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# PERMANENTLY WEAK AMENABILITY OF REES SEMIGROUP ALGEBRAS

**HASAN HOSSEINZADEH[1] AND ALI JABBARI[2]**

[1]*Department of Mathematics, Ardabil Branch, Islamic Azad University, Ardabil, Iran*

[2]*Department of Mathematics, Payame Noor University, Tehran, Iran*

[*]*Corresponding author:* jabbari_al@yahoo.com

Abstract. In this paper, we consider $n$-weak amenability of full matrix algebras and we prove that the Rees semigroup algebra is permanently weakly amenable.

## 1. Introduction

Let $A$ be a Banach algebra, and let $X$ be a Banach $A$-bimodule. Then a linear map $D : A \longrightarrow X$ is a derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b$$

for every $a, b \in A$. Let $x \in X$, and set $\delta_x(a) = a \cdot x - x \cdot a$ for every $a \in A$. Then $\delta_x$ is a derivation; these derivations are inner derivations. The space of continuous derivations from $A$ into $X$ is denoted by $\mathcal{Z}^1(A, X)$, and the subspace consisting of the inner derivations is $\mathcal{N}^1(A, X)$; the first cohomology group of $A$ with coefficients in $X$ is $\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X)$.

A Banach algebra $A$ is *weakly amenable* if $\mathcal{H}^1(A, A^*) = \{0\}$. For example, the group algebra $L^1(G)$ is weak amenable for each locally compact group $G$ [7].

Let $k \in \mathbb{N}$; a Banach algebra $A$ is called *$k$-weakly amenable* if $\mathcal{H}^1(A, A^{(k)}) = \{0\}$. Dales, Ghahramani and Grønbæk brought the concept of $k$-weak amenability of Banach algebras [5]. A Banach algebra $A$ is called

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permanently weakly amenable if $H^1(A, A^{(k)}) = \{0\}$, for each $k \in \mathbb{N}$. In [5], authors showed that for a locally compact group $G$, $L^1(G)$ is $n$-weakly amenable for all odd numbers $n$, but for even case this was open. This open problem solved in [4] and a new prove introduced by Zhang [9].

The above mentioned problem open for semigroups and semigroup algebras. For Rees semi group algebras, Mewomo [8], proved that these algebras are $(2k+1)$-weakly amenable, in this paper, we investigate permanent weak amenability of $n \times n$ matrix Banach algebras. Finally, we prove that the Rees semigroup algebras are permanently weak amenable.

## 2. Characterization of Derivations

Consider the algebra $M_n$ of $n \times n$ matrices. Let $A$ be a Banach algebra. The Banach algebra $M_n(A)$ is the collection of $n \times n$ matrices with components in $A$. We identify the dual of $M_n(A)$ with $M_n(A^*)$ and we have

$$(a \cdot \Lambda)_{ij} = \sum_{s=1}^{n} a_{js} \cdot \lambda_{is}, \quad (\Lambda \cdot a)_{ij} = \sum_{s=1}^{n} \lambda_{sj} \cdot a_{si}, \tag{2.1}$$

for each $a = (a_{ij}) \in M_n(A)$ and $\Lambda = (\lambda_{ij}) \in M_n(A^*)$.

Derivations from $M_n(A)$ into $M_n(A^*)$ is studied in [1]. Set $E_{ij}$ which it is a $n \times n$ matrix, such that whose $(i, j)^{th}$ entry is 1 and other entries are 0. For each $a \in A$, the matrix $a \otimes E_{ij}$ is a matrix that whose $(i, j)^{th}$ entry is $a$ and others entries are 0.

**Lemma 2.1.** *Let $A$ be a Banach algebra and let $D : A \longrightarrow A^*$ be a continuous derivation, then $D$ induces a continuous derivation $\mathfrak{D} : M_n(A) \longrightarrow M_n(A^*)$. Moreover, if $\mathfrak{D}$ is an inner derivation, then $D$ is inner derivation.*

*Proof.* Define $\mathfrak{D} : M_n(A) \longrightarrow M_n(A^*)$ by $\mathfrak{D}((a)_{ij}) = (D(a_{ij}))$ or $\mathfrak{D}((a)_{ij}) = (D(a_{ji}))$. Clearly, continuity of $D$ implies continuity of $\mathfrak{D}$. Similar to argumentation in [6, pp. 17], we have $\mathfrak{D}(ab) = a \cdot \mathfrak{D}(b) + \mathfrak{D}(a) \cdot b$ for every $a, b \in M_n(A)$. Thus, $\mathfrak{D}$ is a module derivation. As well as, if $\mathfrak{D}$ is inner, by a similar method in proof of Theorem 2.7 of [6], $D$ is inner. □

By (2.1),

$$\begin{aligned} \langle \lambda \otimes E_{kl}, (\Lambda_{ij}) \cdot (a_{ij}) \rangle &= \langle (a_{ij}) \cdot (\lambda \otimes E_{kl}), (\Lambda_{ij}) \rangle \\ &= \langle \sum_{s=1}^{n} (a_{sl} \cdot \lambda \otimes E_{kl}), (\Lambda_{ij}) \rangle = \sum_{s=1}^{n} \langle a_{sl} \cdot \lambda, \Lambda_{ks} \rangle \\ &= \langle \lambda, \sum_{s=1}^{n} \Lambda_{ks} \cdot a_{sl} \rangle, \end{aligned} \tag{2.2}$$

for each $\lambda \in A^*, (\Lambda_{ij}) \in M_n(A^{**})$, $(a_{ij}) \in M_n(A)$ and $0 \le k, l \le n$. Hence, (2.2) implies that

$$((\Lambda_{ij}) \cdot (a_{ij}))_{kl} = \sum_{s=1}^{n} \Lambda_{ks} \cdot a_{sl}, \tag{2.3}$$

for each $(\Lambda_{ij}) \in M_n(A^{**})$, $(a_{ij}) \in M_n(A)$ and $0 \leq k, l \leq n$. Similarly

$$((a_{ij}) \cdot (\Lambda_{ij}))_{kl} = \sum_{s=1}^{n} a_{ks} \cdot \Lambda_{sl}, \tag{2.4}$$

for each $(\Lambda_{ij}) \in M_n(A^{**})$, $(a_{ij}) \in M_n(\mathcal{A})$ and $0 \leq k, l \leq n$.

By induction on $m$, for each $(a_{ij}) \in M_n(\mathcal{A})$ and $(\lambda_{ij}) \in M_n(A^{(m)})$ we have

$$((\lambda_{ij}) \cdot (a_{ij}))_{kl} = \sum_{s=1}^{n} \lambda_{sl} \cdot a_{sk}, \quad ((a_{ij}) \cdot (\lambda_{ij}))_{kl} = \sum_{s=1}^{n} a_{ls} \cdot \lambda_{ks}, \tag{2.5}$$

when $m$ is odd and in the case where $m$ is even, we have the following actions:

$$((\lambda_{ij}) \cdot (a_{ij}))_{kl} = \sum_{s=1}^{n} \lambda_{ks} \cdot a_{sl}, \quad ((a_{ij}) \cdot (\lambda_{ij}))_{kl} = \sum_{s=1}^{n} a_{ks} \cdot \lambda_{sl}. \tag{2.6}$$

Now; we are ready to prove the following Lemma that plays an important role in our main results.

**Lemma 2.2.** *Let $A$ be a unital Banach algebra. Then every derivation from $M_n(A)$ into $M_n(A^{(m)})$ ($A^{(m)}$ is the $m$-th dual of $A$) is the sum of an inner derivation and a derivation induced by a derivation from $A$ into $A^{(m)}$.*

*Proof.* Let $e_A$ be the identity element of $A$. Suppose that $\mathfrak{D} : M_n(A) \longrightarrow M_n(A^{(m)})$ is a continuous derivation. For each $i, j$ and $k, l$, define $D_{ij}^{kl} : A \longrightarrow A^{(m)}$ by $D_{ij}^{kl}(a) := (\mathfrak{D}(a \otimes E_{ij}))_{kl}$, for each $a \in A$. Clearly, $D_{ij}^{kl}$ is linear. We prove this Lemma in two cases.

**Case 1.** Let $m$ be an odd positive number. For every $a, b \in A$ and each $1 \leq t \leq n$, we have

$$\begin{aligned} (\big[\mathfrak{D}(a \otimes E_{it})\big] \cdot (b \otimes E_{tj}))_{kl} &= \sum_{s=1}^{n} (\mathfrak{D}(a \otimes E_{it}))_{sl} \cdot (b \otimes E_{tj})_{sk} \\ &= \sum_{s=1}^{n} D_{it}^{sl}(a) \cdot b\delta_{ts}\delta_{jk} = D_{it}^{tl}(a) \cdot b\delta_{jk}, \end{aligned}$$

and

$$\begin{aligned} ((a \otimes E_{it}) \cdot \big[\mathfrak{D}(b \otimes E_{tj})\big])_{kl} &= \sum_{s=1}^{n} (a \otimes E_{it})_{ls} \cdot (\mathfrak{D}(b \otimes E_{tj}))_{ks} \\ &= \sum_{s=1}^{n} a\delta_{il}\delta_{ts} \cdot D_{tj}^{ks}(b) = a\delta_{il} \cdot D_{tj}^{kt}(b), \end{aligned}$$

where $\delta$ is the Kronecker's delta. Then

$$D_{ij}^{kl}(ab) = a\delta_{il} \cdot D_{tj}^{kt}(b) + D_{it}^{tl}(a) \cdot b\delta_{jk}. \tag{2.7}$$

Thus, $D_{ii}^{ii}$ is a derivation from $A$ into $A^{(m)}$. From (2.5) and (2.7), the following statements hold

$$D_{ij}^{jl}(a) = D_{ii}^{il}(e_A) \cdot a \quad (i \neq l), \qquad D_{ij}^{ki}(a) = a \cdot D_{jj}^{kj}(e_A) \quad (j \neq k), \tag{2.8}$$

and again by (2.7) and for $1 \leq i, j, l \leq n$, we have

$$\begin{aligned} D^{jj}_{jj}(a) &= D^{ij}_{ji}(e_A) \cdot a + D^{ji}_{ij}(a) = D^{ij}_{ji}(e_A) \cdot a + D^{li}_{il}(e_A) \cdot a + D^{jl}_{lj}(a) \\ &= D^{ij}_{ji}(e_A) \cdot a + D^{li}_{il}(e_A) \cdot a + D^{ll}_{ll}(a) + a \cdot D^{jl}_{lj}(e_A), \end{aligned} \tag{2.9}$$

and

$$D^{ij}_{ji}(a) = a \cdot D^{ij}_{ji}(e_A) + D^{jj}_{jj}(a). \tag{2.10}$$

Hence $D^{ij}_{ji}(e_A) = -D^{ji}_{ij}(e_A)$ for every $1 \leq i, j \leq n$, and consequently by (2.9), the following relation holds

$$D^{ji}_{ij}(a) = D^{li}_{il}(e_A) \cdot a - a \cdot D^{lj}_{jl}(e_A) + D^{ll}_{ll}(a). \tag{2.11}$$

Together with (2.9) and (2.10) we have

$$D^{ji}_{ij}(a) = D^{ij}_{ji}(a) - D^{ij}_{ji}(e_A) \cdot a - a \cdot D^{ij}_{ji}(e_A), \tag{2.12}$$

for every $a \in A$. By (2.7) and (2.10) the following equality holds

$$\begin{aligned} D^{ij}_{kl}(a) &= D^{ij}_{ki}(e_A) \cdot a + D^{ii}_{il}(a) = D^{ij}_{ki}(e_A) \cdot a + D^{ji}_{ij}(e_A) \cdot a + D^{ij}_{jl}(a) \\ &= D^{ij}_{ki}(e_A) \cdot a + D^{ji}_{ij}(e_A) \cdot a + a \cdot D^{ij}_{jl}(e_A) + D^{jj}_{jj}(a) \\ &= D^{ij}_{ki}(e_A) \cdot a + a \cdot D^{ij}_{jl}(e_A) - D^{ij}_{ji}(e_A) \cdot a - a \cdot D^{ij}_{ji}(e_A) + D^{ij}_{ji}(a), \end{aligned} \tag{2.13}$$

for every $a \in A$. Then by (2.8), (2.12) and (2.13), we have

$$\begin{aligned} (\mathfrak{D}(a_{rs}))_{ij} &= \sum_{k,l=1}^{n} D^{ij}_{kl}(a_{kl}) = \sum_{k=1}^{n} D^{ij}_{ki}(e_A) \cdot a_{ki} + \sum_{l=1}^{n} D^{ii}_{il}(a_{il}) \\ &= \sum_{k=1}^{n} D^{ij}_{ki}(e_A) \cdot a_{ki} + \sum_{l=1}^{n} a_{jl} \cdot D^{ij}_{jl}(e_A) \\ &\quad - D^{ij}_{ji}(e_A) \cdot a_{ji} - a_{ji} \cdot D^{ij}_{ji}(e_A) + D^{ij}_{ji}(a_{ji}) \\ &= \sum_{k=1}^{n} D^{kj}_{kk}(e_A) \cdot a_{ki} + \sum_{k=1}^{n} a_{jk} \cdot D^{ik}_{kk}(e_A) + D^{ji}_{ij}(a_{ji}), \end{aligned} \tag{2.14}$$

for every $(a_{rs}) \in M_n(\mathcal{A})$. As well as,

$$(\mathfrak{D}(E_{kk}E_{ii}))_{ik} = \sum_{k=1}^{n} D^{sk}_{kk}(e_A)\delta_{si} + \sum_{k=1}^{n} \delta_{ks} D^{is}_{ii}(e_A) = D^{ik}_{kk}(e_A) + D^{ik}_{ii}(e_A) = 0.$$

This shows that $D^{ik}_{kk}(e_A) = -D^{ik}_{ii}(e_A)$. Now; for every $1 \leq k, j \leq n$ define $D_{kj} = D^{kj}_{kk}$. By the above obtained results we have

$$\begin{aligned} (\mathfrak{D}(a_{rs}))_{ij} &= \sum_{k=1}^{n} D_{kj}(e_A) \cdot a_{ki} - \sum_{k=1}^{n} a_{jk} \cdot D_{ik}(e_A) + D^{ji}_{ij}(a_{ji}) \\ &= ((D_{rs}(e_A)) \cdot (a_{rs}) - (a_{rs}) \cdot (D_{rs}(e_A)))_{ij} + D^{ji}_{ij}(a_{ji}). \end{aligned} \tag{2.15}$$

Set

$$\mathcal{D}(e_A) = \begin{bmatrix} D^{l1}_{1l}(e_A) & \dots & 0 \\ \vdots & D^{l2}_{2l}(e_A) & \vdots \\ 0 & \dots & D^{ln}_{nl}(e_A) \end{bmatrix}_{n\times n}.$$

Then by (2.11) and (2.15) we have

$$\begin{aligned}\mathfrak{D}((a_{rs})) &= \Big(D_{ij}(e_A) + \mathcal{D}(e_A)\Big)\cdot(a_{ij}) - (a_{ij})\cdot\Big(D_{ij}(e_A) + \mathcal{D}(e_A)\Big) \\ &\quad + (D^{ll}_{ll}(a_{ij})),\end{aligned}$$

where $(D^{ll}_{ll}(a_{ij}))$ is a diagonal matrix.

**Case 2.** Now; let $m$ be an even positive number. Then by (2.6) we have

$$\begin{aligned}([\mathfrak{D}(a\otimes E_{it})]\cdot(b\otimes E_{tj}))_{kl} &= \sum_{s=1}^{n}(\mathfrak{D}(a\otimes E_{it}))_{ks}\cdot(b\otimes E_{tj})_{sl} \\ &= \sum_{s=1}^{n} D^{ks}_{it}(a)\cdot b\delta_{ts}\delta_{jl} = D^{kt}_{it}(a)\cdot b\delta_{jl},\end{aligned}$$

and

$$\begin{aligned}((a\otimes E_{it})\cdot[\mathfrak{D}(b\otimes E_{tj})])_{kl} &= \sum_{s=1}^{n}(a\otimes E_{it})_{ks}\cdot(\mathfrak{D}(b\otimes E_{tj}))_{sl} \\ &= \sum_{s=1}^{n} a\delta_{ik}\delta_{ts}\cdot D^{sl}_{tj}(b) = a\delta_{ik}\cdot D^{tl}_{tj}(b),\end{aligned}$$

for every $a, b \in A$. Then

$$D^{kl}_{ij}(ab) = a\delta_{ik}\cdot D^{tl}_{tj}(b) + D^{kt}_{it}(a)\cdot b\delta_{jl}. \tag{2.16}$$

Thus, $D^{ii}_{ii}$ is a derivation from $A$ into $A^{(m)}$. By (2.6) and (2.16), the following equalities hold

$$D^{kj}_{ij}(a) = D^{ki}_{ii}(e_A)\cdot a \quad (k\neq i), \qquad D^{il}_{ij}(a) = a\cdot D^{jl}_{jj}(e_A) \quad (j\neq l), \tag{2.17}$$

and for $1\leq i,j,l\leq n$, (2.16) follows

$$\begin{aligned}D^{ii}_{ii}(a) &= D^{ji}_{ji}(a) + D^{ij}_{ij}(e_A)\cdot a = D^{ij}_{ij}(e_A)\cdot a + D^{jl}_{jl}(e_A)\cdot a + D^{li}_{li}(a) \\ &= D^{ij}_{ij}(e_A)\cdot a + D^{jl}_{jl}(e_A)\cdot a + D^{ll}_{ll}(a) + a\cdot D^{li}_{li}(e_A),\end{aligned} \tag{2.18}$$

and

$$D^{ji}_{ji}(a) = D^{ii}_{ii}(a) + D^{ji}_{ji}(e_A)\cdot a, \tag{2.19}$$

for every $a\in A$. Therefore $D^{ij}_{ij}(e_A) = -D^{ji}_{ji}(e_A)$, for every $1\leq i,j\leq n$. Then (2.18) implies that

$$D^{ji}_{ji}(a) = D^{jl}_{jl}(e_A)\cdot a - a\cdot D^{il}_{il}(e_A) + D^{ll}_{ll}(a). \tag{2.20}$$

As well as,

$$D^{ij}_{kl}(a) = D^{ij}_{kj}(e_A)\cdot a + D^{jj}_{il}(a) = D^{ij}_{kj}(e_A)\cdot a + a\cdot D^{ij}_{il}(e_A) + D^{ji}_{ji}(a), \tag{2.21}$$

for every $a \in A$. By using the relations (2.17) and (2.21), for every $(a_{rs}) \in M_n(A)$, we have

$$\begin{aligned}(\mathfrak{D}(a_{rs}))_{ij} &= \sum_{k,l=1}^{n} D_{kl}^{ij}(a_{kl}) = \sum_{l=1}^{n} D_{kj}^{ij}(e_A) \cdot a_{kj} + \sum_{k=1}^{n} D_{il}^{jj}(a_{il}) \\ &= \sum_{k=1}^{n} D_{kj}^{ij}(e_A) \cdot a_{kj} + \sum_{k=1}^{n} a_{il} \cdot D_{il}^{ij}(e_A) + D_{ji}^{ji}(a_{ji}) \\ &= \sum_{k=1}^{n} D_{kk}^{ik}(e_A) \cdot a_{kj} + \sum_{k=1}^{n} a_{ik} \cdot D_{kk}^{kj}(e_A) + D_{ji}^{ji}(a_{ji}). \end{aligned} \tag{2.22}$$

Since

$$(\mathfrak{D}(E_{kk}E_{ii}))_{ik} = \sum_{k=1}^{n} D_{kk}^{ks}(e_A)\delta_{is} + \sum_{k=1}^{n} \delta_{is} D_{ii}^{sk}(e_A) = D_{kk}^{ki}(e_A) + D_{ii}^{ik}(e_A) = 0, \tag{2.23}$$

$D_{kk}^{ki}(e_A) = -D_{ii}^{ik}(e_A)$. Now; define $D_{kj} = D_{kk}^{kj}$ for every $1 \leq j, k \leq n$. Then by the above obtained results we have

$$\begin{aligned}(\mathfrak{D}(a_{rs}))_{ij} &= \sum_{k=1}^{n} D_{ik}(e_A) \cdot a_{kj} - \sum_{k=1}^{n} a_{ik} \cdot D_{jk}(e_A) + D_{ji}^{ji}(a_{ji}) \\ &= ((D_{rs}(e_A)) \cdot (a_{rs}) - (a_{rs}) \cdot (D_{rs}(e_A)))_{ij} + D_{ji}^{ji}(a_{ji}). \end{aligned} \tag{2.24}$$

Similar to Case 1, set

$$\mathcal{D}(e_A) = \begin{bmatrix} D_{1l}^{1l}(e_A) & \dots & 0 \\ \vdots & D_{2l}^{2l}(e_A) & \vdots \\ 0 & \dots & D_{nl}^{nl}(e_A) \end{bmatrix}_{n \times n}.$$

Now; by applying (2.20) and (2.24) the following holds

$$\begin{aligned}\mathfrak{D}((a_{rs})) &= \Big(D_{ij}(e_A) + \mathcal{D}(e_A)\Big) \cdot (a_{ij}) - (a_{ij}) \cdot \Big(D_{ij}(e_A) + \mathcal{D}(e_A)\Big) \\ &\quad + (D_{ll}^{ll}(a_{ij})). \end{aligned}$$

Hence proof is complete. □

Weak amenability and $(2k+1)$-weak amenability of $M_n(A)$ considered in [3,8]. Now; by above Lemma we have the following result:

**Theorem 2.1.** *Let $A$ be a unital Banach algebra. Then $A$ is permanently weakly amenable if and only if $M_n(A)$ is permanently weakly amenable.*

*Proof.* Let $M_n(A)$ be permanently weakly amenable and let $D : A \longrightarrow A^{(k)}$ be a continuous derivation, $k \in \mathbb{N}$. Then by Lemma 2.1, $D$ induces a continuous derivation $\mathfrak{D} : M_n(A) \longrightarrow M_n(A^{(k)})$. Hence, by our assumption $\mathfrak{D}$ is inner and Lemma 2.1, implies that $D$ is inner.

Conversely, suppose that $A$ is permanently weakly amenable. Let $\mathfrak{D} : M_n(A) \longrightarrow M_n(A^{(k)})$ be a continuous module derivation, $k \in \mathbb{N}$. Then by Lemma 2.2, it is equal to the sum of an inner derivation and a derivation induced by a derivation from $A$ into $A^{(k)}$. Since $A$ is permanently weakly module, $\mathfrak{D}$ is equal to sum of two inner derivations. Thereby, $M_n(A)$ is permanently weakly module amenable. □

**Example 2.1.** *Let $G$ be a discrete group. Then by [4], [5] and Theorem 2.1, $M_n(\ell^1(G))$ is permanently weakly amenable.*

**Example 2.2.** *Let $A$ be a unital $C^*$-algebra. Then $M_n(A)$ is permanently weakly amenable.*

## 3. Rees semigroup algebras

Let $G$ be a group, and $m, n \in \mathbb{N}$; the zero adjoined to $G$ is $o$. A Rees semigroup has the form $S = \mathcal{M}(G, P, m, n)$; here $P = (a_{ij}) \in M_{n,m}(G)$ is the collection of $n \times m$ matrices with components in $G$. For $x \in G$, $1 \leq i \leq m$ and $1 \leq j \leq n$, let $(x)_{ij}$ be the element of $M_{m,n}(G^o)$ with $x$ in the $(i, j)$-th place and $o$ elsewhere. As a set, $S$ consists of the collection of all these matrices $(x)_{ij}$. Multiplication in $S$ is given by the formula

$$(x)_{ij}(y)_{kl} = (xa_{jk}y)_{il} \qquad (x, y \in G, 1 \leq i, k \leq m,\ 1 \leq j, l \leq n).$$

It is known that $S$ is a semigroup. Now; consider the semigroup $\mathcal{M}^o(G, P, m, n)$, where the elements of this semigroup are those of $\mathcal{M}(G, P, m, n)$, together with the element $o$, identified with the matrix that has $o$ in each place (so that $o$ is the zero of $\mathcal{M}^o(G, P, m, n)$), and the components of $P$ are belong to $G^o$. The matrix $P$ is called the sandwich matrix in each case. The semigroup $\mathcal{M}^o(G, P, m, n)$ is a Rees matrix semigroup with a zero over $G$. We write $\mathcal{M}^o(G, P, n)$ for $\mathcal{M}^o(G, P, n, n)$ in the case where $m = n$. As well as, $P$ is called regular if every row and column contains at least one entry in $G$. The semigroup $\mathcal{M}^o(G, P, m, n)$ is regular as a semigroup if and only if the sandwich matrix $P$ is regular.

According to [6] we have the following equalities as Banach spaces

$$\ell^1(S) = \mathcal{M}^o(\ell^1(G), P, m, n) = \mathcal{M}(\ell^1(G), P, m, n) \oplus \mathbb{C}\delta_0.$$

Bowling and Duncan proved that for any Rees semigroup $S$, $\ell^1(S)$ is weakly amenable [3, Theorem 2.5] and after them Mewomo in [8], proved that $\ell^1(S)$ is $(2k+1)$-weakly amenable where $S = \mathcal{M}^o(G, P, n)$, for $k, n \in \mathbb{N}$. Now; we are completing them works as follows:

**Theorem 3.1.** *Let $S = \mathcal{M}^o(G, P, n)$, $n \in \mathbb{N}$. Then $\ell^1(S)$ is permanently weakly amenable.*

*Proof.* It is sufficient we show that $\ell^1(S)$ is $(2k)$-weakly amenable, for $k \in \mathbb{N}$. For any locally compact group $G$, $\ell^1(G)$ is permanently weakly amenable ( [4, pp. 3179] and [5, Theorem 4.1]). Theorem 2.1 implies that $M_n(\ell^1(G))$ is $(2k)$-weakly amenable. Since $M_n(\ell^1(G)) = \ell^1(S)$, $\ell^1(S)$ is $(2k)$-weakly amenable. □

Let $S$ be a semigroup. The weak amenability of $\ell^1(S)$ is considered by Blackmore in [2]. He proved that $\ell^1(S)$ to be weakly amenable whenever $S$ is completely regular, in the sense that, for each $s \in S$, there exists

$t \in S$ with $sts = s$ and $st = ts$. Suppose that $S$ has a zero $o$. Then $S$ is $o$-simple if $S_{[2]} \neq \{o\}$ and the only ideals in $S$ are $\{o\}$ and $S$. The semigroup $S$ is called completely $o$-simple if it is $o$-simple and contains a primitive idempotent.

**Corollary 3.1.** *Let $S$ be an infinite, completely $o$-simple semigroup with finitely many idempotents. Then $\ell^1(S)$ is permanently weakly amenable.*

*Proof.* By Corollary 4.2 of [8], it suffices to show that $\ell^1(S)$ is $(2k)$-weakly amenable, for $k \in \mathbb{N}$. The semigroup $S$ is isomorphic as a semigroup to a regular Rees matrix semigroup with a zero $\mathcal{M}^o(G, P, n)$, $n \in \mathbb{N}$ [6, Theorem 3.13]. Now; apply Theorem 3.1. □

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