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IMPLICIT SUMMATION FORMULA FOR 2-VARIABLE LAGUERRE-BASED POLY-GENOCCHI POLYNOMIALS

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Abstract. The main object of this paper is to introduce a new class of Laguerre-based poly-Genocchi polynomials and investigate some properties for these polynomials and related to the Stirling numbers of the second kind. We derive summation formulae and general symmetry identities by using different analytical means and applying generating functions.

1. Introduction

The generalized Bernoulli, Euler and Genocchi polynomials of (real or complex) order α are usually defined by means of the following generating functions (see [1-16]):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, (\mid t \mid < 2\pi; 1^{\alpha}=1), \tag{1.1}$$

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!}, (\mid t \mid < \pi; 1^{\alpha}=1) \tag{1.2}$$

and

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{t^n}{n!}, (\mid t \mid < \pi; 1^{\alpha}=1). \tag{1.3}$$

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So that obviously

$$B_n(x) = B_n^1(x), E_n(x) = E_n^1(x) \ and \ G_n(x) = G_n^1(x), (n \in \mathbb{N}),$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ $(\mathbb{N} = 1, 2, 3, \cdots)$.

The classical polylogarithmic function $\mathrm{Li}_k(z)$ is defined by (see [2], [10]):

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, (k \in \mathbb{Z}). \tag{1.4}$$

The poly-Bernoulli numbers and polynomials are defined by following generating functions (see [7], [8], [9]):

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \tag{1.5}$$

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}. \tag{1.6}$$

In the case $k = 1$ in (1.5) and (1.6), we have

$$B_n^{(1)} = B_n, B_n^{(1)}(x) = B_n(x).$$

The poly-Genocchi numbers and polynomials are defined by following generating functions (see [14]):

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1} = \sum_{n=0}^{\infty} G_n^{(k)} \frac{t^n}{n!}, \tag{1.7}$$

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!}. \tag{1.8}$$

In the case $k = 1$ in (1.7) and (1.8), we have

$$G_n^{(1)} = G_n, G_n^{(1)}(x) = G_n(x).$$

The 2-variable Laguerre polynomials (2-VLP) $Ł_n(x, y)$, which is defined by (see [5]):

$$\frac{1}{(1-yt)} \exp\left(\frac{-xt}{1-yt}\right) = \sum_{n=0}^{\infty} Ł_n(x,y) t^n, (\mid yt \mid < 1) \tag{1.9}$$

It is equivalently given by (see [6]).

$$\exp(yt) C_0(xt) = \sum_{n=0}^{\infty} Ł_n(x,y) \frac{t^n}{n!}, \tag{1.10}$$

where $C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined as:

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!}, (n \in \mathbb{N}_0) \tag{1.11}$$

with the following generating function:

$$\exp\left(t - \frac{x}{t}\right) = \sum_{n=0}^{\infty} C_n(x)t^n, \tag{1.12}$$

for $t \neq 0$ and for all finite x.

From (1.9) and (1.10), we get

$$L_n(x,y) = n!\sum_{s=0}^{n} \frac{(-1)^s x^s y^{n-s}}{(s!)^2(n-s)!} = y^n L_n(x/y). \tag{1.13}$$

Thus, we have

$$Ł_n(x,0) = \frac{(-1)^n x^n}{n!}, Ł_n(0,y) = y^n, Ł_n(x,1) = Ł_n(x), \tag{1.14}$$

where $L_n(x)$ are the classical Laguerre polynomials (see [1]).

Now, we recall here the following definition as follows:

The Stirling number of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, (n \geq 0) \tag{1.15}$$

and the Stirling number of the second kind is defined by generating function:

$$(e^t - 1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}. \tag{1.16}$$

2. 2-Variable Laguerre-based poly-Genocchi polynomials

Let $k \in \mathbb{Z}$, we inroduce 2-variable Laguerre-based poly-Genocchi polynomials by the following generating function:

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}\exp(yt)C_0(xt) = \sum_{n=0}^{\infty} {}_LG_n^{(k)}(x,y)\frac{t^n}{n!}, \tag{2.1}$$

so that

$${}_LG_n^{(k)}(x,y) = \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} G_{n-m}^{(k)} L_m(x,y). \tag{2.2}$$

When $x = y = 0$, ${}_LG_n^{(k)} = G_n^{(k)}(0,0)$ are called the poly-Genocchi numbers. For $k = 1$ in (2.1), we have

$$\frac{2\mathrm{Li}_1(1-e^{-t})}{e^t+1}\exp(yt)C_0(xt) = \sum_{n=0}^{\infty} {}_LG_n(x,y)\frac{t^n}{n!}, \tag{2.3}$$

where ${}_LG_n(x,y)$ is Laguerre-based Genocchi polynomials (see [13]).

Thus, we have

$${}_LG_n^{(k)}(x,y) = {}_LG_n(x,y), (n \geq 0).$$

On setting $x=0$, (2.1) reduces to the known result of Kim et al. [14.,p.Eq.(4)4776]:

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}\exp(yt)=\sum_{n=0}^{\infty}G_n^{(k)}(y)\frac{t^n}{n!},(k\in\mathbb{Z}). \tag{2.4}$$

Theorem 2.1. The following explicit summation formulae for Laguerre-based poly-Genocchi polynomials holds true:

$${}_LG_n^{(2)}(x,y)=\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)\frac{B_m m!}{m+1}{}_LG_{n-m}(x,y). \tag{2.5}$$

Proof. Using generating function for Laguerre-based poly-Genocchi polynomials (2.1), we have

$$\sum_{n=0}^{\infty}{}_LG_n^{(k)}(x,y)\frac{t^n}{n!}=\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}\exp(yt)C_0(xt)$$

$$=\frac{2}{e^t+1}\exp(yt)C_0(xt)\int_0^t\frac{1}{e^z-1}\int_0^t\frac{1}{e^z-1}\cdots\frac{1}{e^z-1}\int_0^t\frac{z}{e^z-1}dz\cdots dz.$$

In particular $k=2$, we have

$$\sum_{n=0}^{\infty}{}_LG_n^{(2)}(x,y)\frac{t^n}{n!}=\frac{2}{e^t+1}\exp(yt)C_0(xt)\int_0^t\frac{z}{e^z-1}dz=\left(\sum_{m=0}^{\infty}\frac{t^mB_m}{m+1}\right)\frac{2t}{e^t+1}\exp(yt)C_0(xt)$$

$$=\left(\sum_{m=0}^{\infty}\frac{t^mB_m m!}{m+1m!}\right)\left(\sum_{n=0}^{\infty}{}_LG_n(x,y)\frac{t^n}{n!}\right).$$

Replacing n by $n-m$ in the r.h.s of above equation, we have

$$\sum_{n=0}^{\infty}{}_LG_n^{(2)}(x,y)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)\frac{B_m m!}{m+1}{}_LG_{n-m}(x,y)\right)\frac{t^n}{n!}.$$

On equating the coefficients of the like powers of t in both sides, we get (2.5).

Remark 2.1. On setting $x=0$, Theorem (2.1) reduces to the known result of Kim et al. [14.,p. 4777, Theorem (2.1)].

Corollary 2.1. For $n\geq 0$, we have

$$G_n^{(2)}(y)=\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)\frac{B_m m!}{m+1}G_{n-m}(y). \tag{2.6}$$

Theorem 2.2. For $n\geq 1$, the degree of ${}_LG_n^{(k)}(x,y)$ is n-1. We have

$$\frac{{}_LG_n^{(k)}(x,y)}{n}=\sum_{m=0}^{n-1}\left(\begin{array}{c}n-1\\m\end{array}\right)\frac{G_{m+1}^{(k)}}{m+1}L_{n-m-1}(x,y). \tag{2.7}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty}{}_LG_n^{(k)}(x,y)\frac{t^n}{n!}=\frac{2\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\exp(yt)C_0(xt)$$

$$= \left(\sum_{m=0}^{\infty} G_m^{(k)} \frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} L_n(x,y)\frac{t^n}{n!}\right)$$

Replacing n by $n-m$ in above equation and comparing the coefficients of t^n, we get

$$_LG_n^{(k)}(x,y) = \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} G_m^{(k)} L_{n-m}(x,y), (n \geq 0). \tag{2.8}$$

From (2.8), we have

$$\frac{_LG_n^{(k)}(x,y)}{n} = \sum_{m=0}^{n-1} \begin{pmatrix} n-1 \\ m \end{pmatrix} \frac{G_{m+1}^{(k)}}{m+1} L_{n-m-1}(x,y), (n \geq 1) \tag{2.9}$$

Therefore by (2.9), we obtain the result (2.7).

Remark 2.2. For $x = 0$, Theorem (2.2) reduces to the known result of Kim et al. [14.,p. 4778, Theorem (2.2)].

Corollary 2.2. For $n \geq 1$, the degree of $G_n^{(k)}(x)$ is n-1. We have

$$\frac{G_n^{(k)}(y)}{n} = \sum_{m=0}^{n-1} \begin{pmatrix} n-1 \\ m \end{pmatrix} \frac{G_{m+1}^{(k)}}{m+1} y^{n-m-1}. \tag{2.10}$$

Theorem 2.3. For $n \geq 0$, we have

$$_LG_n^{(k)}(x,y) = \sum_{p=0}^{n}\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1,l)}{l^k(p+1)} \begin{pmatrix} n \\ p \end{pmatrix} {}_LG_{n-p}(x,y). \tag{2.11}$$

Proof. By using (2.1), we can be written as

$$\sum_{n=0}^{\infty} {}_LG_n^{(k)}(x,y)\frac{t^n}{n!} = \left(\frac{\mathrm{Li}_k(1-e^{-t})}{t}\right)\left(\frac{2t}{e^t+1}\exp(yt)C_0(xt)\right). \tag{2.12}$$

Now

$$\frac{1}{t}\mathrm{Li}_k(1-e^{-t}) = \frac{1}{t}\sum_{l=1}^{\infty}\frac{(1-e^{-t})^l}{l^k} = \frac{1}{t}\sum_{l=1}^{\infty}\frac{(-1)^l}{l^k}(1-e^{-t})^l$$

$$= \frac{1}{t}\sum_{l=1}^{\infty}\frac{(-1)^l}{l^k} l! \sum_{p=l}^{\infty}(-1)^p S_2(p,l)\frac{t^p}{p!}$$

$$= \frac{1}{t}\sum_{p=1}^{\infty}\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k} l! S_2(p,l)\frac{t^p}{p!}$$

$$= \sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k} l! \frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}. \tag{2.13}$$

From equations (2.12) and (2.13), we get

$$\sum_{n=0}^{\infty} {}_LG_n^{(k)}(x,y)\frac{t^n}{n!} = \sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k} l! \frac{S_2(p+1,l)}{p+1}\frac{t^p}{p!}\right)\left(\sum_{n=0}^{\infty} {}_LG_n(x,y)\frac{t^n}{n!}\right).$$

Replacing n by $n-p$ in the r.h.s. of above equation and comparing the coefficients of t^n in both sides, we arrive at the desired result (2.11).

Remark 2.3. For $x=0$, Theorem (2.3) reduces to the known result of Kim et al. [14.,p. 4779, Theorem (2.3)].

Corollary 2.3. For $n \geq 0$, we have

$$G_n^{(k)}(y)=\sum_{p=0}^{n}\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}l!S_2(p+1,l)}{l^k(p+1)}\left(\begin{array}{c} n \\ p \end{array}\right)G_{n-p}(y). \tag{2.14}$$

Theorem 2.4. For $n \geq 1$, we have

$${}_LG_n^{(k)}(x,y+1)+{}_LG_n^{(k)}(x,y)=2\sum_{p=1}^{n}\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\left(\begin{array}{c} n \\ p \end{array}\right)L_{n-p}(x,y). \tag{2.15}$$

Proof. By using definition (2.1), we have

$$\sum_{n=0}^{\infty}{}_LG_n^{(k)}(x,y+1)\frac{t^n}{n!}+\sum_{n=0}^{\infty}{}_LG_n^{(k)}(x,y)\frac{t^n}{n!}$$

$$=\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}\exp((y+1)t)C_0(xt)+\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}\exp(yt)C_0(xt)$$

$$=2\mathrm{Li}_k(1-e^{-t})\exp(y)tC_0(xt)$$

$$=\sum_{p=1}^{\infty}\left(2\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\right)\frac{t^p}{p!}\exp(yt)C_0(xt)$$

$$=\left(\sum_{p=1}^{\infty}\left(2\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\right)\frac{t^p}{p!}\right)\left(\sum_{n=0}^{\infty}L_n(x,y)\frac{t^n}{n!}\right).$$

Replacing n by $n-p$ in the above equation and comparing the coefficients of t^n in both sides, we obtain the result (2.15).

Remark 2.4. Taking $x=0$, Theorem 2.4 reduces to the known result of Kim et al. [14.,p. 4780, Theorem (2.4)].

Corollary 2.4. For $n \geq 1$, we have

$$G_n^{(k)}(y+1)+G_n^{(k)}(y)=2\sum_{p=1}^{n}\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\left(\begin{array}{c} n \\ p \end{array}\right)y^{n-p}. \tag{2.16}$$

Theorem 2.5. For $d \in \mathbb{N}$ with $d \equiv 1(mod2)$, we have

$${}_LG_n^{(k)}(x,y)=\sum_{p=0}^{n}\left(\begin{array}{c} n \\ p \end{array}\right)d^{n-p-1}\sum_{l=0}^{p+1}\sum_{a=0}^{d-1}\frac{(-1)^{l+p+1}l!S_2(p+1,l)}{l^k}(-1)^a{}_LG_{n-p}(\frac{a+y}{d},x). \tag{2.17}$$

Proof. From equation (2.1), we can be written as

$$\sum_{n=0}^{\infty} {}_LG_n^{(k)}(x,y)\frac{t^n}{n!} = \frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}\exp(yt)C_0(xt)$$

$$= \left(\frac{2\mathrm{Li}_k(1-e^{-t})}{t}\right)\left(\frac{2t}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\exp((a+y)t)C_0(xt)\right)$$

$$= \left(\sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}\right)\left(\sum_{n=0}^{\infty}d^{n-1}\sum_{a=0}^{d-1}(-1)^a{}_LG_n(\frac{a+y}{d},x)\frac{t^n}{n!}\right).$$

Replacing n by $n-p$ in above equation and comparing the coefficient of t^n in both sides, we get (2.17).

Remark 2.5. For $x=0$, Theorem 2.5 reduces to the known result of Kim et al. [14.,p. 4780].

Corollary 5. For $d \in \mathbb{N}$ with $d \equiv 1(mod 2)$, we have

$$G_n^{(k)}(y) = \sum_{p=0}^{n}\begin{pmatrix} n \\ p \end{pmatrix} d^{n-p-1}\sum_{l=0}^{p+1}\sum_{a=0}^{d-1}\frac{(-1)^{l+p+1}l!S_2(p+1,l)}{l^k}(-1)^a G_{n-p}(\frac{a+y}{d}). \tag{2.18}$$

3. Summation formulae for Laguerre-based poly-Genocchi polynomials

In this section, we establish summation formula for Laguerre-based poly-Genocchi polynomials by using series techniques method.

Theorem 3.1. The following implicit summation formulae for Laguerre-based poly-Genocchi polynomials ${}_LG_n^{(k)}(x,y)$ holds true:

$${}_LG_{l+p}^{(k)}(x,z) \;=\; \sum_{m,n=0}^{l,p}\begin{pmatrix} l \\ m \end{pmatrix}\begin{pmatrix} p \\ n \end{pmatrix}(z-y)^{m+n}{}_LG_{l+p-m-n}^{(k)}(x,y). \tag{3.1}$$

Proof. Replacing t by $t+u$ and rewrite the generating function (2.1) as

$$\frac{2\mathrm{Li}_k(1-(e)^{-(t+u)})}{e^{t+u}+1}C_0(x(t+u)) = e^{-y(t+u)}\sum_{l,p=0}^{\infty}{}_LG_{l+p}^{(k)}(x,y)\frac{t^l}{l!}\frac{u^p}{p!}. \tag{3.2}$$

Replacing y by z in the above equation and equating the resulting equation to the above equation, we get

$$e^{(z-y)(t+u)}\sum_{m,l=0}^{\infty}{}_LG_{l+p}^{(k)}(x,y)\frac{t^l}{l!}\frac{u^p}{p!} = \sum_{l,p=0}^{\infty}{}_LG_{l+p}^{(k)}(x,z)\frac{t^l}{l!}\frac{u^p}{p!}. \tag{3.3}$$

On expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty}\frac{[(z-y)(t+u)]^N}{N!}\sum_{l,p=0}^{\infty}{}_LG_{l+p}^{(k)}(x,y)\frac{t^l}{l!}\frac{u^p}{p!} = \sum_{l,p=0}^{\infty}{}_LG_{l+p}^{(k)}(x,z)\frac{t^l}{l!}\frac{u^p}{p!} \tag{3.4}$$

which on using formula [16, p.52(2)]

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty}f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{3.5}$$

in the left hand side becomes

$$\sum_{m,n=0}^{\infty} \frac{(z-x)^{m+n} t^m u^n}{m!n!} \sum_{l,p=0}^{\infty} {}_HG_{l+p}^{(k)}(x,y) \frac{t^l}{l!} \frac{u^p}{p!} = \sum_{l,p=0}^{\infty} {}_HG_{l+p}^{(k)}(z,y) \frac{t^l}{l!} \frac{u^p}{p!} \tag{3.6}$$

Now replacing l by $l-m$, p by $p-n$ and using the lemma [16, p.100(1)] in the left hand side of (3.6), we get

$$\sum_{m,n=0}^{\infty} \sum_{l,p=0}^{\infty} \frac{(z-x)^{m+n}}{m!n!} {}_LG_{l+p-m-n}^{(k)}(x,y) \frac{t^l}{(l-m)!} \frac{u^p}{(p-n)!}$$

$$= \sum_{l,p=0}^{\infty} {}_LG_{l+p}^{(k)}(x,z) \frac{t^l}{l!} \frac{u^p}{p!}. \tag{3.7}$$

Finally on equating the coefficients of the like powers of t and u in the above equation, we get the required result.

Remark 3.1. Taking $l = 0$ in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.

Corollary 3.1. The following summation formula for Laguerre-based poly-Genocchi polynomials ${}_HG_n^{(k)}(z,y)$ holds true:

$${}_LG_p^{(k)}(x,z) = \sum_{n=0}^{p} \binom{p}{n} (z-y)^n {}_LG_{p-n}^{(k)}(x,y). \tag{3.8}$$

Remark 3.2. Replacing z by $z+y$ in (3.8), we obtain

$${}_LG_p^{(k)}(x,z+y) = \sum_{n=0}^{p} \binom{p}{n} z^n {}_LG_{p-n}^{(k)}(x,y). \tag{3.9}$$

Theorem 3.2. The following summation formula for Laguerre-based poly-Genocchi polynomials ${}_HG_n^{(k)}(z,y)$ holds true:

$${}_LG_n^{(k)}(x,y+u) = \sum_{j=0}^{n} \binom{n}{j} u^j {}_LG_{n-j}^{(k)}(x,y). \tag{3.10}$$

Proof. Using (2.1), we can be written as

$$\sum_{n=0}^{\infty} {}_LG_n^{(k)}(x,y+u) \frac{t^n}{n!} = \frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1} \exp((y+u)t) C_0(xt) = \left(\sum_{n=0}^{\infty} {}_LG_n^{(k)}(x,y) \frac{t^n}{n!} \right) \left(\sum_{j=0}^{\infty} u^j \frac{t^j}{j!} \right)$$

Now replacing n by $n-j$ and comparing the coefficients of t^n in both sides, we obtain (3.10).

Theorem 3.3. The following summation formula for Laguerre-based poly-Genocchi polynomials ${}_HG_n^{(k)}(z,y)$ holds true:

$$ {}_LG_n^{(k)}(x+w,y+u)=\sum_{m=0}^{n}\left(\begin{array}{c}n\\ \\ m\end{array}\right){}_LG_{n-m}^{(k)}(x,y)L_m(u,w). \tag{3.11}$$

Proof. From (2.1) and (1.10), we have

$$\frac{2\mathrm{Li}_k(1-(e)^{-t})}{e^t+1}\exp((y+u)t)C_0((x+w)t)=\left(\sum_{n=0}^{\infty}{}_LG_n^{(k)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}L_m(u,w)\frac{t^m}{m!}\right).$$

Now replacing n by $n-m$ and comparing the coefficients of t^n in both sides, we get (3.11).

Theorem 3.4. The following summation formula for Laguerre-based poly-Genocchi polynomials ${}_LG_n^{(k)}(x,y)$ holds true:

$$ {}_LG_n^{(k)}(x,y+1)=\sum_{m=0}^{n}\left(\begin{array}{c}n\\ \\ m\end{array}\right){}_LG_{n-m}^{(k)}(x,y). \tag{3.12}$$

Proof. Using definition (2.1), we have

$$\sum_{n=0}^{\infty}{}_LG_n^{(k)}(x,y+1)\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_LG_n^{(k)}(x,y)\frac{t^n}{n!}=\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}\exp(yt)C_0(xt)(e^t-1)$$

$$=\left(\sum_{n=0}^{\infty}{}_LG_n^{(k)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}\right)-\sum_{n=0}^{\infty}{}_LG_n^{(k)}(x,y)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}{}_LG_{n-m}^{(k)}(x,y)\frac{t^n}{(n-m)!m!}-\sum_{n=0}^{\infty}{}_LG_n^{(k)}(x,y)\frac{t^n}{n!}.$$

Finally, equating the coefficients of the like powers of t^n, we get (3.12).

4. Identities for 2-variable Laguerre-based poly-Genocchi polynomials

In this section, we derive general symmetry identities for 2-variable Laguerre-based poly-Genocchi polynomials ${}_LG_n^{(k)}(x,y)$ by applying the generating function(2.1). Such type of identities have been introduced by several authors (see [11], [12], [13], [15]).

Theorem 4.1. Let $a,b>0$ and $a\neq b$, $x,y\in\mathbb{R}$, $n\geq 0$, then the following identity holds true:

$$\sum_{m=0}^{n}\left(\begin{array}{c}n\\ \\ m\end{array}\right)a^{n-m}b^m{}_LG_{n-m}^{(k)}(bx,by){}_LG_m^{(k)}(au,aw)$$

$$=\sum_{m=0}^{n}\left(\begin{array}{c}n\\ \\ m\end{array}\right)a^mb^{n-m}{}_LG_{n-m}^{(k)}(ax,ay){}_LG_m^{(k)}(bu,bw). \tag{4.1}$$

Proof. Let

$$G(t)=\left(\frac{(2\mathrm{Li}_k(1-e^{-at})(2\mathrm{Li}_k(1-e^{-bt}))}{(e^{at}+1)(A^{bt}-B^{-bt})}\right)\exp(ab(y+u)t)C_0(abxt)C_0(abwt). \tag{4.2}$$

Since $G(t)$ is symmetric in a and b and $G(t)$ can written as

$$G(t)=\sum_{n=0}^{\infty}{}_LG_n^{(k)}(bx,by)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}{}_LG_m^{(k)}(au,aw)\frac{(bt)^m}{m!}$$

$$G(t)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)a^{n-m}b^m{}_LG_{n-m}^{(k)}(bx,by){}_LG_m^{(k)}(au,aw)\right)\frac{t^n}{n!}. \tag{4.3}$$

Similarly, we can show that

$$G(t)=\sum_{n=0}^{\infty}{}_LG_n^{(k)}(ax,ay)\frac{(bt)^n}{n!}\sum_{m=0}^{\infty}{}_LG_m^{(k)}(bu,bw)\frac{(at)^m}{m!}$$

$$G(t)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)a^{m}b^{n-m}{}_LG_{n-m}^{(k)}(ax,ay){}_LG_m^{(k)}(bu,bw)\right)\frac{t^n}{n!}. \tag{4.4}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in (4.3) and (4.4), we arrive at the desired result.

Remark 4.1. On setting $b = 1$ in Theorem 4.1, we get

$$\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)a^{n-m}{}_LG_{n-m}^{(k)}(x,y){}_LG_m^{(k)}(au,aw)$$

$$=\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)a^{m}{}_LG_{n-m}^{(k)}(ax,ay){}_LG_m^{(k)}(u,w).$$

Theorem 4.2. Let $a,b>0$ and $a\neq b$, $x,y\in\mathbb{R}$ and $n\geq 0$, then the following identity holds true:

$$\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}{}_LG_{n-m}^{(k)}\left(by+\frac{b}{a}i+j,bx\right){}_LG_m^{(k)}(au,aw)b^ma^{n-m}$$

$$=\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}{}_LG_{n-m}^{(k)}\left(ay+\frac{a}{b}i+j,ax\right){}_LG_m^{(k)}(bu,bw)a^mb^{n-m}. \tag{4.5}$$

Proof. Let

$$G(t)=\left(\frac{(2\mathrm{Li}_k(1-e^{-at}))(2\mathrm{Li}_k(1-e^{-bt}))}{(e^{at}+1)^2(e^{bt}+1)^2}\right)(e^{abt}+1)^2\exp(ab(y+u)t)C_0(abxt)C_0(abwt)$$

$$G(t)=\left(\frac{2\mathrm{Li}_k(1-e^{-at})}{e^{at}+1}\right)\exp(abyt)C_0(abxt)\left(\frac{e^{abt}+1}{e^{bt}+1}\right)\left(\frac{2\mathrm{Li}_k(1-e^{-bt})}{e^{bt}+1}\right)$$
$$\times\exp(abut)C_0(abwt)\left(\frac{e^{abt}+1}{e^{at}+1}\right)$$

$$= \left(\frac{2\mathrm{Li}_k(1-e^{-at})}{(e^{at}+1}\right)\exp(abyt)C_0(abxt)\sum_{i=0}^{a-1}(-1)^i e^{bti}\left(\frac{2\mathrm{Li}_k(1-e^{-bt})}{e^{bt}+1}\right)$$

$$\times \exp(abut)C_0(abwt)\sum_{j=0}^{b-1}(-1)^j e^{atj}.$$

$$= \left(\frac{2\mathrm{Li}_k(1-e^{-t})}{e^{at}+1}\right)C_0(abxt)\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-1)^{i+j}e^{(by+\frac{b}{a}i+j)at}\sum_{m=0}^{\infty}{}_LG_m^{(k)}(au,aw)\frac{(bt)^m}{m!}$$

$$= \sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}{}_LG_n^{(k)}\left(by+\frac{b}{a}i+j,bx\right)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}{}_LG_m^{(k)}(au,aw)\frac{(bt)^m}{(m)!}$$

$$G(t) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right)\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-1)^{i+j}{}_LG_{n-m}^{(k)}\left(by+\frac{b}{a}i+j,bx\right){}_LG_m^{(k)}(au,aw)b^m a^{n-m}\right)\frac{t^n}{n!}. \tag{4.6}$$

On the other hand

$$G(t) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right)\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-1)^{i+j}{}_LG_{n-m}^{(k)}\left(ay+\frac{a}{b}i+j,ax\right){}_LG_m^{(k)}(bu,bw)a^m b^{n-m}\right)\frac{t^n}{n!}. \tag{4.7}$$

On comparing the coefficients of $\frac{t^n}{n!}$ in (4.6) and (4.7), we arrive at the desired result (4.5).

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