanson and Mond in [6], but so named by Jeyakumar [7].

Definition 1.1. [1,6] A set $I \subseteq \mathbb{R}$ is *invex with respect to a real bifunction* $\eta : I \times I \to \mathbb{R}$, if

$$x, y \in I, \lambda \in [0, 1] \Rightarrow y + \lambda\eta(x, y) \in I.$$

Also if I is an invex set with respect to η, then a function $f : I \to \mathbb{R}$ is said to be *preinvex* if $x, y \in I$ and $\lambda \in [0, 1]$ implies

$$f\big(y + \lambda\eta(x, y)\big) \leq \lambda f(x) + (1 - \lambda)f(y).$$

The following theorem has been proved in [8] which is preinvex version of Theorem 1.2.

Theorem 1.3. *Let $K \subset \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + a + \eta(b, a)])$ and $w : [a, a + \eta(b, a)] \to [0, +\infty)$ is an integrable mapping and symmetric to $a + \frac{1}{2}\eta(b, a)$. If $|f'|^q$, $q > 1$, is preinvex on K, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ we have the following inequality:*

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}\int_a^{a+\eta(b,a)} w(x)dx - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)w(x)dx\right| \leq \frac{1}{2}\left(\int_0^1 g^p(t)\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}, \tag{1.3}$$

where

$$g(t) = \left|\int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)} w(x)dx\right|,\ t \in [0, 1] \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

Furthermore the concept of η-convex functions (at the beginning was named by φ-convex functions), considered in [4], has been introduced as the following.

Definition 1.2. *Consider a convex set $I \subset \mathbb{R}$ and a bifunction $\eta : f(I) \times f(I) \to \mathbb{R}$. A function $f : I \to \mathbb{R}$ is called convex with respect to η (briefly η-convex), if*

$$f\big(\lambda x + (1-\lambda)y\big) \leq f(y) + \lambda\eta\big(f(x), f(y)\big),$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Geometrically it says that if a function is η-convex on I, then for any $x, y \in I$, its graph is on or under the path starting from $\big(y, f(y)\big)$ and ending at $\Big(x, f(y) + \eta(f(x), f(y))\Big)$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x, y) = x - y$ and the function reduces to a convex one.
The following theorem has been proved in [3], where the absolute value of the derivative of considered function is η-convex.

Theorem 1.4. *Suppose that $f : [a,b] \to \mathbb{R}$ is a differentiable function, $g : [a,b] \to [0, +\infty)$ is a continuous function and symmetric to $\frac{a+b}{2}$ and $|f'|$ is an η-convex function where η is bounded from above on $[a,b]$. Then*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx - \int_a^b f(x)g(x)dx\right| \leq$$
$$\frac{(b-a)}{4}\Big[2\big|f'(b)\big| + \eta\big(|f'(a)|, |f'(b)|\big)\Big]\int_0^1 \int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)dudt. \tag{1.4}$$

For more results about η-convex functions see [3, 4, 10, 11].

Motivated by above works and references therein, we introduce the concept of (η_1, η_2)-convex functions as a generalization of preinvex and η-convex functions. Also we give some Fejér type trapezoid inequalities when the absolute value of the derivative of considered function is (η_1, η_2)-convex but with new face without using of Hölder's inequality. Furthermore we obtain two estimation results when the derivative of considered function is bounded and satisfies a Lipschitz condition.

Definition 1.3. *Let $I \subset \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \to \mathbb{R}$. Consider $f : I \to \mathbb{R}$ and $\eta_2 : f(I) \times f(I) \to \mathbb{R}$. The function f is said to be (η_1, η_2)-convex if*

$$f\big(x + \lambda\eta_1(y,x)\big) \leq f(x) + \lambda\eta_2\big(f(y), f(x)\big),$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Note. An (η_1, η_2)-convex function reduces to

(i) an η-convex function if we consider $\eta_1(x,y) = x - y$ for all $x, y \in I$.

(ii) a preinvex function if we consider $\eta_2(x,y) = x - y$ for all $x, y \in f(I)$.

(iii) a convex function if satisfies (i) and (ii).

We can find an (η_1, η_2)-convex function which is not convex.

Example 1.1. *Consider the function* $f : [0, +\infty) \to [0, +\infty)$ *by*

$$f(x) = \begin{cases} x, & 0 \le x \le 1; \\ 1, & x > 1. \end{cases}$$

Define two bifunction $\eta_1 : [0, +\infty) \times [0, +\infty) \longrightarrow \mathbb{R}$ *and* $\eta_2 : [0, +\infty) \times [0, +\infty) \longrightarrow [0, +\infty)$ *by*

$$\eta_1(x,y) = \begin{cases} -y, & 0 \le y \le 1; \\ x + y, & y > 1, \end{cases}$$

and

$$\eta_2(x,y) = \begin{cases} x + y, & x \le y; \\ 2(x + y), & x > y. \end{cases}$$

Then f *is an* (η_1, η_2)*-convex function. But* f *is not preinvex with respect to* η_1 *and also it is not convex.* (consider $x = 0$, $y = 2$ and $\lambda > 0$).

2. Main results

In this section without using Hölder's inequality we obtain a trapezoid type inequality related to (1.1). The obtained results are different from (1.2), (1.3) and (1.4) in the face and proof. The following lemma is of importance:

Lemma 2.1. *Suppose that* $I \subset \mathbb{R}$ *is an invex set with respect to* η_1*. If for* $a, b \in I$ *with* $\eta_1(b,a) > 0$ *the function* $g : [a, a + \eta_1(b,a)] \to \mathbb{R}$ *is integrable and symmetric to* $a + \frac{1}{2}\eta_1(b,a)$*, then for any* $0 \le t \le \frac{1}{2}$ *we have*

$$\int_t^1 g\big(a + s\eta_1(b,a)\big)ds - \int_0^t g\big(a + s\eta_1(b,a)\big)ds = 2\int_t^{\frac{1}{2}} g\big(a + s\eta_1(b,a)\big)ds. \tag{2.1}$$

Proof. Using the change of variable $x = a + s\eta_1(b,a)$, for $0 \le t \le \frac{1}{2}$ we get

$$\int_t^1 g\big(a + s\eta_1(b,a)\big)ds - \int_0^t g\big(a + s\eta_1(b,a)\big)ds = \tag{2.2}$$
$$\frac{1}{\eta_1(b,a)}\left[\int_u^{a+\eta_1(b,a)} g(x)dx - \int_a^u g(x)dx\right],$$

where $a \le u \le a + \frac{1}{2}\eta_1(b,a)$.

Since g is symmetric to $a + \frac{1}{2}\eta_1(b,a)$ we have

$$\int_{a+\frac{1}{2}\eta_1(b,a)}^{a+\eta_1(b,a)} g(x)dx = \int_a^{a+\frac{1}{2}\eta_1(b,a)} g(x)dx.$$

Then

$$\int_u^{a+\eta_1(b,a)} g(x)dx = \int_u^{a+\frac{1}{2}\eta_1(b,a)} g(x)dx + \int_{a+\frac{1}{2}\eta_1(b,a)}^{a+\eta_1(b,a)} g(x)dx =$$

$$\int_u^{a+\frac{1}{2}\eta_1(b,a)} g(x)dx + \int_a^{a+\frac{1}{2}\eta_1(b,a)} g(x)dx.$$

Also

$$\int_a^{a+\frac{1}{2}\eta_1(b,a)} g(x)dx = \int_a^u g(x)dx + \int_u^{a+\frac{1}{2}\eta_1(b,a)} g(x)dx.$$

So

$$\frac{1}{\eta_1(b,a)}\left[\int_u^{a+\eta_1(b,a)} g(x)dx - \int_a^u g(x)dx\right] = \frac{2}{\eta_1(b,a)}\int_u^{a+\frac{1}{2}\eta_1(b,a)} g(x)dx =$$

$$2\int_t^{\frac{1}{2}} g\big(a+s\eta_1(b,a)\big)ds. \tag{2.3}$$

Using (2.3) in (2.2) we get (2.1). □

With the same argument used in the proof of Lemma 2.1 we can drive the following lemma.

Lemma 2.2. *Suppose that $I \subset \mathbb{R}$ is an invex set with respect to η_1. If for $a, b \in I$ with $\eta_1(b,a) > 0$ the function $g : [a, a+\eta_1(b,a)] \to \mathbb{R}$ is integrable and symmetric to $a+\frac{1}{2}\eta_1(b,a)$, then for any $\frac{1}{2} \le t \le 1$ we have*

$$\int_0^t g\big(a+s\eta_1(b,a)\big)ds - \int_t^1 g\big(a+s\eta_1(b,a)\big)ds = 2\int_{\frac{1}{2}}^t g\big(a+s\eta_1(b,a)\big)ds. \tag{2.4}$$

From Lemma 2.1 and Lemma 2.2, if g is symmetric nonnegative function, we can obtain two integral inequalities that are useful for our next results.

Corollary 2.1. *Suppose that $I \subset \mathbb{R}$ is an invex set with respect to η_1. If for $a, b \in I$ with $\eta_1(b,a) > 0$ the function $g : [a, a+\eta_1(b,a)] \to [0,+\infty)$ is integrable and symmetric to $a+\frac{1}{2}\eta_1(b,a)$, then*

$$\int_t^1 g\big(a+s\eta_1(b,a)\big)ds - \int_0^t g\big(a+s\eta_1(b,a)\big)ds \ge 0, \qquad 0 \le t \le \frac{1}{2} \tag{2.5}$$

and

$$\int_0^t g\big(a+s\eta_1(b,a)\big)ds - \int_t^1 g\big(a+s\eta_1(b,a)\big)ds \ge 0, \qquad \frac{1}{2} \le t \le 1. \tag{2.6}$$

Also the following lemma has been proved in [8] which is needed.

Lemma 2.3. *Suppose that $I^\circ \subset \mathbb{R}$ is an invex set with respect to η_1 and $f : I^\circ \to \mathbb{R}$ is a differentiable mapping on I°. For any $a, b \in I^\circ$ with $\eta_1(b,a) > 0$, if $g : [a, a+\eta_1(b,a)] \to [0,+\infty)$ is differentiable mapping on I° and $f' \in L^1[a, a+\eta_1(b,a)]$, then*

$$\frac{1}{\eta_1(b,a)}\left(\int_a^{a+\eta_1(b,a)} f(x)g(x)dx - \frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx\right) =$$

$$\frac{\eta_1(b,a)}{2}\int_0^1 P(t)f'\big(a+t\eta_1(b,a)\big)dt,$$

where

$$P(t) = \int_t^1 g\big(a + s\eta_1(b,a)\big)ds - \int_0^t g\big(a + s\eta_1(b,a)\big)ds.$$

Now we are ready to give our main result of the paper which is a trapezoid type inequality related to (1.1) with a new face.

Theorem 2.1. *Suppose that $I^\circ \subset \mathbb{R}$ is an invex set with respect to η_1 and $f : I^\circ \to \mathbb{R}$ is a differentiable mapping on I°. For any $a, b \in I^\circ$ with $\eta_1(b,a) > 0$, let $g : [a, a + \eta_1(b,a)] \to [0, +\infty)$ be a differentiable mapping on I° and $f' \in L^1[a, a + \eta_1(b,a)]$. If $|f'|$ is a (η_1, η_2)-convex mapping on $[a, a + \eta_1(b,a)]$, then*

$$\left|\frac{f(a) + f\big(a + \eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx - \int_a^{a+\eta_1(b,a)} f(x)g(x)dx\right| \leq \tag{2.7}$$
$$\Big[2|f'(a)| + \eta_2\big(|f'(b)|, |f'(a)|\big)\Big]\int_{a+\frac{1}{2}\eta_1(b,a)}^{a+\eta_1(b,a)} g(x)\big(a + \eta_1(b,a) - x\big)dx.$$

Proof. From Lemma 2.3, Corollary 2.1 and (η_1, η_2)-convexity of $|f'|$ we have

$$\left|\frac{f(a) + f\big(a + \eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx - \int_a^{a+\eta_1(b,a)} f(x)g(x)dx\right| =$$
$$\frac{\eta_1^2(b,a)}{2}\left|\int_0^1 \Big[\int_t^1 g\big(a + s\eta_1(b,a)\big)ds - \int_0^t g\big(a + s\eta_1(b,a)\big)ds\Big] f'\big(a + t\eta_1(b,a)\big)dt\right| \leq$$
$$\frac{\eta_1^2(b,a)}{2}\Big\{\int_0^{\frac{1}{2}}\Big|\int_t^1 g\big(a + s\eta_1(b,a)\big)ds - \int_0^t g\big(a + s\eta_1(b,a)\big)ds\Big|\Big|f'\big(a + t\eta_1(b,a)\big)\Big|dt +$$
$$\int_{\frac{1}{2}}^1\Big|\int_t^1 g\big(a + s\eta_1(b,a)\big)ds - \int_0^t g\big(a + s\eta_1(b,a)\big)ds\Big|\Big|f'\big(a + t\eta_1(b,a)\big)\Big|dt\Big\} \leq$$
$$\frac{\eta_1^2(b,a)}{2}\Big\{\int_0^{\frac{1}{2}}\Big(\int_t^1 g\big(a + s\eta_1(b,a)\big)ds - \int_0^t g\big(a + s\eta_1(b,a)\big)ds\Big)$$
$$\times\Big[|f'(a)| + t\eta_2\big(|f'(b)|, |f'(a)|\big)\Big]dt +$$
$$\int_{\frac{1}{2}}^1\Big(\int_0^t g\big(a + s\eta_1(b,a)\big)ds - \int_t^1 g\big(a + s\eta_1(b,a)\big)ds\Big)$$
$$\times\Big[|f'(a)| + t\eta_2\big(|f'(b)|, |f'(a)|\big)\Big]dt\Big\} = I.$$

If we change the order of integration in I, then

$$I = \frac{\eta_1^2(b,a)}{2}\Big\{\int_0^{\frac{1}{2}}\int_0^s g\big(a + s\eta_1(b,a)\big)\Big[|f'(a)| + t\eta_2\big(|f'(b)|, |f'(a)|\big)\Big]dtds +$$
$$\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} g\big(a + s\eta_1(b,a)\big)\Big[|f'(a)| + t\eta_2\big(|f'(b)|, |f'(a)|\big)\Big]dtds -$$
$$\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} g\big(a + s\eta_1(b,a)\big)\Big[|f'(a)| + t\eta_2\big(|f'(b)|, |f'(a)|\big)\Big]dtds +$$

$$\int_{\frac{1}{2}}^{1}\int_{s}^{1} g\big(a+s\eta_1(b,a)\big)\Big[|f'(a)|+t\eta_2\big(|f'(b)|,|f'(a)|\big)\Big]dtds+$$
$$\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} g\big(a+s\eta_1(b,a)\big)\Big[|f'(a)|+t\eta_2\big(|f'(b)|,|f'(a)|\big)\Big]dtds-$$
$$\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} g\big(a+s\eta_1(b,a)\big)\Big[|f'(a)|+t\eta_2\big(|f'(b)|,|f'(a)|\big)\Big]dtds\Big\}.$$

Calculating all inner integrals in I we get

$$I=\frac{\eta_1^2(b,a)}{2}\Big\{\int_0^{\frac{1}{2}} g\big(a+s\eta_1(b,a)\big)\Big(s|f'(a)|+\frac{1}{2}s^2\eta_2\big(|f'(b)|,|f'(a)|\big)\Big)ds+$$
$$\int_{\frac{1}{2}}^{1} g\big(a+s\eta_1(b,a)\big)\Big(\frac{1}{2}|f'(a)|+\frac{1}{8}\eta_2\big(|f'(b)|,|f'(a)|\big)\Big)ds-$$
$$\int_0^{\frac{1}{2}} g\big(a+s\eta_1(b,a)\big)\Big((\frac{1}{2}-s)|f'(a)|+(\frac{1}{8}-\frac{1}{2}s^2)\eta_2\big(|f'(b)|,|f'(a)|\big)\Big)ds+$$
$$\int_{\frac{1}{2}}^{1} g\big(a+s\eta_1(b,a)\big)\Big((1-s)|f'(a)|+(\frac{1}{2}-\frac{1}{2}s^2)\eta_2\big(|f'(b)|,|f'(a)|\big)\Big)ds+$$
$$\int_0^{\frac{1}{2}} g\big(a+s\eta_1(b,a)\big)\Big(\frac{1}{2}|f'(a)|+\frac{3}{8}\eta_2\big(|f'(b)|,|f'(a)|\big)\Big)ds-$$
$$\int_{\frac{1}{2}}^{1} g\big(a+s\eta_1(b,a)\big)\Big((s-\frac{1}{2})|f'(a)|+(\frac{1}{2}s^2-\frac{1}{8})\eta_2\big(|f'(b)|,|f'(a)|\big)\Big)ds\Big\}.$$

Simple form of I can be obtained as the following.

$$I=\frac{\eta_1^2(b,a)}{2}\Big\{\int_0^{\frac{1}{2}} g\big(a+s\eta_1(b,a)\big)\Big(2s|f'(a)|+(s^2+\frac{1}{4})\eta_2\big(|f'(b)|,|f'(a)|\big)\Big)ds+$$
$$\int_{\frac{1}{2}}^{1} g\big(a+s\eta_1(b,a)\big)\Big((-2s+2)|f'(a)|+(\frac{3}{4}-s^2)\eta_2\big(|f'(b)|,|f'(a)|\big)\Big)ds\Big\}.$$

If we apply the change of variable $x=a+s\eta_1(b,a)$ in I, we get

$$I=\frac{\eta_1(b,a)}{2}\Big\{\int_a^{a+\frac{1}{2}\eta_1(b,a)} g(x)\bigg(\Big[2\Big(\frac{x-a}{\eta_1(b,a)}\Big)\Big]|f'(a)|+$$
$$\Big[\Big(\frac{x-a}{\eta_1(b,a)}\Big)^2+\frac{1}{4}\Big]\eta_2\big(|f'(b)|,|f'(a)|\big)\bigg)dx+$$

$$\int_{a+\frac{1}{2}\eta_1(b,a)}^{a+\eta_1(b,a)} g(x)\bigg(\Big[-2\Big(\frac{x-a}{\eta_1(b,a)}\Big)+2\Big]|f'(a)|+$$
$$\Big[\frac{3}{4}-\Big(\frac{x-a}{\eta_1(b,a)}\Big)^2\Big]\eta_2\big(|f'(b)|,|f'(a)|\big)\bigg)dx\Big\}.$$

On the other hand since g is symmetric to $a+\frac{1}{2}\eta_1(b,a)$ then we have

$$\int_a^{a+\frac{1}{2}\eta_1(b,a)} g(x)\Big(\Big[2\Big(\frac{x-a}{\eta_1(b,a)}\Big)\Big]|f'(a)|+$$
$$\Big[\Big(\frac{x-a}{\eta_1(b,a)}\Big)^2+\frac{1}{4}\Big]\eta_2\big(|f'(b)|,|f'(a)|\big)\Big)dx=$$
$$\int_{a+\frac{1}{2}\eta_1(b,a)}^{a+\eta_1(b,a)} g(x)\Big(\Big[2\Big(\frac{a+\eta_1(b,a)-x}{\eta_1(b,a)}\Big)\Big]|f'(a)|+$$
$$\Big[\Big(\frac{a+\eta_1(b,a)-x}{\eta_1(b,a)}\Big)^2+\frac{1}{4}\Big]\eta_2\big(|f'(b)|,|f'(a)|\big)\Big)dx.$$

So

$$I=\frac{\eta_1(b,a)}{2}\Big\{\int_{a+\frac{1}{2}\eta_1(b,a)}^{a+\eta_1(b,a)} g(x)\Big(\Big[2\Big(\frac{x-a}{\eta_1(b,a)}\Big)+2\Big(\frac{a+\eta_1(b,a)-x}{\eta_1(b,a)}\Big)\Big]|f'(a)|$$
$$+\Big[\Big(\frac{x-a}{\eta_1(b,a)}\Big)^2+\frac{1}{4}+\Big(\frac{a+\eta_1(b,a)-x}{\eta_1(b,a)}\Big)^2+\frac{1}{4}\Big]\eta_2\big(|f'(b)|,|f'(a)|\big)\Big)dx\Big\}$$
$$=\frac{\eta_1(b,a)}{2}\Big\{\int_{a+\frac{1}{2}\eta_1(b,a)}^{a+\eta_1(b,a)} g(x)\Big(4\Big[\frac{a+\eta_1(b,a)-x}{\eta_1(b,a)}\Big]|f'(a)|$$
$$+2\Big[\frac{a+\eta_1(b,a)-x}{\eta_1(b,a)}\Big]\eta_2\big(|f'(b)|,|f'(a)|\big)\Big)dx\Big\}$$
$$=\Big[2|f'(a)|+\eta_2\big(|f'(b)|,|f'(a)|\big)\Big]\int_{a+\frac{1}{2}\eta_1(b,a)}^{a+\eta_1(b,a)} g(x)\big(a+\eta_1(b,a)-x\big)dx.$$

□

Corollary 2.2. *If in Theorem 2.1 we consider $\eta_2(x,y)=x-y$ for all $x,y\in f(I^\circ)$, then*

$$\left|\frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx-\int_a^{a+\eta_1(b,a)} f(x)g(x)dx\right|\le \tag{2.8}$$
$$\Big[|f'(a)|+|f'(b)|\Big]\int_{a+\frac{1}{2}\eta_1(b,a)}^{a+\eta_1(b,a)} g(x)\big(a+\eta_1(b,a)-x\big)dx.$$

Also if we put $\eta_1(x,y)=x-y$ for all $x,y\in f(I^\circ)$ in (2.8) we get

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx-\int_a^b f(x)g(x)dx\right|\le\Big[|f'(a)|+|f'(b)|\Big]\int_{\frac{a+b}{2}}^b g(x)(b-x)dx. \tag{2.9}$$

Furthermore if in (2.9) we set $g\equiv 1$, then we recapture Theorem 2.2 in [2].

$$\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)dx\right|\le\frac{(b-a)^2}{8}\big(|f'(a)|+|f'(b)|\big).$$

Remark 2.1. *Inequalities (2.8) and (2.9), obtained in Corollary 2.2 are new inequalities in the literature.*

3. Estimation type results

In this section we give two estimation results when the derivative of considered function is bounded and satisfies a Lipschitz condition respectively. If the derivative of the considered function is bounded from below and above, then we can drive an estimation type result related to Fejér inequality.

Theorem 3.1. *Suppose that $I^\circ \subset \mathbb{R}$ is an invex set with respect to η_1 and $f : I^\circ \to \mathbb{R}$ is a differentiable mapping on I°. For any $a, b \in I^\circ$ with $\eta_1(b,a) > 0$, let $g : [a, a+\eta_1(b,a)] \to [0, +\infty)$ be a differentiable mapping on I°. Assume that $f' \in L^1[a, a+\eta_1(b,a)]$ and there exist constants $m < M$ such that*

$$-\infty < m \le f'(x) \le M < +\infty \qquad \text{for all} \;\; x \in [a, a+\eta_1(b,a)].$$

Then

$$\begin{aligned}&\frac{1}{\eta_1(b,a)}\left|\frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx - \int_a^{a+\eta_1(b,a)} f(x)g(x)dx\right. \\ &\left. - \frac{\eta_1(b,a)(m+M)}{4}\int_0^1 P(t)dt\right| \le \frac{\eta_1(b,a)(M-m)}{4}\int_0^1 |P(t)|dt,\end{aligned} \tag{3.1}$$

where $P(t)$ is defined in Lemma 2.3.

Proof. From Lemma 2.3 we have

$$\begin{aligned}&\frac{1}{\eta_1(b,a)}\left(\int_a^{a+\eta_1(b,a)} f(x)g(x)dx - \frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx\right) = \\ &\frac{\eta_1(b,a)}{2}\int_0^1 P(t)\Big[f'\big(a+t\eta_1(b,a)\big) - \frac{m+M}{2} + \frac{m+M}{2}\Big]dt = \\ &\frac{(m+M)\eta_1(b,a)}{4}\int_0^1 P(t)dt + \frac{\eta_1(b,a)}{2}\int_0^1 P(t)\Big[f'\big(a+t\eta_1(b,a)\big) - \frac{m+M}{2}\Big]dt.\end{aligned}$$

So

$$\begin{aligned}I = &\frac{1}{\eta_1(b,a)}\left(\int_a^{a+\eta_1(b,a)} f(x)g(x)dx - \frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx\right) \\ &- \frac{(m+M)\eta_1(b,a)}{4}\int_0^1 P(t)dt = \frac{\eta_1(b,a)}{2}\int_0^1 P(t)\Big[f'\big(a+t\eta_1(b,a)\big) - \frac{m+M}{2}\Big]dt.\end{aligned}$$

Taking the modulus on I we obtain

$$|I| \le \frac{\eta_1(b,a)}{2}\int_0^1 \Big|P(t)\Big|\Big|f'\big(a+t\eta_1(b,a)\big) - \frac{m+M}{2}\Big|dt \le \frac{(M-m)\eta_1(b,a)}{4}\int_0^1 |P(t)|dt,$$

since from $m \le f'\big(a+t\eta_1(b,a)\big) \le M$ we have

$$m - \frac{m+M}{2} \le f'\big(a+t\eta_1(b,a)\big) - \frac{m+M}{2} \le M - \frac{m+M}{2},$$

which implies that

$$\left|f'\big(a+t\eta_1(b,a)\big) - \frac{m+M}{2}\right| \le \frac{M-m}{2}.$$

□

Corollary 3.1. *In Theorem 3.1 if we set* $\eta_1(x,y) = x - y$ *for all* $x, y \in I^\circ$ *and* $g \equiv 1$*, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{m(1+a-b)+M(1+b-a)}{8}.$$

Proof. If we consider $g \equiv 1$, then the relations $||g||_\infty = 1$ and $\int_0^1 |p(t)|dt \leq \frac{1}{2}$ imply that

$$\frac{1}{b-a}\left|\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)dx\right| \leq$$
$$\left|\frac{m+M}{4}\int_0^1 p(t)dt\right| + \frac{(M-m)(b-a)}{8} \leq$$
$$\frac{m+M}{8} + \frac{(M-m)(b-a)}{8} = \frac{m(1+a-b)+M(1+b-a)}{8}.$$

□

Estimation for difference between the right and middle terms of (1.1) when the derivative of considered function satisfies a Lipschitz condition is our next aim.

Definition 3.1. [9] A function $f : [a,b] \to \mathbb{R}$ is said to satisfy the *Lipschitz condition* on $[a,b]$ if there is a constant K so that for any two points $x, y \in [a,b]$, $|f(x) - f(y)| \leq K|x-y|$.

Theorem 3.2. *Suppose that* $I^\circ \subset \mathbb{R}$ *is an invex set with respect to* η_1 *and* $f : I^\circ \to \mathbb{R}$ *is a differentiable mapping on* I°*. For any* $a, b \in I^\circ$ *with* $\eta_1(b,a) > 0$*, let* $g : [a, a+\eta_1(b,a)] \to [0,+\infty)$ *be a differentiable mapping on* I°*. Assume that* f' *is integrable on* $[a, a+\eta_1(b,a)]$ *and satisfies a Lipschitz condition for some* $K > 0$*. Then*

$$\left|\frac{1}{\eta_1(b,a)}\left(\frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx - \int_a^{a+\eta_1(b,a)} f(x)g(x)dx\right) -\right.$$
$$\left.\frac{\eta_1(b,a)}{2} f'\Big(\frac{2a+\eta_1(b,a)}{2}\Big)\int_0^1 P(t)dt\right| \leq K\frac{\eta_1(b,a)}{2}\int_0^1 |t-\frac{1}{2}||P(t)|dt,$$

where $P(t)$ *is defined in Lemma 2.3.*

Proof. From Lemma 2.3 we have

$$\frac{1}{\eta_1(b,a)}\left(\int_a^{a+\eta_1(b,a)} f(x)g(x)dx - \frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx\right) =$$
$$\frac{\eta_1(b,a)}{2}\int_0^1 P(t)\Big[f'\big(a+t\eta_1(b,a)\big) - f'\left(\frac{2a+\eta_1(b,a)}{2}\right) + f'\left(\frac{2a+\eta_1(b,a)}{2}\right)\Big]dt =$$
$$\frac{\eta_1(b,a)}{2}\int_0^1 P(t)\Big[f'\big(a+t\eta_1(b,a)\big) - f'\left(\frac{2a+\eta_1(b,a)}{2}\right)\Big]dt +$$
$$\frac{\eta_1(b,a)}{2} f'\left(\frac{2a+\eta_1(b,a)}{2}\right)\int_0^1 P(t)dt.$$

Then

$$\frac{1}{\eta_1(b,a)}\left(\int_a^{a+\eta_1(b,a)} f(x)g(x)dx - \frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx\right) -$$
$$\frac{\eta_1(b,a)}{2} f'\left(\frac{2a+\eta_1(b,a)}{2}\right)\int_0^1 P(t)dt =$$
$$\frac{\eta_1(b,a)}{2}\int_0^1 P(t)\Big[f'\big(a+t\eta_1(b,a)\big) - f'\left(\frac{2a+\eta_1(b,a)}{2}\right)\Big]dt.$$

Since f' satisfies a Lipschitz condition for $K > 0$, we have

$$\left|f'\big(a+t\eta_1(b,a)\big) - f'\left(\frac{2a+\eta_1(b,a)}{2}\right)\right| \leq K\left|a+t\eta_1(b,a) - \frac{2a+\eta_1(b,a)}{2}\right| =$$
$$K\left|t-\frac{1}{2}\right|\eta_1(b,a).$$

So

$$\left|\frac{1}{\eta_1(b,a)}\left(\int_a^{a+\eta_1(b,a)} f(x)g(x)dx - \frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx\right) -\right.$$
$$\left.\frac{\eta_1(b,a)}{2} f'\left(\frac{2a+\eta_1(b,a)}{2}\right)\int_0^1 P(t)dt\right| \leq K\frac{\eta_1(b,a)}{2}\int_0^1 \left|t-\frac{1}{2}\right|\left|P(t)\right|dt.$$

□

Corollary 3.2. *If in Theorem 3.2 we consider $g \equiv 1$, then*

$$\left|\frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2} - \frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)dx\right| \leq \eta_1(b,a)\left(\frac{K}{12}+\frac{1}{4}\left|f'\left(\frac{2a+\eta_1(b,a)}{2}\right)\right|\right).$$

Furthermore if we consider $\eta_1(x,y) = x - y$ for all $x, y \in I^\circ$ we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\left(\frac{K}{12}+\frac{1}{4}\left|f'\left(\frac{a+b}{2}\right)\right|\right).$$

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