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SOME RESULTS ON CONTROLLED $K-$FRAMES IN HILBERT SPACES

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Abstract. Controlled frames have been introduced to improve the numerical efficiency of iterative algorithms for inverting the frame operator. Also K-frames have been introduced to study atomic systems with respect to bounded linear operator. In this paper, the notion of controlled K-frames will be studied and it will be shown that controlled K-frames are equivalent to K-frames under some mild conditions. Finally, the stability of controlled K-Bessel sequences under perturbation will be discussed with some examples.

1. Introduction

Frames in Hilbert spaces were first proposed by Duffin and Schaeffer to deal with nonharmonic Fourier series in 1952 [9] and widely studied from 1986 by Daubechies et al. [10]. Now, frames play an important role not only in the theoretics also in many kinds of applications and have been widely applied in signal processing [13], sampling [11,12], coding and communications [19], filter bank theory [3], system modeling [8] and so on.

Over the years, various extentions of the frame theory have been investigated and proposed, such as the fusion frames [5, 6] to deal with hierarchical data processing, g-frames [20] by Sun to deal with all existing frames as united object, oblique dual frames [11] by Elder to deal with sampling reconstructions, and etc.

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The notion of K-frames were recently introduced by L. Găvruta to study the atomic systems with respect to a bounded linear operator K in Hilbert spaces. K-frames are more general than ordinary frames in sense that the lower frame bound only holds for the elements in the range of the K, where K is a bounded linear operator in a separable Hilbert Space H.

Recent addition to these generalized frames are the controlled frames [1]. Controlled frames have been introduced to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces, however, they were used earlier just as a tool for spherical wavelets [2]. The main advantage of these frames lies in the fact that they retain all the advantages of standard frames but additionally they give a generalized way to check the frame condition while offering a numerical advantage in the sense of preconditioning. Recent developments in this direction can be found in [14–18] and the references therein.

In this paper, the concept of controlled K-frame will be defined and it will be shown that any controlled K-frame is equivalent to a K-frame. Finally, we will discuss the stability of controlled K-Bessel sequences under perturbation.

Throughout this paper, H is a separable Hilbert space, $B(H)$ is the family of all bounded linear operators on H and $K \in B(H)$. $GL(H)$ denotes the set of all bounded linear operators which have bounded inverses and $GL^+(H)$ denotes the set of all positive operators in $GL(H)$.

The paper is organized as follows:

Section 2 contains some preliminary result. In section 3, we define the concept of controlled K-frame and we will show that controlled K-frames are equivalent to K-frames. In section 4, we discuss the stability of a more general perturbation for controlled K-Bessel sequence. In section 5, we will examine with some examples the perturbation of controlled K-Bessel sequences.

2. Preliminaries and notations

In this section, some necessary definitions and theorems are presented.

Definition 2.1. *A bounded operator $T \in B(H)$ is called positive (respectively, non-negative), if $\langle Tf, f\rangle > 0$ for all $f \neq 0$ (respectively, $\langle Tf, f\rangle \geq 0$ for all f).*

Every non-negative operator is clearly self-adjoint.

If $A \in B(H)$ is non-negative, then there exists a unique non-negative operator B such that $B^2 = A$. Furthermore, B commutes with every operator that commutes with A. This will be denoted by $B = A^{\frac{1}{2}}$.

Let $B^+(H)$ be the set of positive operators on H. For self-adjoint operators T_1 and T_2, the notation $T_1 \leq T_2$ or $T_2 - T_1 \geq 0$ means

$$\langle T_1 f, f\rangle \leq \langle T_2 f, f\rangle \quad , \forall f \in H.$$

The following result is needed in the sequel, but straightforward to prove:

Proposition 2.1. *[1] Let $T : H \to H$ be a linear operator. Then the following conditions are equivalent:*

a. *There exist $m > 0$ and $M < \infty$, such that $mI \leq T \leq MI$,*

b. *T is positive and there exist $m > 0$ and $M < \infty$, such that $m\|f\|^2 \leq \|T^{\frac{1}{2}} f\|^2 \leq M\|f\|^2$ for all $f \in H$,*

c. *T is positive and $T^{\frac{1}{2}} \in GL(H)$,*

d. *There exists a self-adjoint operator $A \in GL(H)$, such that*
$$A^2 = T,$$

e. *$T \in GL^+(H)$,*

f. *There exist constants $m > 0$ and $M < \infty$ and operator $C \in GL^+(H)$, such that $m'C \leq T \leq M'C$,*

g. *For every $C \in GL^+(H)$, there exist constants $m > 0$ and $M < \infty$, such that $m'C \leq T \leq M'C$.*

It is well-known that all bounded operators U on a Hilbert space H are not invertible: an operator U needs to be injective and surjective in order to be invertible. For doing this, one can use right-inverse operator. The following lemma shows that if an operator U has closed range, there exists a right-inverse operator $U^\dagger$ in the following sense:

Lemma 2.1. *[7] Let H_1 and H_2 be Hilbert spaces and suppose that $U : H_2 \to H_1$ is a bounded operator with closed range R_U. Then there exists a bounded operator $U^\dagger : H_1 \to H_2$ which*
$$UU^\dagger x = x, \quad \forall x \in R_U.$$

The operator $U^\dagger$ in the Lemma 2.3 is called the pseudo-inverse of U.

In the literature, one will often see the pseudo-inverse of an operator U with closed range defined as the unique operator $U^\dagger$ satisfying that
$$N_{U^\dagger} = R_U^\perp, \quad R_{U^\dagger} = N_U^\perp, \quad UU^\dagger x = x, \quad \forall x \in R_U.$$

Definition 2.2. *A sequence $\{f_i\}_{i\in I}$ in H is called a frame for H, if there exist constants $0 < A \leq B < \infty$ such that*
$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2 , \quad \forall f \in H.$$
If $A = B$, then $\{f_i\}_{i\in I}$ is called a tight frame and if $A = B = 1$, it is called a Parseval frame.

If only the right inequality of the above inequality holds,$\{f_n\}_{n\in I}$ is called a Bessel sequence.

Remark 2.1. *The frame operator $Sf = \sum_{i\in I}\langle f, f_i\rangle f_i$ associated with a frame $\{f_i\}_{i\in I}$ is a bounded, invertible and positive operator on H. This provides the reconstruction formulas*

$$f = S^{-1}Sf = \sum_{i\in I}\langle f, f_i\rangle S^{-1}f_i = \sum_{i\in I}\langle f, S^{-1}f_i\rangle f_i, \quad \forall f \in H.$$

Furthermore, $AI \leq S \leq BI$ and $B^{-1}I \leq S^{-1} \leq A^{-1}I$.

Definition 2.3. *Let $C \in GL(H)$. A frame controlled by the operator C or C-controlled frame is a family of vectors $\{f_i\}_{i\in I}$ in H, such that there exist constants $0 < m_C \leq M_C < \infty$, verifying*

$$m_C\|f\|^2 \leq \sum_{i\in I}\langle f, f_i\rangle\langle Cf_i, f\rangle \leq M_C\|f\|^2 \ , \quad \forall f \in H.$$

The controlled frame operator S is defined by

$$Sf = \sum_{i\in I}\langle f, f_i\rangle Cf_i, \quad \forall f \in H.$$

Definition 2.4. *Let $K \in B(H)$. A sequence $\{f_n\}_{n=1}^{\infty} \subset H$ is called a K-frame for H, if there exist constants $A, B > 0$ such that*

$$A\|K^*f\|^2 \leq \sum_{n=1}^{\infty}|\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad \forall f \in H. \tag{2.1}$$

We call A and B the lower and upper frame bounds for K-frame, respectively.

If only the right inequality of the above inequality holds, $\{f_n\}_{n=1}^{\infty} \subset H$ is called a K-Bessel sequence.

Remark 2.2. *If $K = I$, then K-frame are just the ordinary frame.*

Remark 2.3. *In the following, we will assume that $R(K)$ is closed, since this can assume that the pseudo-inverse $K^\dagger$ of K exists.*

Because of the higher generality of K-frames, some properties of ordinary frames can not hold for K-frames, such as the frame operator of a K-frame is not an isomorphism. For more differences between K-frames and ordinary frames, we refer to [21].

Definition 2.5. *Let $K \in B(H)$. A sequence $\{f_n\}_{n=1}^{\infty} \subset H$ is called an atomic system for K, if the following conditions are satisfied:*

1. *$\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence.*
2. *For any $x \in H$, there exists $a_x = \{a_n\} \in l^2$ such that*

$$Kx = \sum_{n=1}^{\infty} a_n f_n$$

where $\|a_x\|_{l^2} \leq C\|x\|$, C is positive constant.

Suppose that $\{f_n\}_{n=1}^{\infty}$ is a K-frame for H. Obviously it is a Bessel sequence, so we can define the following operator

$$T : l^2 \to H, \quad Ta = \sum_{n=1}^{\infty} a_n f_n, \quad a = \{a_n\} \in l^2,$$

it follows that

$$T^* : H \to l^2$$

$$T^* f = \{\langle f, f_n \rangle\}_{n=1}^{\infty}, \quad \forall f \in H.$$

Let $S = TT^*$, we obtain

$$Sf = \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n, \quad \forall f \in H.$$

we call T, T^* and S the synthesis operator, analysis operator and frame operator for K-frame $\{f_n\}_{n=1}^{\infty}$, respectively.

The following theorem gives a characterization of K-frames in Hilbert spaces.

Proposition 2.2. *Let $\{f_n\}_{n=1}^{\infty}$ be a Bessel sequence in H. Then $\{f_n\}_{n=1}^{\infty}$ is a K-frame for H, if and only if there exists $A > 0$ such that*

$$S \geq AKK^*,$$

where S is the frame operator for $\{f_n\}_{n=1}^{\infty}$.

Proof. The sequence $\{f_n\}_{n=1}^{\infty}$ is a K-frame for H with frame bounds A, B and frame operator S if and only if

$$A\|K^* f\|^2 \leq \sum_{K=1}^{\infty} |\langle f, f_n \rangle|^2 = \langle Sf, f \rangle \leq B\|f\|^2 \ , \ \forall f \in H, \tag{2.2}$$

that is

$$\langle AKK^* f, f \rangle \leq \langle Sf, f \rangle \leq \langle Bf, f \rangle \ , \ \forall f \in H.$$

so the conclusion holds. □

Remark 2.4. *Frame operator of a K-frames is not invertible on H in general, but we can show that it is invertible on the subspace $R(K) \subset H$. In fact, since $R(K)$ is closed, there exists a pseudo-inverse $K^{\dagger}$ of K, such that $KK^{\dagger} f = f$, $\forall f \in R(K)$, namely $KK^{\dagger}|_{R(K)} = I_{R(K)}$, so we have $I^*_{R(K)} = (K^{\dagger}|_{R(K)})^* K^*$. Hence for any $f \in R(K)$, we obtain*

$$\|f\| = \|(K^{\dagger}|_{R(K)})^* K^* f\| \leq \|K^{\dagger}\|.\|K^* f\|,$$

that is, $\|K^* f\|^2 \geq \|K^{\dagger}\|^{-2}\|f\|^2$. Combined with (2.2), we have

$$\langle Sf, f \rangle \geq A\|K^* f\|^2 \geq A\|K^{\dagger}\|^{-2}\|f\|^2 \ , \ \forall f \in R(K). \tag{2.3}$$

So, from the definition of K-frame we have

$$A\|K^{\dagger}\|^{-2}\|f\| \leq \|Sf\| \leq B\|f\| \ , \ \forall f \in R(K), \tag{2.4}$$

which implies that $S: \ R(K) \to S(R(K))$ is a homeomorphism. Furthermore, we have

$$B^{-1}\|f\| \leq \|S^{-1}f\| \leq A^{-1}\|K^{\dagger}\|^{2}\|f\| \ , \ \forall f \in S(R(K)).$$

3. Controlled K-frames

Controlled frames for spherical wavelets were introduced in [2] to get a numerically more efficient approximation algorithm and the related theory. For general frames, it was developed in [1]. For getting a numerical solution of a linear system of equations $Ax = b$, we can solve the system of equations $PAx = Pb$, where P is a suitable preconditioning matrix. It was the main motivation for introducing controlled frames in [2]. Controlled frames extended to g-frames in [17] and for fusion frames in [15]. In this section, the concept of controlled frames and controlled Bessel sequences will be extended to K-frames and it will be shown that controlled K-frames are equivalent K-frames.

Definition 3.1. *Let $C \in GL^{+}(H)$ and let $CK = KC$. The family $\{f_n\}_{n=1}^{\infty}$ is called C-controlled K-frame for H, if $\{f_n\}_{n=1}^{\infty}$ is a K-Bessel sequence and there exist constants $A > 0$ and $B < \infty$ such that*

$$A\|C^{\frac{1}{2}}K^{*}f\|^{2} \leq \sum_{n=1}^{\infty}\langle f, f_n\rangle\langle f, Cf_n\rangle \leq B\|f\|^{2} \ , \ \forall f \in H.$$

The constants A and B are called C-controlled K-frame bounds. If $C = I$, the C-controlled K-frame $\{f_n\}_{n=1}^{\infty}$ is a K-frame for H with bounds A and B.

If the second part of the above inequality holds, it called C-controlled K-Bessel sequence with bound B.

Definition 3.2. *Let $C \in GL^{+}(H)$. A sequence $\{f_n\}_{n=1}^{\infty} \in H$ is a C-controlled Bessel sequence for H if and only if the operator*

$$L_C : H \to H \ , \ L_C f = \sum_{n=1}^{\infty}\langle f, f_n\rangle Cf_n, \quad \forall f \in H.$$

is well defined and there exists constant $B < \infty$ such that

$$\sum_{n=1}^{\infty}\langle f, f_n\rangle\langle f, Cf_n\rangle \leq B\|f\|^{2} \ , \ \forall f \in H.$$

Definition 3.3. *The operator $L_C : H \to H$ and $L_C f = \sum_{n=1}^{\infty}\langle f, f_n\rangle Cf_n$ where $f \in H$ is called the C-controlled Bessel sequence operator, also $L_C f = CSf$.*

The following lemma characterizes C-controlled K-frames in term of their operators.

Lemma 3.1. *Let $\{f_n\}_{n=1}^{\infty}$ be a C-controlled K-frame in H, for $C \in GL^{+}(H)$. Then*

$$AI\|C^{\frac{1}{2}}K^{\dagger}\|^{2} \leq L_C \leq BI.$$

Proof. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a C-controlled K-frame with bounds A and B. Then

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \sum_{n=1}^{\infty}\langle f, f_n\rangle\langle f, Cf_n\rangle \leq B\|f\|^2 \ , \ \forall f \in H.$$

For $f \in H$

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \langle f, L_C f\rangle \leq B\|f\|^2$$

i.e.

$$A\|C^{\frac{1}{2}}K^*\|^2 I \leq L_C \leq BI.$$

□

The following proposition shows that for evaluation a family $\{f_n\}_{n=1}^{\infty} \subset H$ to be a controlled K-frame it is sufficient to check just a simple operator inequality.

Proposition 3.1. *Let $\{f_n\}_{n=1}^{\infty}$ be a Bessel sequence in H and $C \in GL^+(H)$. Then $\{f_n\}_{n=1}^{\infty}$ is a C-controlled K-frame for H if and only if there exists $A > 0$ such that $CS \geq CAKK^*$.*

Proof. The sequence $\{f_n\}_{n=1}^{\infty}$ is a controlled K-frame for H with frame bounds A, B and frame operator S if and only if

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \sum_{n=1}^{\infty}\langle f, f_n\rangle\langle f, Cf_n\rangle \leq B\|f\|^2 \ , \ \forall f \in H.$$

That is,

$$\langle CAKK^*f, f\rangle \leq \langle CSf, f\rangle \leq \langle Bf, f\rangle, \ \forall f \in H.$$

□

The following proposition shows that any controlled K-frame is a K-frame.

Proposition 3.2. *Let $\{f_n\}_{n=1}^{\infty}$ be a C-controlled K-frame and $C \in GL^+(H)$. Then $\{f_n\}_{n=1}^{\infty}$ is a K-frame for H.*

Proof. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a controlled K-frame with bounds A and B. Then for any $f \in H$,

$$\begin{aligned}
A\|K^*f\|^2 &= A\|C^{-\frac{1}{2}}C^{\frac{1}{2}}K^*f\|^2 \\
&\leq A\|C^{\frac{1}{2}}\|^2\|C^{-\frac{1}{2}}K^*f\|^2 \\
&\leq \|C^{\frac{1}{2}}\|^2\sum_{n=1}^{\infty}\langle f, f_n\rangle\langle f, C^0 f_n\rangle \\
&= \|C^{\frac{1}{2}}\|^2\sum_{n=1}^{\infty}|\langle f, f_n\rangle|^2.
\end{aligned}$$

Hence for $f \in H$,

$$A\|C^{\frac{1}{2}}\|^{-2}\|K^*f\|^2 \leq \sum_{n=1}^{\infty}|\langle f, f_n\rangle|^2$$

On the other hand for every $f \in H$,

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 &= \langle f, Sf\rangle \\
&= \langle f, C^{-1}CSf\rangle \\
&= \langle (C^{-1}CS)^{\frac{1}{2}} f, (C^{-1}CS)^{\frac{1}{2}} f\rangle \\
&= \|(C^{-1}CS)^{\frac{1}{2}} f\|^2 \\
&\leq \|C^{-\frac{1}{2}}\|^2 \|(CS)^{\frac{1}{2}} f\|^2 \\
&= \|C^{-\frac{1}{2}}\|^2 \langle f, CSf\rangle \\
&\leq \|C^{-\frac{1}{2}}\|^2 B\|f\|^2.
\end{aligned}$$

These inequalities yields that $\{f_n\}_{n=1}^{\infty}$ is a K-frame with bounds $A\|C^{\frac{1}{2}}\|^{-2}$ and $B\|C^{-\frac{1}{2}}\|^2$. □

The following proposition show that any K-frame is a controlled K-frame under some conditions.

Proposition 3.3. *Let $C \in GL^+(H)$ be a self adjoint and $KC = CK$, if $\{f_n\}_{n=1}^{\infty}$ is K-Frame for H, then $\{f_n\}_{n=1}^{\infty}$ is a C-controlled K-frame for H.*

Proof. Suppose that $\{f_n\}_{n=1}^{\infty}$ be a K-frame with bounds A' and B'. Then for all $f \in H$,

$$A'\|K^* f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B'\|f\|^2.$$

$$\begin{aligned}
A'\|C^{\frac{1}{2}} K^* f\|^2 = A'\|K^* C^{\frac{1}{2}} f\|^2 &\leq \sum_{n=1}^{\infty} \langle C^{\frac{1}{2}} f, f_n\rangle \langle C^{\frac{1}{2}} f, f_n\rangle \\
&= \langle C^{\frac{1}{2}} f, \sum_{n=1}^{\infty} \langle f_n, C^{\frac{1}{2}} f\rangle f_n\rangle \\
&= \langle C^{\frac{1}{2}} f, C^{\frac{1}{2}} Sf\rangle = \langle f, CSf\rangle.
\end{aligned}$$

Hence $A'\|C^{\frac{1}{2}} K^* f\|^2 \leq \langle f, CSf\rangle$ for every $f \in H$. On the other hand for every $f \in H$,

$$|\langle f, CSf\rangle|^2 = |\langle C^* f, Sf\rangle|^2 = |\langle Cf, Sf\rangle|^2 \leq \|Cf\|^2\|Sf\|^2 \leq \|C\|^2\|f\|^2 B\|f\|^2.$$

Hence,

$$A'\|C^{\frac{1}{2}} K^* f\|^2 \leq \langle f, CSf\rangle \leq B'\|C\|\|f\|^2.$$

Therefore $\{f_n\}_{n=1}^{\infty}$ is a C-controlled K-frame with bounds A' and $B'\|C\|$. □

4. Perturbation for Controlled K-Bessel Sequences

One of the most important problems in the studying of frames and its applications specially on wavelet and Gabor systems is the invariance of these systems under perturbation. At the first, the problem of perturbation studied by Paley and Wiener for bases and then extended to frames. There are many versions of perturbation of frames in Hilbert spaces, Banach space, Hilbert C^*-modules and etc. In the last decade, several authors have generalized the Paley-Wiener perturbation theorem to the perturbation of frames in Hilbert spaces. The most general result of these was the following obtained by Casazza and Christensen [4].

In this section, we mainly give an important on stability of perturbation for K-frames.To do this, we have to introduce tree lemmas below first.

Lemma 4.1. *[7] A sequence $\{f_n\}_{n=1}^{\infty} \subset H$ is a Bessel sequence with bound B in H, if and only if the operator*

$T: \ L^2 \to H, \ Ta = \sum a_n f_n$ is well- defined and bounded operator with $\|T\| \leq \sqrt{B}$.

Lemma 4.2. *[7] If $\{f_n\}_{n=1}^{\infty}$ is an ordinary frame for H, then $\{Kf_n\}_{n=1}^{\infty}$ is a $K-$Frames for H.*

Lemma 4.3. *[7] Let $T_1 \in L(X,Y)$ and let $T_2: \ X \to Y$ be linear. If there exist two constants $\lambda_1, \lambda_2 \in [0,1]$ such that*

$$\|T_1x - T_2x\| \leq \lambda_1\|T_1x\| + \lambda_2\|T_2x\| \ , \ \forall x \in X$$

then $T_2 \in L(X,Y)$. Moreover, if T_1 is invertible on X, then T_2 is also invertible on X, and we have

$$\frac{1-\lambda_1}{1+\lambda_2}\|T_1x\| \leq \|T_2x\| \leq \frac{1+\lambda_1}{1-\lambda_2}\|T_1x\| \ , \ \forall x \in X$$

and

$$\frac{1-\lambda_2}{1+\lambda_1}.\frac{1}{\|T_1\|}\|y\| \leq \|T_2^{-1}y\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|T_1^{-1}\|\|y\| \ , \ \forall y \in Y.$$

Theorem 4.1. *[4] Let $\{x_j\}_{j\in J}$ be a frame for a Hilbert space H with frame bounds C and D. Assume that $\{y_j\}_{j\in J}$ is a sequence of H and that there exist $\lambda_1, \lambda_2, \mu > 0$ such that $max\{\lambda_1 + \frac{\mu}{\sqrt{C}}, \lambda_2\} < 1$. Suppose one of the following conditions holds for any finite scalar sequence $\{c_j\}$ and every $x \in H$. Then $\{y_j\}_{j\in J}$ is also a frame for H;*

(1) $(\sum_{j\in J}|\langle x, x_j - y_j\rangle|^2)^{\frac{1}{2}} \leq \lambda_1(\sum_{j\in J}|\langle x, x_j\rangle|^2)^{\frac{1}{2}} + \lambda_2(\sum_{j\in J}|\langle x, y_j\rangle|^2)^{\frac{1}{2}} + \mu\|x\|$,

(2) $\|\sum_{i=1}^{n} c_j(x_j - y_j)\| \leq \lambda_1\|\sum_{i=1}^{n} c_jx_j\| + \lambda_2\|\sum_{i=1}^{n} c_jy_j\| + \mu(\sum_{i=1}^{n}|c_j|^2)^{\frac{1}{2}}$.

Moreover, if $\{x_j\}_{j\in J}$ is a Riesz basis for H and $\{y_j\}_{j\in J}$ satisfies (2), then $\{y_j\}_{j\in J}$ is also a Riesz basis for H.

The perturbation theorem investigated by X. Xiao, Y. Zhu, L. Găvruta to K-frames [21]:

Theorem 4.2. *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a K-frame for H, and $\alpha, \beta \in [0, \infty]$, such that max$\{\alpha+\gamma\sqrt{A^{-1}}\|K^+\|, \beta\} < 1$.*

If $\{g_n\}_{n=1}^{\infty} \subset H$ and satisfy,

$$\|\sum_{k=1}^{n} c_k(f_k - g_k)\| \leq \alpha\|\sum_{k=1}^{n} c_k f_k\| + \beta\|\sum_{k=1}^{n} c_k g_k\| + \gamma(\sum_{k=1}^{n} |c_k|^2)^{\frac{1}{2}},$$

for any c_i $(i \in \mathbb{N})$, then $\{g_n\}_{n=1}^{\infty}$ is a $P_{Q(R(K))}K$-Frame for H, with frame bounds

$$\frac{[\sqrt{A}\|K^+\|^{-1}(1-\alpha) - \gamma]^2}{(1+\beta)^2\|K\|^2}, \frac{[\sqrt{B}(1+\alpha)+\gamma]^2}{(1-\beta)^2},$$

where $P_{Q(R(K))}$ is a orthogonal projection operator for H to $Q(R(K))$, $Q = UT^*$, T, U are synthesis operator for $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ respectively.

Motivating the above theorems, we prove perturbation for controlled K-Bessel sequences.

Theorem 4.3. *Suppose that $\{f_n\}_{n=1}^{+\infty} \subset H$ is a C-controlled K-frame for H, with frame bounds A, B, and $\alpha, \beta, \gamma \in [0, \infty)$, such that*

$$Max\{\alpha + \gamma\sqrt{A^{-1}}\|K^+\|, \beta\} < 1.$$

If $\{g_n\}_{n=1}^{+\infty} \subset H$ and satisfy,

$$\begin{aligned} \|\sum_{k=1}^{n} c_k f_k - \sum_{k=1}^{n} c_k g_k\| &\leq \alpha\|c\|\|\sum_{k=1}^{n} c_k f_k\| + \beta\|\sum_{k=1}^{n} c_k g_k\| \\ &+\gamma(\sum_{k=1}^{n} |c_k|^2)^{\frac{1}{2}}, \end{aligned} \tag{4.1}$$

for any c_i, then $\{g_n\}_{n=1}^{+\infty}$ is a controlled K-Bessel sequence for H with bound $(\frac{(1+\alpha\|c\|)\sqrt{B}+\gamma}{1-\beta})^2$, where T, U are the synthesis operator for $\{f_n\}_{n=1}^{+\infty}$ and $\{g_n\}_{n=1}^{+\infty}$, respectively.

Proof. Let $\{f_n\}_{n=1}^{+\infty}$ be a frame for H, so by lemma 4.1, the frame operator T is bounded and $\|T\| \leq \sqrt{B}$.

The condition (4.1) implies that for all finite sequences $\{c_k\}$,

$$\|\sum_{k=1}^{n} c_k g_k\| = \| - \sum_{k=1}^{n} c_k(f_k - g_k) + \sum_{k=1}^{n} c_k f_k\| \leq \| - \sum_{k=1}^{n} c_k(f_k - g_k)\| + \|\sum_{k=1}^{n} c_k f_k\|$$

$$\leq (1 + \alpha\|c\|)\|\sum_{k=1}^{n} c_k f_k\| + \beta\|\sum_{k=1}^{n} c_k g_k\| + \gamma(\sum_{k=1}^{n} |c_k|^2)^{\frac{1}{2}}.$$

This calculation actually holds for all $\{c_k\}_{k=1}^{+\infty} \in l^2(\mathbb{N})$. To see this, at the first we have to prove that $\sum_{k=1}^{\infty} c_k g_k$ is convergent for any given $\{c_k\}_{k=1}^{+\infty} \in l^2(\mathbb{N})$.

Given $n, m \in \mathbb{N}$ with $n > m$,

$$\|\sum_{k=1}^{n} c_k f_k - \sum_{k=1}^{m} c_k f_k\| = \|\sum_{k=m+1}^{n} c_k f_k\|$$

$$\leq (1+\alpha\|c\|)\|\sum_{k=m+1}^{n} c_k f_k\| + \beta\|\sum_{k=m+1}^{n} c_k g_k\| + \gamma(\sum_{k=m+1}^{n} |c_k|^2)^{\frac{1}{2}}.$$

Since $\{c_k\}_{k=1}^{\infty} \in l^2(\mathbb{N})$ and $\sum_{k=1}^{\infty} c_k f_k$ is convergent, this implies that $\{\sum_{k=1}^{n} c_k f_k\}_{n=1}^{\infty}$ is a Cauchy sequence in H and therefore convergent, thus the pre-frame operator U is well defined on $l^2(\mathbb{N})$; it follows that for all $\{c_k\}_{k=1}^{\infty} \in l^2(\mathbb{N})$,

$$\|\sum_{k=1}^{\infty} c_k g_k\| \leq (1+\alpha\|c\|)\|\sum_{k=1}^{\infty} c_k f_k\| + \beta\|\sum_{k=1}^{\infty} c_k g_k\| + \gamma(\sum_{k=1}^{\infty} |c_k|^2)^{\frac{1}{2}}, \tag{4.2}$$

In terms of the operator T, U, (4.3) states that

$$\|U\{c_k\}_{k=1}^{\infty}\| \leq (1+\alpha\|c\|)\|T\{c_k\}_{k=1}^{\infty}\| + \beta\|U\{c_k\}_{k=1}^{\infty}\| + \gamma(\sum_{k=1}^{\infty} |c_k|^2)^{\frac{1}{2}}$$

$$\leq ((1+\alpha\|c\|)\sqrt{B}+\gamma)(\sum_{k=1}^{\infty} |c_k|^2)^{\frac{1}{2}} + \beta\|U\{c_k\}_{k=1}^{\infty}\| \ , \ \forall \{c_k\}_{k=1}^{\infty} \in l^2(\mathbb{N}).$$

So

$$\|U\{c_k\}_{k=1}^{\infty}\| \leq \frac{((1+\alpha\|c\|)\sqrt{B}+\gamma)}{1-\beta}(\sum_{k=1}^{\infty} |c_k|^2)^{\frac{1}{2}}. \tag{4.3}$$

Via lemma 4.1, this estimation shows that $\{g_k\}_{k=1}^{\infty}$ is a Bessel sequence with bound $\dfrac{((1+\alpha\|c\|)\sqrt{B}+\gamma)^2}{(1-\beta)^2}$. □

5. Examples

In this section, we give some examples that examines the stability of the controlled K-Bessel sequences under perturbation.

Example 5.1. *Suppose that $H = \mathbb{C}^3$, $\{g_n\}_{n=1}^{3} = \{e_1, e_2, e_3\}$, where*

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now define $K \in L(H)$ as follows

$$K : H \to H, \quad Ke_1 = e_1, \ Ke_2 = e_1, \ Ke_3 = e_2.$$

Obviously, $\{g_n\}_{n=1}^{3}$ is an ordinary frame for H. By Lemma 4.2, we know that $\{f_n\}_{n=1}^{3} = \{Kg_n\}_{n=1}^{3}$ is a K-frame for H. By Proposition 3.3, we can show that $\{f_n\}_{n=1}^{+\infty}$ is a controlled K-frame for H.

Example 5.2. *Let $H = \mathbb{C}^3, \{e_i\}_{i=1}^{3}$ be the orthonormal basis for H. Now define $K \in L(H)$ as follows*

$$K : H \to H, \quad Ke_1 = 2e_1, \ Ke_2 = 2e_1, \ Ke_3 = 6e_2.$$

Let $\{f_i\}_{i=1}^3 = \{2e_1, 2e_1, 6e_2\}$, *we know* $\{f_i\}_{i=1}^3$ *is a* $K-$*Frame for* H *by Lemma 4.2. We can take* K^+ *as follows*

$$K^+e_1 = \frac{e_1+e_2}{4},\ K^+e_2 = \frac{e_3}{6},\ K^+e_3 = 0. \tag{5.1}$$

It's easy the calculate the adjoint of K *as follows*

$$K^+e_1 = 2e_1 + 2e_2,\ K^+e_2 = 6e_3,\ K^+e_3 = 0.$$

Since $\{f_i\}_{i=1}^3$ *is a* $K-$*frame for* H, *by the definition of* $K-$*frame we can get* $0 < A \leq 1$, *so let me take* $A = 1$. *From* (5.1) *we can obtain* $\|K^+\| = \dfrac{1}{2\sqrt{2}}$. *Let* $\alpha = 0.95, \beta = 0.96, \gamma = 0.001$, *it's easy the check that* $\max\{\alpha + \gamma\sqrt{A^{-1}}\|K^+\|, \beta\} < 1$ *holds. Now, take* $g_1 = e_1, g_2 = e_1, g_3 = 5e_2$, *for any* $c_i (i = 1, 2, 3)$, *we have*

$$\begin{aligned}
&\left\|\sum_{k=1}^{3} c_k(f_k - g_k)\right\| = \sqrt{(c_1+c_2)^2 + c_3^2},\\
&\gamma\left(\sum_{i=1}^{n} c_k^2\right)^{\frac{1}{2}} = 0.001\sqrt{c_1^2 + c_2^2 + c_3^2} \geq 0,\\
&\alpha\left\|\sum_{k=1}^{n} c_k f_k\right\| = 0.95\sqrt{4(c_1+c_2)^2 + 36c_3^2} = \sqrt{3.16(c_1+c_2)^2 + 32.49c_3^2},\\
&\left\|\sum_{k=1}^{n} c_k g_k\right\| = \sqrt{3.6864(c_1+c_2)^2 + 23.04c_3^2} \geq 0.
\end{aligned}$$

It is trivial to check that (4.1) *holds. Moreover, we can show that* $\{g_i\}_{i=1}^3 = \{e_1, e_1, 5e_2\}$ *is a* $P_{Q(R(K))}K-$*Frame for* H. *In fact, by the definitions of* T, U *we get* $Qf = UT^*f = \sum_{k=1}^{3}\langle f, f_k\rangle g_k, f \in H$. *For any* $f \in H$, *suppose that* $f = c_1e_1 + c_2e_2 + c_3e_3$, *then*

$$Qf = \sum_{k=1}^{3}\langle c_1e_1 + c_2e_2 + c_3e_3, f_k\rangle g_k = 4c_1e_1 + 30c_2e_2,$$

which implies that $R(Q) = \overline{\mathrm{span}}\{e_1, e_2\}$. *By the definition of* K *we can calculate* $\|K\| = 6$, *now we leave the reader to verify that* (4.3) *holds.*

6. Conclusion

In this article, controlled K-frames is first defined. Then, we examined the conditions that controlled K-frames are equivalent to K-frames (under certain conditions). At the end, the stability of the controlled K-Bessel sequences were checked under perturbation.

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