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ON THE LIMITED p-SCHUR PROPERTY OF SOME OPERATOR SPACES

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Abstract. We introduce and study the notion of limited p-Schur property ($1 \leq p \leq \infty$) of Banach spaces. Also, we establish some necessary and sufficient conditions under which some operator spaces have the limited p-Schur property. In particular, we prove that if X and Y are two Banach spaces such that X contains no copy of ℓ_1 and Y has the limited p-Schur property, then $K(X, Y)$ (the space of all compact operators from X into Y) has the limited p-Schur property.

1. Introduction

A non-empty subset K of a Banach space X is said to be limited (resp Dunford-Pettis (DP)), if for every $weak^*$-null (resp. weakly null) sequence (x_n^*) in the dual space X^* of X converges uniformly on K, that is,

$$\lim_{n\to\infty} \sup_{x\in K} |\langle x, x_n^*\rangle| = 0$$

where $\langle x, x^*\rangle$ denotes the duality between $x \in X$ and $x^* \in X^*$. In particular, a sequence $(x_n) \subset X$ is limited if and only if $\langle x_n, x_n^*\rangle \to 0$, for all $weak^*$-null sequences (x_n^*) in X^*.

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A subset K of a Banach space X is a limited set if and only if for any Banach space Y, every pointwise convergent sequence $(T_n) \subset L(X,Y)$ converges uniformly on K, where $L(X,Y)$ denoted the space of all bounded operators from X into Y [17, Corollary 1.1.2].

It is easily seen that every relatively compact subset of a Banach space is limited. But the converse is not true, in general. If every limited subset of Banach space X is relatively compact, then X has the Gelfand-Phillips (GP) property. For example, the classical Banach space c_0 and ℓ_1 have the GP property and every reflexive space and dual space containing no copy of ℓ_1 have the same property.

A sequence (x_n) in Banach space X is called weakly p-summable with $1 \leq p < \infty$, if for each $x^* \in X^*$, the sequence $(\langle x_n, x^* \rangle) \in \ell_p$ and a sequence (x_n) in X is said to be weakly p-convergent to $x \in X$ if the sequence $(x_n - x) \in \ell_p^{weak}(X)$, where $\ell_p^{weak}(X)$ denoted the space of all weakly p-summable sequence in X. Also a bounded set K in a Banach space is said to be relatively weakly p-compact, $1 \leq p \leq \infty$ if every sequence in K has a weakly p-convergent subsequence. If the limit point of each weakly p-convergent subsequence is in K, then we call K weakly p-compact set. Also, a Banach space X is weakly p-compact if the closed unit ball B_X of X is a weakly p-compact set. An operator $T \in L(X,Y)$ is said to be p-converging if it transfers weakly p-summable sequence into norm null sequences. The class of all p-converging operators from X into Y is denoted by $C_p(X,Y)$.

An operator $T \in L(X,Y)$ is limited p-converging if it transfers limited and weakly p-summable sequences into norm null sequences. we denote the space of all limited p-converging operators from X into Y by $C_{lp}(X,Y)$ [7].

A Banach space X has the Schur property if every weakly null sequence in X converges in norm. The simplest Banach space with the Schur property is ℓ_1. Also a banach space X has the p-Schur property ($1 \leq p \leq \infty$) if every weakly p-summable subset of X is compact. In other words, if $1 \leq p < \infty$, X has the p-Schur property if and only if every sequence $(x_n) \in \ell_p^{weak}(X)$ is a norm null sequence, for example, ℓ_p has the 1-Schur property. Moreover, X has the ∞-Schur property if and only if every sequence in $c_0^{weak}(X)$ in norm null where $c_0^{weak}(X)$ containing all weakly null sequences in X. So ∞-Schur property coincides with the Schur property. Also one note that every Schur space has the p-Schur property [6].

The reader is referred to [2, 11, 14–16] for more information about these concepts.

In this note, we study the limited p-Schur property of some operator spaces, specially, the space of compact operators. We prove that if X and Y are two Banach spaces such that X contains no copy of ℓ_1 and Y has the limited p-Schur property, then $K(X,Y)$ has the limited p-Schur property. Finally, we conclude that if $(X_\alpha)_{\alpha \in I}$ are Banach spaces and $X = (\oplus_{\alpha \in I} X_\alpha)_1$ their ℓ_1-sum, then the space X has the p-Schur property if and only if each factor X_α has the same property.

2. Main results

Recall that the Banach space X has the limited p-Schur property if every limited weakly p-compact subset of X is relatively compact. More precisely, the Banach space X has the limited p-Schur property if and only if every limited sequence $(x_n) \in \ell_p^{weak}(X)$ is norm null. It is easy to see that every Banach space with the p-Schur property and every Banach space with GP property is limited p-Schur [7]. Moreover, a Banach space X has the GP property if and only if every limited weakly null sequence in X is norm null [2, Proposition 6.8]. Therefore the limited Schur (i.e., limited ∞-Schur) property is equivalent to the GP property. Also, if a Banach space X have the limited p-Schur and DP_p^* properties, then it has the p-Schur property. Indeed, a Banach space X is said to have the DP^*-property of order p (DP_p^*) if all weakly p-compact sets in X are limited [10].

Recall that if M is a closed subspace of $L(X, Y)$, then for arbitrary elements $x \in X$ and $y^* \in Y^*$, the evaluation operators $\phi_x : M \to Y$ and $\psi_{y^*} : M \to X^*$ on M are defined by

$$\phi_x(T) = Tx, \quad \psi_{y^*}(T) = T^*y^*, \quad (T \in M).$$

Theorem 2.1. *Let X and Y be two Banach spaces such that X is weakly p-compact and Y has the p-Schur property. Then $L(X, Y)$ has the limited p-Schur property.*

Proof. Suppose that (T_n) is a limited weakly p-summable sequence in $L(X, Y)$. We have to prove that (T_n) is norm null. We first observe that for every $x \in X$ the evaluation operator ϕ_x from $L(X, Y)$ to Y maps the sequence (T_n) to the sequence $(T_n x)$. So the latter is also a limited weakly p-summable sequence in Y. Therefore $\|T_n x\| \to 0$, since Y has the limited p-Schur property.

Now, suppose that (T_n) is not norm null. Then there is a sequence (x_n) in X and $\varepsilon > 0$ such that

$$\|T_n x_n\| > 2\varepsilon,$$

for all $n \in \mathbb{N}$. Since X is weakly p-compact we may assume that there exists $x \in X$ such that $(x_n - x) \in \ell_p^{weak}(X)$. As $\|T_n x\| \to 0$, we may finally suppose that $f_n = \|T_n x_n - T_n x\| > \varepsilon$ for all $n \in \mathbb{N}$. Now, choose functional y_n^* in B_{Y^*} so that $\langle T_n x_n - T_n x, y_n^* \rangle = f_n$, and define $\Lambda_n \in L(X, Y)^*$ by

$$\langle T, \Lambda_n \rangle = \langle T x_n - T x, y_n^* \rangle,$$

for all $T \in L(X, Y)$. Since $|\langle T, \Lambda_n \rangle| \leq \|T x_n - T x\| \to 0$, because $(x_n - x) \in \ell_p^{weak}(X)$, we see that (Λ_n) is a weak*-null sequence. But $\langle T_n, \Lambda_n \rangle = f_n > \varepsilon > 0$ for all $n \in \mathbb{N}$. Contradicting the assumption that (T_n) is limited. □

Corollary 2.1. *Let X and Y be two Banach spaces. If X is reflexive and Y has the Schur property, then $L(X, Y)$ has the GP property.*

Proof. Let $p = \infty$ in Theorem 2.1. □

Corollary 2.2. *Let X and Y be two Banach spaces. If X is a weakly p-compact and Y^* has the p-Schur property, then $(X\widehat{\otimes}_\pi Y)^*$ has the limited p-Schur property.*

Proof. It follows easily from the fact that $L(X, Y^*) = (X\widehat{\otimes}_\pi Y)^*$. □

Corollary 2.3. *Let X and Y be two Banach spaces. If X^* has the p-Schur property and Y^* is weakly p-compact, then $L(X, Y)$ has the limited p-Schur property.*

Proof. The mapping $T \mapsto T^*$ maps $L(X, Y)$ onto a closed subspace of $L(Y^*, X^*)$, which has the limited p-Schur property by virtue of Theorem 2.1. □

In the following theorem we give a necessary and sufficient condition for which a Banach space has the limited p-Schur property.

Theorem 2.2. *The Banach space X has the limited p-Schur property if and only if $L(X, Y) = C_{lp}(X, Y)$, for every Banach space Y.*

Proof. Suppose that X has the limited p-Schur property. If $T \in L(X, Y)$ and $(x_n) \in \ell_p^{weak}(X)$ is a limited sequence, then $\|x_n\| \to 0$. Hence $\|Tx_n\| \to 0$.

Conversely, If $Y = X$, then the identity operator on X is belongs to C_{lp}. Therefore X has the limited p-Scuhr property. □

Similarly, we can prove that the Banach space X has the limited p-Schur property if and only if $L(Y, X) = C_{lp}(Y, X)$ for every Banach space Y.

Theorem 2.3. *If X^* has the limited p-Schur property and Y has the Schur property, then $L(X, Y)$ has the limited p-Schur property.*

Proof. Since X^* has the limited p-Schur property, Theorem 2.2 implies that each $\psi_{y^*} : L(X, Y) \to X^*$ is limited p-converging. It follows that $L(X, Y)$ has the limited p-Schur property. In fact, if $L(X, Y)$ does not have the limited p-Schur property, then there exists a limited weakly p-summable sequence $(T_n) \subseteq L(X, Y)$ such that $\|T_n\| > \varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. Choose a sequence $x_n \in B_X$ such that $\|T_n x_n\| > \varepsilon$. On the other hand, ψ_{y^*} is limited p-converging, for all $y^* \in Y^*$. Therefore $\|T_n^* y^*\| = \|\psi_{y^*} T_n\| \to 0$. It follows that

$$|\langle T_n x_n, y^*\rangle| \leq \|T_n^* y^*\|\|x_n\| \to 0.$$

Hence $(T_n x_n)$ is weakly null and so is norm null. This contradiction shows that $L(X, Y)$ has the limited p-Schur property. □

Example 2.1. *If X^* has the limited p-Schur property, then $\ell_1^{weak}(X^*)$ has the same property. Indeed, if one denote $\ell_1^{weak^*}(X^*)$ as the space of all sequences $(x_n^*) \subset X^*$ such that $(\langle x, x_n^* \rangle) \in \ell_1$, for all $x \in X$, then by [5, P. 427], $\ell_1^{weak}(X^*) = \ell_1^{weak^*}(X^*)$. Also, $\ell_1^{weak^*}(X^*)$ is isometrically isomorphism to $L(X, \ell_1)$; see e.g., [8, Proposition 19.4.3]. Since ℓ_1 has the Schur property, it follows that $L(X, \ell_1) = \ell_1^{weak}(X^*)$ has the limited p-Schur property.*

If we take $p = \infty$ in Theorem 2.3 we obtain the following result.

Corollary 2.4. *If X^* has the GP property and Y has the Schur property, then $L(X, Y)$ has the GP property.*

Theorem 2.4. *Let X and Y be Banach spaces. If X has the limited p-Schur property and Y has the GP property, then the space $K_{w^*}(X^*, Y)$ of all compact weak*-weak continuous operators from X^* into Y has the limited p-Schur property.*

Proof. Let (T_n) be a limited weakly p-summable sequence in $K_{w^*}(X^*, Y)$. We have to show that $\|T_n\| \to 0$. We can choose a sequence (x_n^*) in X^* such that $\|x_n^*\| = 1$ and $\|T_n x_n^*\| \geq \frac{1}{2}\|T_n\|$ for all $n \in \mathbb{N}$. Now, we prove that $(T_n x_n^*)$ is weakly null limited sequence in Y. Fix any $y^* \in Y^*$. Then for all $T \in K_{w^*}(X^*, Y)$, the operator $y^* \circ T$ is a weak* continuous linear functional on X^* so that $y^* \circ T \in X \subset X^{**}$. Thus the operator $T \mapsto y^* \circ T$ from $K_{w^*}(X^*, Y)$ into X shows that the sequence $(y^* \circ T_n)$ is limited weakly p-summable in X. So $\|y^* \circ T_n\| \to 0$ and for each $y^* \in Y^*$ we have

$$\langle y^*, T_n x_n^* \rangle = \langle y^* \circ T_n, x_n^* \rangle \to 0$$

and so $(T_n x_n^*)$ is weakly null.

Now, assume that (y_n^*) is a weak*-null sequence in Y^* and define a sequence (Λ_n) in $K_{w^*}(X^*, Y)^*$ by $\langle T, \Lambda_n \rangle = \langle T x_n^*, y_n^* \rangle$. If $T \in K_{w^*}(X^*, Y)$, then $T(B_{X^*})$ is relatively compact and so it is a limited set in Y. It follows that

$$\lim_{n\to\infty} \sup_{x^* \in B_{X^*}} \langle T x^*, y_n^* \rangle = 0.$$

Therefore $\|y_n^* \circ T\| \to 0$. Thus $\langle y_n^* \circ T, x_n^* \rangle \to 0$ and so (Λ_n) is weak*-null in $K_{w^*}(X^*, Y)^*$. Since (T_n) is limited, we have

$$\langle T_n x_n^*, y_n^* \rangle = \langle T_n, \Lambda_n \rangle \to 0$$

and so $(T_n x_n^*)$ is limited. Finally, the GP property of Y yields that $\|T_n x_n^*\| \to 0$ which implies $\|T_n\| \to 0$. □

Note that the map $T \mapsto T^{**}$ is an isometric isomorphism from $K(X, Y)$ into $K_{w^*}(X^*, Y)$. Therefore we have the following result.

Corollary 2.5. *Let X and Y be two Banach spaces. If X^* has the limited p-Schur property and Y has the GP property, then $K(X, Y)$ has the limited p-Schur property.*

Since $X\widehat{\otimes}_\varepsilon Y$ may be identified with a closed subspace of $K_{w^*}(X^*,Y)$ via the isometric embedding $X\widehat{\otimes}_\varepsilon Y \hookrightarrow K_{w^*}(X^*,Y)$ which is defined by $x\otimes y \mapsto \theta_{x\otimes y}$, where $\theta_{x\otimes y}(x^*) = \langle x, x^*\rangle y$, we have the following corollary.

Corollary 2.6. *If X has the limited p-Schur property and Y has the GP property, then injective tensor product $X\widehat{\otimes}_\varepsilon Y$ has the limited p-Schur property.*

Theorem 2.5. *[9, 13] Let X and Y be two Banach spaces and $M \subseteq K(X,Y)$ such that for all $x \in X$, $M(x) := \{Tx : T \in M\}$ is relatively compact in Y. Then under each of the following conditions, M is a relatively compact subset of $K(X,Y)$.*

- (a) *X^{**} has the GP property and for every weak*-null sequence $(x_n^{**}) \subseteq X^{**}$, $(T^{**}x_n^{**})$ is norm null uniformly with respect $T \in M$.*
- (b) *X contains no copy of ℓ_1 and for every weakly null sequence $(x_n) \subseteq X$, (Tx_n) is norm null uniformly with respect $T \in M$.*

Recall that the operator $T \in L(X,Y)$ is said to be limited operator if $T(B_X)$ is a limited set in Y. The class of all limited operator from X into Y is denoted by $L(X,Y)$. On the other hand, $T \in L(X,Y)$ if and only if $T^* : Y^* \to X^*$ is weak*-norm sequential continuous cf. [2].

Theorem 2.6. *Let X be a Banach space such that X^* has the GP property. If F is a closed subspace of $K(X,Y)$ and for every $x^{**} \in X^{**}$, the evaluation operator $\phi_{x^{**}}$ on F is limited p-converging, then F has the limited p-Schur property.*

Proof. First, observe that the evaluation operator $\phi_{x^{**}}$, as a generalization of ϕ_x is denoted by $\phi_{x^{**}}(T) = T^{**}x^{**}$, for all $T \in M$ and $x^{**} \in X^{**}$.

Let $M \subset F$ be a limited weakly p-compact set. Since for every $x \in X$, the evaluation map ϕ_x is limited p-converging, we conclude that $M(x) = \{Tx : T \in M\}$ is relatively compact. Since the adjoint of every limited operator is weak*-norm sequentially continuous, it follows that for every compact operator $T \in K(X,Y)$, the operator T^* is also compact and so is limited. This shows that T^{**} is weak*-norm sequentially continuous and therefore for each weak*-null sequence (x_n^{**}) in X^{**}, the sequence $(T^{**}x_n^{**})$ is norm null, that is $\phi_{x^{**}}$ is a pointwise norm null sequence of bounded linear operators. Hence $(\phi_{x_n^{**}})$ converges uniformly on the limited set M [17, Corollary 1.1.2]. It follows that

$$\lim_{n\to\infty} \sup_{T\in M} \|\phi_{x_n^{**}}(T)\| = 0.$$

Then by Theorem 2.5 (a) M is relatively compact and so F has the p-Schur property. □

If one use Theorem 2.5 (b) instead of Theorem 2.5 (a), we can prove the following theorem.

Theorem 2.7. *Let X be a Banach space containing no copy of ℓ_1. If F is a closed subspace of $K(X,Y)$ such that for each $x \in X$, the evaluation operator ϕ_x is limited p-converging, then F has the limited p-Schur property.*

Recall that a subset H of $L(X,Y)$ is uniformly completely continuous, if for every weakly null sequence (x_n) in X,

$$\lim_{n\to\infty} \sup_{T\in H} \|Tx_n\| = 0.$$

We remember the following theorem, which has a main role in the proof of the Theorem 2.9.

Theorem 2.8. *[13] If X contains no copy of ℓ_1, then a subset $H \subseteq K(X,Y)$ is relatively compact if and only if H is uniformly completely continuous and for each $x \in X$, the set $\phi_x(H)$ is relatively compact in Y.*

Theorem 2.9. *If X contains no copy of ℓ_1 and Y has the limited p-Schur property, then $K(X,Y)$ has the limited p-Schur property.*

Proof. If Y has the limited p-Schur property, then Theorem 2.2 shows that each $\phi_x : K(X,Y) \to Y$ is limited p-converging. Now, suppose that $H \subset K(X,Y)$ is a limited weakly p-compact set. Therefore $\phi_x(H)$ is relatively compact for all $x \in X$. On the other hand, if (x_n) is weakly null in X, then complete continuity of each operator $T \in H$ implies that $\|\phi_{x_n}(T)\| = \|Tx_n\| \to 0$. Therefore (ϕ_{x_n}) is a norm null sequence at each element $T \in H$ and then it is uniformly convergent on the limited set H [17, Corolarry 1.1.2]. Hence

$$\lim_{n\to\infty} \sup_{T\in H} \|Tx_n\| = \lim_{n\to\infty} \sup_{T\in H} \|\phi_{x_n}(T)\| = 0.$$

This shows that H is uniformly completely continuous. Hence Theorem 2.5 (a) shows that H is relatively compact in $K(X,Y)$ and so $K(X,Y)$ has the limited p-Schur property. □

Recall that if $1 \le p \le \infty$, the Banach space X has the Dunford-Pettis property of order p (DP_p) if for each Banach space Y, every weakly compact operator $T : X \to Y$ is p-converging. For more information about DP_p property of Banach spaces the reader is referred to [3].

Corollary 2.7. *If $2 < q < \infty$ and $\frac{1}{q} + \frac{1}{q^*} = 1$, then $(\ell_q \widehat{\otimes}_\varepsilon \ell_q)^*$ and $(\ell_q \widehat{\otimes}_\pi \ell_q)^*$ have the limited p-Schur property, for all $1 < p < q$.*

Proof. Since $1 < q^* < 2$ and $q^* < q < \infty$, by Pitt's Theorem; (see [1, Theorem 2.1.4]), every bounded operator $T : \ell_q \to \ell_{q^*}$ is compact. Therefore $(\ell_q \widehat{\otimes}_\pi \ell_q)^* = L(\ell_q, \ell_{q^*}) = K(\ell_q, \ell_{q^*})$ and $(\ell_q \widehat{\otimes}_\varepsilon \ell_q)^* = I(\ell_q, \ell_{q^*}) \subset K(\ell_q, \ell_{q^*})$, where $I(\ell_q, \ell_{q^*})$ is the space of all integral operators from ℓ_q into ℓ_{q^*} [5, P. 119]. Hence it is enough to show that $K(\ell_q, \ell_{q^*})$ has the limited p-Schur property, for all $1 < p < q$. In fact, by [3, Example 3.3] ℓ_{q^*} has the DP_p property, for all $1 < p < q$. It follows from [6, Theorem 2.31] that ℓ_{q^*} has the (limited)

p-Schur property, for all $1 < p < q$. On the other hand, ℓ_q contains no copy of ℓ_1. Therefore Theorem 2.9 (or Corollary 2.5) shows that $K(\ell_q, \ell_{q^*})$ has the limited p-Schur property, for all $1 < p < q$. □

We also notice that by Theorem 2.2, if the closed subspace M of $L(X, Y)$ has the limited p-Schur property, then all operators on M, such as evaluation operators, are limited p-converging. Therefore the converse of Theorem 2.6 is also true. Moreover, in the following two theorems 2.11 and 2.12, we will give another sufficient conditions for the limited p-Schur property of closed subspace M of some operator spaces with respect to the limited p-converging of evaluation operators.

To obtain our next result we need the following well known theorem.

Theorem 2.10. *[9] Let X and Y be two Banach spaces and H be a subset of $L(X, Y)$ such that*

(1) *$H(B_X) = \{Tx : T \in H, x \in B_X\}$ is relatively compact.*
(2) *$\psi_{y^*}(H)$ is relatively compact for all $y^* \in Y^*$.*

Then H is relatively compact.

Theorem 2.11. *Let M be a closed linear subspace of $L(X, Y)$ such that the closed linear span of the set $M(X) = \{Tx : T \in M, x \in X\}$ has the GP property. If all evaluation operator ψ_{y^*} are limited p-converging, then M has the limited p-Schur property.*

Proof. Suppose that H is a limited weakly p-compact subset of M. By Theorem 2.10, it is enough to show that $H(B_X)$ and all $\psi_{y^*}(H)$ are relatively compact in Y and X^*, respectively. For every $y^* \in Y^*$, the evaluation operator ψ_{y^*} is limited p-converging. Therefore $\psi_{y^*}(H)$ is relatively compact. On the other hand, if (y_n^*) is a weak*-null sequence in Y^*, then the weak*-norm sequential continuity of the adjoint of eah $T \in H$ implies that $\|\psi_{y_n^*}(T)\| = \|T^* y_n^*\| \to 0$ as $n \to \infty$. Therefore $(\psi_{y_n^*})$ converges pointwise on H an so it is converges uniformly on the subset H of M. Hence

$$\sup\{|\langle Tx, y_n^*\rangle| : T \in H, x \in B_X\} = \sup\{|\langle x, T^* y_n^*\rangle| : T \in H, x \in B_X\}$$
$$= \sup_{T \in H} \|T^* y_n^*\| \to 0.$$

Thus $H(B_X)$ is limited and so is relatively compact. □

Now, we give a sufficient condition for the limited p-Schur property of subspaces of $L_{w^*}(X^*, Y)$ of all bounded weak*-weak continuous operator from X^* to Y. Clearly, if $T \in L_{w^*}(X^*, Y)$, then T^* transfers Y^* into X. The proof of this theorem is similar to the proof of Theorem 3.6 of [6]. So we omit its proof.

Theorem 2.12. *Let X and Y be Banach spaces such that X has the Schur property. If M is a closed subspace of $L_{w^*}(X^*, Y)$ such that every evaluation operator ϕ_{x^*} is limited p-converging on M, then M has the limited p–Schur property.*

Recall that according to [6], a bounded subset K of a Banach space X is p-Limited if

$$\lim_n \sup_{x\in K} |\langle x, x_n^*\rangle| = 0,$$

for every $(x_n^*) \in \ell_p^{weak}(X^*)$.

A subset K of a dual space X^* of X is L_p-set if $\lim_n \sup_{x^*\in K} |\langle x_n, x^*\rangle| = 0$ for every sequence $(x_n) \in \ell_p^{weak}(X)$. Also, a sequence (x_n^*) in X^* is an L_p-set if and only if $\lim_{n\to\infty}\langle x_n, x_n^*\rangle = 0$ for all $(x_n) \in \ell_p^{weak}(X)$ [7]. It is clear that for every limited subset and every p-limited subset of a dual space is an L_p-set. Moreover, the following result has been proved in [7].

Theorem 2.13. *A Banach space X is weakly p-compact if and only if every L_p-set in X^* is relatively compact.*

Theorem 2.14. *Let X and Y be Banach spaces. If X contains no copy of ℓ_1, Y^* is weakly p-compact and for every $h \in L(X, Y^{**})$, for every weakly null sequence $(x_n) \subset X$, the sequence (hx_n) is an L_p-set, then $K(X, Y)$ has the GP property and so has the limited p-Schur property.*

Proof. Let $M \subset K(X, Y)$ be a limited set. We have to prove that M is relatively compact. Since $M(x) = \{Tx : T \in M\}$ is a limited set in Y and so is an L_p-set, therefore $M(x)$ is a relatively compact set, by Theorem 2.13. Assume that condition (b) of Theorem 2.5 in not verified. So there are a positive number ε, a weakly null sequence $(x_n) \subset X$ and a sequence $(T_n) \subset M$ such that for all $n \in \mathbb{N}$, $||T_n x_n|| > \varepsilon$. Now we prove that $(T_n x_n)$ is weakly null. For every $y^* \in Y^*$, the set $\{T_n^* y^* : n \in \mathbb{N}\}$ is a Dunford-Pettis subset of X^*. Since (x_n) is weakly null, it follows that

$$\langle T_n x_n, y^*\rangle = \langle T_n^* y^*, x_n\rangle \to 0$$

for every $y^* \in Y^*$. So the sequence $(T_n x_n)$ is weakly null.

Now, we prove that $(T_n x_n)$ is a p-limited set. Suppose that $(y_n^*) \in \ell_p^{weak}(Y^*)$ and $h \in (X\widehat{\otimes}_\pi Y^*)^* = L(X, Y^{**})$. As (hx_n) is an L_p-set in Y^{**} we have $h(x_n \otimes y_n^*) = \langle hx_n, y_n^*\rangle \to 0$ and so $(x_n \otimes y_n^*)$ is weakly null in $X \otimes_\pi Y^*$. Since $X\widehat{\otimes}_\pi Y^*$ embeds into $K(X, Y)^*$, it follows that $(x_n \otimes y_n^*)$ is also weakly null in space $K(X, Y)^*$. Then it must be that

$$\lim_{n\to\infty}\langle T_n x_n, y_n^*\rangle = \lim_{n\to\infty}\langle T_n, x_n \otimes_\pi y_n^*\rangle = 0,$$

because (T_n) is a limited set and so is a DP set. So we have actually proved that $(T_n x_n)$ is a p-limited set and so L_p-set. It follows from Theorem 2.13 that it must be a relatively compact set. Since it is a weakly null sequence, there is a norm null subsequence and it is a contradiction. □

In [18] the authors have been proved that for Banach spaces $(X_\alpha)_{\alpha\in I}$, if $X = (\oplus_{\alpha\in I} X_\alpha)_1$ is their ℓ_1-direct sum, then X has the Schur property if and only if each factor X_α has the same property. Here, by a similar idea, we prove that the same condition holds for (limited) p-Schur property.

Theorem 2.15. *Let $(X_\alpha)_{\alpha\in I}$ be Banach spaces and $X = (\oplus_{\alpha\in I} X_\alpha)_1$. Then the space X has the p-Schur property if and only if each X_α has the p-Schur property.*

Proof. If $X = (\oplus_{\alpha\in I} X_\alpha)_1$ has the p-Schur property, then clearly, every closed subspace of X has the p-Schur property. Hence each X_α has the p-Schur property. On the other hand, a straightforward computations shows that a Banach space has the p-Schur property if and only if all of its closed separable subspaces have the p-Schur property. Therefore we can assume that each X_α is separable and take $I = \mathbb{N}$. Hence $X = (\oplus X_k)_1$ is separable and so has the GP property.

If $(x_n) \in \ell_p^{weak}(X)$, where $x_n = (b_{n,k})_{k\in\mathbb{N}}$, then $(b_{n,k}) \in \ell_p^{weak}(X_k)$ for all $k \in \mathbb{N}$. Since X_k has the p-Schur property, therefore $||b_{n,k}|| \to 0$ as $n \to \infty$, for all $k \in \mathbb{N}$. We have to prove that $||x_n|| \to 0$ or the weakly null sequence (x_n) is relatively compact. Let $\{f_n\}_{n\in\mathbb{N}}$ be a w^*-null sequence in B_{X^*}. If we show that $\lim\limits_{n\to\infty} \langle x_n, f_n\rangle = 0$, then the proof is completed, thanks to the GP property of X.

Each f_n is of the form $f_n = (a_{n,k})_{k\in\mathbb{N}}$ and for all $k \in \mathbb{N}$, $a_{n,k} \xrightarrow{w^*} 0$ in X_k^* as $n \to \infty$. To prove that $\lim\limits_{n\to\infty} \langle x_n, f_n\rangle = 0$, it is enough to show that

$$\sup_n \sum_{k>M} |\langle a_{n,k}, b_{n,k}\rangle| \to 0 \; as \; M \to \infty.$$

Therefore we have to show that for each $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that

$$\sum_{k>M} |\langle a_{n,k}, b_{n,k}\rangle| < \varepsilon, \tag{2.1}$$

for all sufficiently large enough $n \in \mathbb{N}$. Let (2.1) is false. Then there is an $\varepsilon > 0$ such that

$$\sum_{k>M} |\langle a_{n,k}, b_{n,k}\rangle| \geq \varepsilon, \tag{2.2}$$

for all $M \in \mathbb{N}$ and some sufficiently large enough $n \in \mathbb{N}$. Consider a sequence of positive number, (δ_k) such that $\sum_{k=1}^{\infty} \delta_k < \frac{\varepsilon}{4}$. By the technique given in the proof of main theorem of [18] one can construct two strictly increasing sequences, $(n_k)_{k\geq 1}$ and $(M_k)_{k\geq 0}$ such that

(1) $\sum\limits_{j>M_k} ||b_{n_k,j}|| \leq \delta_k$ for each $k \geq 1$
(2) $\sum\limits_{j=1}^{M_{k-1}} |\langle a_{n,j}, b_{n_{k-1}}, j\rangle| \leq \delta_k$ for each $n \geq n_k$
(3) $\sum\limits_{j>M_{k-1}} |\langle a_{n_k,j}, b_{n_k}, j\rangle| \geq \varepsilon$.

Now, let us choose a sequence (λ_j) such that $|\lambda_j| = 1$, for all j and

$$\lambda_j \langle a_{n_k,j}, b_{n_k,j}\rangle = |\langle a_{n_k,j}, b_{n_k,j}\rangle|,$$

where $k \geq 1$ and $M_{k-1} + 1 \leq j \leq M_k$. Let

$$h = (h_j)_{j\geq 1} = (\lambda_1 a_{n_1,1}, \lambda_2 a_{n_1,2}, ..., \lambda_{M_1} a_{n_1,M_1}, \lambda_{M_1+1} a_{n_2,M_1+1}, ...).$$

Then $||h|| = \sup_{j\geq 1} ||h_j|| \leq 1$ and

$$\langle h, x_{n_k}\rangle = \sum_{j=1}^{\infty}\langle h_j, b_{n_k,j}\rangle = \sum_{i=1}^{k-1}\sum_{j=M_{i-1}+1}^{M_i} \lambda_j\langle a_{n_i,j}, b_{n_i,j}\rangle$$
$$+ \sum_{j=M_{k-1}+1}^{M_k} |\langle a_{n_k,j}, b_{n_k,j}\rangle| + \sum_{j=M_k}^{\infty} \lambda_j\langle a_{n_k,j}, b_{n_k,j}\rangle.$$

with due attention to $||a_{n_k,j}|| \leq 1$ and inequalities (1), (2) and (3):

$$\begin{aligned}
|\langle h, x_{n_k}\rangle| &\geq -\sum_{i=1}^{k-1}\delta_i + \sum_{j=M_{k-1}+1}^{M_k} |\langle a_{n_k,j}, b_{n_k,j}\rangle| - \delta_k \\
&\geq -\sum_{i=1}^{k-1}\delta_i + \sum_{j>M_{k-1}} |\langle a_{n_k,j}, b_{n_k,j}\rangle| - \sum_{j\leq M_{k-1}} |\langle a_{n_k,j}, b_{n_k,j}\rangle| - \delta_k \\
&\geq -\sum_{i=1}^{k-1}\delta_i + \sum_{j>M_{k-1}} |\langle a_{n_k,j}, b_{n_k,j}\rangle| - 2\delta_k \\
&\geq \varepsilon - \sum_{i=1}^{k-1}\delta_i - 2\delta_k \geq \varepsilon - 2\sum_{i=1}^{\infty}\delta_i > \frac{\varepsilon}{2}.
\end{aligned}$$

This contradiction shows that (2.1) is true. So

$$\lim_{M\to\infty}\sup_{n\in\mathbb{N}}\sum_{k=1}^{M} |\langle a_{n,k}, b_{n,k}\rangle| = 0$$

Therefore $\lim_{n\to\infty}\sum_{k=1}^{\infty} |\langle a_{n,k}, b_{n,k}\rangle| = \sum_{k=1}^{\infty}\lim_{n\to\infty} |\langle a_{n,k}, b_{n,k}\rangle| = 0$ Since $|\langle f_n, x_n\rangle| \leq \sum_{k=1}^{\infty} |\langle a_{n,k}, b_{n,k}\rangle|$ we conclude that $\lim_{n\to\infty} |\langle f_n, x_n\rangle| = 0$ and so $||x_n|| \to 0$. □

By a similar technique we have the following theorem.

Theorem 2.16. *Suppose that $(X_\alpha)_{\alpha\in I}$ are Banach spaces and $X = (\oplus_{\alpha\in I} X_\alpha)_1$. Then the space X has the limited p-Schur property if and only if each X_α has the limited p-Schur property.*

References

[1] F. Albiac and N.J. Kalton, *Topics in Banach Space Theory, Graduate Texts in Mathematics*, 233, Springer, New York, 2006.

[2] J. Bourgain and J. Diestel, *Limited operators and strict consingularity*, Math. Nachr. **119** (1984) 55-58.

[3] J. Castillo and F. Sanchez, *Dunford-Pettis-like properties of continuous vector function spaces*, Rev. Mat. Univ. Complut. Madrid **6** (1993), no. 1, 43-59.

[4] D. Chen, J. Alejandro Chvez-Domnguez, and Li. Lei. *Unconditionally p-converging operators and Dunford-Pettis Property of order p*, arXiv preprint arXiv:1607.02161 (2016).

[5] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Mathematics Studies, 176, North-Holland Publishing Co., Amsterdam, 1993.

[6] Mohammad B. Dehghani and S. Mohammad Moshtaghioun, *On the p-Schur property of Banach spaces*, Ann. Funct. Anal. (2017), 14 pages.

[7] M.B. Dehghani and S.M. Moshtaghioun, *Limited p-converging operators and its relation with some geometric properties of Banach spaces*, (2017), Submitted.

[8] J. Diestel, H. Jachowr and A. Tonge, *Absolutely summing operators*, Cambrigde University Press, 1995.

[9] G. Emmanuele, *On relative compactness in $K(X, Y)$*, J. Math. Anal. Appl. **379** (2013) 88-90.

[10] J. H. Fouire and E. D. Zeekoei, *DP^* properties of order p on Banach spaces*, Quaest. Math. **37** (2014), no. 3, 349-358.

[11] I. Ghencia and P. Lewis, *The Dunford-Pettis property, the Gelfand-Phillips property and L-set*, *Colloq. Math.* **1.6** (2006), 311-324.

[12] H. Jarchow, *Locally convex spaces*, B.G. Teubner, 1981.

[13] F. Mayoral, *Compact sets of compact operators in absence of ℓ_1*, Proc. Amer. Math. Soc. **129** (2001), 7982.

[14] S.M. Moshtaghioun and J. Zafarani, *Weak sequentional convergence in the dual of operator ideas*, J. Oper. Theory **49** (2003), 143-151.

[15] A. Pelczynski, *Banach Spaces in which every unconditionally converging operator is weakly compact*, Bull. L'Acad. Polon. Sci. **10** no. 2, (1962), 641-648.

[16] R. Ryan,*The Dunford-Pettis property and projective tensor products*, Bull. Polish Acad. Sci. **35** no. 11-12, (1987), 785-792.

[17] T. Schlumprecht, *Limited sets in Banach spaces*, Ph. D. Dissertation, München, (1987).

[18] B. Tanbay, *Direct sums and the Schur property*, Turk. J. Math. **22** (1999), 349-354.