



## NEW SUBCLASS OF ANALYTIC FUNCTIONS IN CONICAL DOMAIN ASSOCIATED WITH RUSCHEWEYH $q$ -DIFFERENTIAL OPERATOR

SHAHID KHAN<sup>1,\*</sup>, SAQIB HUSSAIN<sup>2</sup>, MUHAMMAD ASAD ZAIGHUM<sup>1</sup>, MUHAMMAD MUMTAZ KHAN<sup>3</sup>

<sup>1</sup>*Department of Mathematics, Riphah International University Islamabad, Pakistan*

<sup>2</sup>*Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad, Pakistan*

<sup>3</sup>*Department of Public Health Riphah University of Haripur, Pakistan*

\*Corresponding author: shahidmath761@gmail.com

ABSTRACT. In this paper, we consider a new class of analytic functions which is defined by means of a Ruscheweyh  $q$ -differential operator. We investigated some new results such as coefficients inequalities and other interesting properties of this class. Comparison of new results with those that were obtained in earlier investigation are given as Corollaries.

### 1. INTRODUCTION

Let  $\mathcal{A}$  denote the class of functions  $f$  analytic in the open unit disk

$$E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and satisfying the normalization condition

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

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Thus, the functions in  $\mathcal{A}$  are represented by the Taylor-Maclaurin series expansion given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in E). \tag{1.1}$$

Let  $\mathcal{S}$  be the subset of  $\mathcal{A}$  consisting of the functions that are univalent in  $E$ . Given functions  $f, g \in \mathcal{A}$ ,  $f$  is said to be subordinate to  $g$  in  $E$ , denoted by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in E),$$

if there exists a function  $w \in \mathcal{P}_0$  where

$$\mathcal{P}_0 = \{w \in \mathcal{A} : w(0) = 0, \quad \text{and} \quad |w(z)| < 1 \quad (z \in E)\},$$

such that

$$f(z) = g(w(z)) \quad (z \in E).$$

If  $g$  is univalent in  $E$ , then it follows that

$$f(z) \prec g(z) \quad (z \in E), \Rightarrow f(0) = 0 \text{ and } f(E) \subset g(E).$$

Kanas and Wiśniowska [5, 6] introduced the conic domain  $\Omega_k, k \geq 0$  as

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}.$$

We note that  $\Omega_k$  is a region in the right half-plane, symmetric with respect to real axis, and contains the point  $(1, 0)$ . More precisely for  $k = 0$ ,  $\Omega_0$  is the right half-plane, for  $0 < k < 1$ ,  $\Omega_k$  is an unbounded region having boundary  $\partial\Omega_k$ , a rectangular hyperbola; for  $k = 1$ ,  $\Omega_1$  is still an unbounded region where  $\partial\Omega_1$  is a parabola, and for  $k > 1$ ,  $\Omega_k$  is a bounded region enclosed by an ellipse. The extremal functions for these conic regions are

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi^2} \left( \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, & k = 1, \\ \frac{1}{1-k^2} \cosh \left\{ \left( \frac{2}{\pi} \arccos k \right) \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right\} - \frac{k^2}{1-k^2}, & 0 < k < 1, \\ \frac{1}{k^2-1} \sin \left( \frac{\pi}{2K(\kappa)} \int_0^{\frac{u(z)}{\sqrt{\kappa}}} \frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}} \right) + \frac{k^2}{k^2-1}, & k > 1, \end{cases} \tag{1.2}$$

where

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa}z} \quad (z \in E)$$

and  $\kappa \in (0, 1)$  is chosen such that  $k = \cosh(\pi K'(\kappa)/(4K(\kappa)))$ . Here  $K(\kappa)$  is Legendre's complete elliptic integral of first kind and  $K'(\kappa) = K(\sqrt{1-\kappa^2})$  and  $K'(t)$  is the complementary integral of  $K(t)$  for details

see [1, 5, 6] and more recently [9, 12, 14]. If  $p_k(z) = 1 + L_1(k)z + L_2(k)z^2 + \dots, z \in E$ . Then it was shown in [6] that for (1.2) one can have

$$L_1(k) = \begin{cases} \frac{2A^2}{1-k^2}, & 0 \leq k < 1 \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{\pi^2}{4K^2(t)^2(1+t)\sqrt{t}}, & k > 1, \end{cases} \tag{1.3}$$

$$L_2(k) = D(k)L_1(k),$$

where

$$D(k) = \begin{cases} \frac{A^2+2}{3}, & 0 \leq k < 1 \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{(4K(t))^2(t^2+6t+1)-\pi^2}{24K(t)^2(1+t)\sqrt{t}}, & k > 1, \end{cases} \tag{1.4}$$

with  $A = \frac{2}{\pi} \arccos k$ .

Furthermore a function  $p$  is said to be in the class  $k - P[A, B]$ , if and only if

$$p(z) \prec \frac{(A+1)p_k(z) - (A-1)}{(B+1)p_k(z) - (B-1)}, \quad k \geq 0,$$

where  $p_k$  is defined in (1.2) and  $-1 \leq B < A \leq 1$ . Geometrically, the function  $p \in k - P[A, B]$  takes all values from the domain  $\Omega_k[A, B]$ ,  $-1 \leq B < A \leq 1, k \geq 0$  which is defined as:

$$\Omega_k[A, B] = \left\{ w : \Re \left( \frac{(B-1)w - (A-1)}{(B+1)w - (A+1)} \right) > k \left| \frac{(B-1)w - (A-1)}{(B+1)w - (A+1)} - 1 \right| \right\},$$

or equivalently  $\Omega_k[A, B]$  is a set of numbers  $w = u + iv$  such that

$$\begin{aligned} & [(B^2 - 1)(u^2 + v^2) - 2(AB - 1)u + (A^2 - 1)]^2 \\ & > k \left[ (-2(B+1)(u^2 + v^2) + 2(A+B+2)u - 2(A+1))^2 + 4(A-B)^2v^2 \right]. \end{aligned}$$

This domain represents the conic type regions for detail see [11]. It can be easily seen that  $0 - P[A, B] = P[A, B]$  introduced in [4] and  $k - P[1, -1] = P(p_k)$  introduced in [5].

We now recall some basic concept details of the  $q$ -calculus which are used in this paper. Throughout this paper we assume  $q$  to be a fixed number between 0 and 1.

For any non-negative integer  $n$ , the  $q$ -integer number  $n$ ,  $[n, q]$  is defined by:

$$[n, q] = \frac{1 - q^n}{1 - q} = 1 + q + \dots + q^{n-1}, \quad [0, q] = 0. \tag{1.5}$$

The  $q$ -number shifted factorial is defined by  $[0, q]! = 1$  and  $[n, q]! = [1, q][2, q] \dots [n, q]$ . Clearly,  $\lim_{q \rightarrow 1} [n, q] = n$  and  $\lim_{q \rightarrow 1} [n, q]! = n!$ . In general we will denote  $[t, q] = \frac{1 - q^t}{1 - q}$  also for a non-integer number. Let  $f \in \mathcal{A}$ , and let the  $q$ -derivative operator or  $q$ -difference operator be defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \quad (z \in E).$$

It is easy to observed that for  $n \in \mathbb{N} := \{1, 2, 3, \dots\}$  and  $z \in E$

$$\partial_q z^n = [n, q] z^{n-1}.$$

Let the  $q$ -generalized Pochhammer symbol be defined as

$$[t, q]_n = [t, q] [t + 1, q] [t + 2, q] \dots [t + n - 1, q],$$

and for  $t > 0$  let the  $q$ -gamma function be defined as

$$\Gamma_q(t + 1) = [t] \Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1.$$

The study of opretors play in important role in Geomatic Functions Theory. Several differiential and integral operators were introduced and studied, see for example [2, 3, 13]. Kannas et al. define Ruscheweyh  $q$ -differential operator as follow:

**Definition 1.1.** [7] For  $f \in \mathcal{A}$ , let the Ruscheweyh  $q$ -differential operator be defined as follows:

$$R_q^\lambda f(z) = f(z) * F_{q, \lambda+1}(z), \quad (z \in E, \lambda > -1) \tag{1.6}$$

where

$$\begin{aligned} F_{q, \lambda+1}(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n + \lambda)}{[n - 1, q]! \Gamma_q(1 + \lambda)} z^n, \\ &= z + \sum_{n=2}^{\infty} \frac{[\lambda + 1, q]_{n-1}}{[n - 1, q]!} z^n, \\ &= z + \sum_{n=2}^{\infty} \varphi_{n-1} z^n. \end{aligned} \tag{1.7}$$

Where

$$\varphi_{n-1} = \frac{\Gamma_q(n + \lambda)}{[n - 1, q]! \Gamma_q(1 + \lambda)} = \frac{[\lambda + 1, q]_{n-1}}{[n - 1, q]!}.$$

From (1.6) we obtain that

$$R_q^0 f(z) = f(z), \quad R_q^1 f(z) = z \partial_q f(z)$$

and

$$R_q^m f(z) = \frac{z \partial_q^m (z^{m-1} f(z))}{[m, q]!}, \quad (m \in \mathbb{N}).$$

Making use of (1.6) and (1.7), the power series of  $R_q^\lambda f(z)$  is given by

$$R_q^\lambda f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n + \lambda)}{[n - 1, q]! \Gamma_q(1 + \lambda)} a_n z^n = z + \sum_{n=2}^{\infty} \frac{[\lambda + 1, q]_{n-1}}{[n - 1, q]!} a_n z^n. \tag{1.8}$$

Note that

$$\lim_{q \rightarrow 1} F_{q, \lambda+1}(z) = \frac{z}{(1 - z)^{\lambda+1}}$$

and

$$\lim_{q \rightarrow 1} R_q^\lambda f(z) = f(z) * \frac{z}{(1-z)^{\lambda+1}}.$$

Thus, we can say that Ruscheweyh  $q$ -differential operator reduces to the differential operator defined by Ruscheweyh [16] in the case when  $q \rightarrow 1$ . It is easy to check that

$$z\partial (F_{q,\lambda+1}(z)) = \left(1 + \frac{[\lambda, q]}{q^\lambda}\right) F_{q,\lambda+2}(z) - \frac{[\lambda, q]}{q^\lambda} F_{q,\lambda+1}(z). \tag{1.9}$$

Making use of (1.6), (1.9) and the properties of Hadamard product, we obtain the following equality

$$z\partial (R_q^\lambda f(z)) = \left(1 + \frac{[\lambda, q]}{q^\lambda}\right) R_q^{\lambda+1} f(z) - \frac{[\lambda, q]}{q^\lambda} R_q^\lambda f(z). \tag{1.10}$$

If  $q \rightarrow 1$ , the equality (1.10) implies

$$z (R^\lambda f(z))' = (1 + \lambda) R^{\lambda+1} f(z) - \lambda R^\lambda f(z).$$

which is the well known recurrent formula for Ruscheweyh differential operator.

Using Ruscheweyh differential operator various new classes of convex and starlike functions have been defined.

Now by using Ruscheweyh  $q$ -differential operator we introduce the following class of functions.

**Definition 1.2.** A function  $f(z) \in A$  is said to be in the class  $k - \mathcal{US}_q(\lambda, A, B, \beta)$ ,  $k \geq 0, -1 \leq B < A \leq 1$ , if and only if

$$\Re \left( \frac{(B-1)G(z) - (A-1)}{(B+1)G(z) - (A+1)} \right) > k \left| \frac{(B-1)G(z) - (A-1)}{(B+1)G(z) - (A+1)} - 1 \right|,$$

where

$$G(z) = \frac{z\partial_q R_q^\lambda f(z)}{R_q^\lambda f(z)} + \beta \frac{z^2 \partial_q^2 R_q^\lambda f(z)}{R_q^\lambda f(z)},$$

or equivalently

$$\frac{z\partial_q R_q^\lambda f(z)}{R_q^\lambda f(z)} + \beta \frac{z^2 \partial_q^2 R_q^\lambda f(z)}{R_q^\lambda f(z)} \in k - P[A, B]. \tag{1.11}$$

**Remark 1.1.** It is easily see that

$$\lim_{q \rightarrow 1^-} k - \mathcal{US}_q(0, A, B, 0) = k - \mathcal{ST}(A, B)$$

where  $k - \mathcal{ST}(A, B)$  is a functions class, intriduced and studied by Noor and Sarfraz [11].

Each of the following lemmas will be needed in our present investigation.

**Lemma 1.1.** [15] Let  $h(z) = 1 + \sum_{n=1}^\infty c_n z^n$  be subordinate to  $H(z) = 1 + \sum_{n=1}^\infty C_n z^n$ . If  $H(z)$  is univalent in  $E$  and  $H(E)$  is convex, then

$$|c_n| \leq |C_1|, \quad n \geq 1.$$

**Lemma 1.2.** ([8], [10]) If  $q(z) = 1 + c_1z + c_2z^2 + \dots$  is an analytic function with positive real part in  $E$ , then

$$|c_2 - vc_1^2| \leq 2 \max \{1, |2v - 1|\}.$$

The result is sharp for the functions

$$q(z) = \frac{1 + z^2}{1 - z^2}, \text{ or } q(z) = \frac{1 + z}{1 - z}.$$

**Lemma 1.3.** [8] Let the function  $w \in E$  be given by

$$w(z) = c_1z + c_2z^2 + \dots \quad z \in E.$$

Then for every complex number  $v$ ,

$$|c_2 - vc_1^2| \leq 1 + (|v| - 1) |c_1|^2.$$

**Lemma 1.4.** [11] Let  $k \in [0, \infty)$  be a fixed and

$$q_k(z) = \frac{(A + 1)p_k(z) - (A - 1)}{(B + 1)p_k(z) - (B - 1)},$$

then

$$q_k(z) = 1 + H_1(k)z + H_2(k)z^2 + \dots, z \in E.$$

and

$$H_1 := H_1(k) = \frac{A - B}{2} L_1(k),$$

$$H_2 := H_2(k) = \frac{A - B}{4} \{2D(k) - (B + 1)H_1\} L_1(k)$$

where  $L_1(k)$  and  $D(k)$  are defined in (1.3) and (1.4).

## 2. Main Results

**Theorem 2.1.** A function  $f \in \mathcal{A}$  and of the form (1.1) is in the class  $k - \mathcal{US}_q(\lambda, A, B, \beta)$ , if it satisfies the condition

$$\sum_{n=2}^{\infty} \left\{ \begin{array}{l} \{2(k + 1) \{1 - [n, q] - \beta[n, q][n - 1, q]\} \\ + |\{(B + 1)[n, q] + \beta[n, q][n - 1, q] - (A + 1)\}| \} \end{array} \right\} \varphi_{n-1} |a_n| \leq |B - A|. \quad (2.1)$$

where  $-1 \leq B < A \leq 1$ ,  $\beta \geq 0$  and  $k \geq 0$ .

*Proof.* Assume (2.1) is hold, then it suffices to show that

$$\left\{ \begin{array}{l} k \left| \frac{(B-1) \left( \frac{z \partial_q R_q^\lambda f(z)}{R_q^\lambda f(z)} + \beta \frac{z^2 \partial_q^2 R_q^\lambda f(z)}{R_q^\lambda f(z)} \right) - (A-1)}{(B+1) \left( \frac{z \partial_q R_q^\lambda f(z)}{R_q^\lambda f(z)} + \beta \frac{z^2 \partial_q^2 R_q^\lambda f(z)}{R_q^\lambda f(z)} \right) - (A+1)} - 1 \right| \\ -\Re \left\{ \frac{(B-1) \left( \frac{z \partial_q R_q^\lambda f(z)}{R_q^\lambda f(z)} + \beta \frac{z^2 \partial_q^2 R_q^\lambda f(z)}{R_q^\lambda f(z)} \right) - (A-1)}{(B+1) \left( \frac{z \partial_q R_q^\lambda f(z)}{R_q^\lambda f(z)} + \beta \frac{z^2 \partial_q^2 R_q^\lambda f(z)}{R_q^\lambda f(z)} \right) - (A+1)} - 1 \right\} \end{array} \right\} < 1,$$

We have

$$\begin{aligned} & \left\{ \begin{array}{l} k \left| \frac{(B-1) \left( \frac{z \partial_q R_q^\lambda f(z)}{R_q^\lambda f(z)} + \beta \frac{z^2 \partial_q^2 R_q^\lambda f(z)}{R_q^\lambda f(z)} \right) - (A-1)}{(B+1) \left( \frac{z \partial_q R_q^\lambda f(z)}{R_q^\lambda f(z)} + \beta \frac{z^2 \partial_q^2 R_q^\lambda f(z)}{R_q^\lambda f(z)} \right) - (A+1)} - 1 \right| \\ -\Re \left\{ \frac{(B-1) \left( \frac{z \partial_q R_q^\lambda f(z)}{R_q^\lambda f(z)} + \beta \frac{z^2 \partial_q^2 R_q^\lambda f(z)}{R_q^\lambda f(z)} \right) - (A-1)}{(B+1) \left( \frac{z \partial_q R_q^\lambda f(z)}{R_q^\lambda f(z)} + \beta \frac{z^2 \partial_q^2 R_q^\lambda f(z)}{R_q^\lambda f(z)} \right) - (A+1)} - 1 \right\} \end{array} \right\} \\ & \leq (k+1) \left| \frac{(B-1) (z \partial_q R_q^\lambda f(z) + \beta z^2 \partial_q^2 R_q^\lambda f(z)) - (A-1) R_q^\lambda f(z)}{(B+1) (z \partial_q R_q^\lambda f(z) + \beta z^2 \partial_q^2 R_q^\lambda f(z)) - (A+1) R_q^\lambda f(z)} - 1 \right| \\ & = 2(k+1) \left| \frac{R_q^\lambda f(z) - z \partial_q R_q^\lambda f(z) - \beta z^2 \partial_q^2 R_q^\lambda f(z)}{(B+1) (z \partial_q R_q^\lambda f(z) + \beta z^2 \partial_q^2 R_q^\lambda f(z)) - (A+1) R_q^\lambda f(z)} \right| \\ & = 2(k+1) \frac{\sum_{n=2}^\infty (1 - [n, q] - \beta [n, q][n-1, q]) \varphi_{n-1} a_n z^n}{(B-A)z + \sum_{n=2}^\infty \left\{ (B+1) [n, q]_q + \beta [n, q][n-1, q] - (A+1) \right\} \varphi_{n-1} a_n z^n} \\ & \leq 2(k+1) \frac{\sum_{n=2}^\infty (1 - [n, q] - \beta [n, q][n-1, q]) \varphi_{n-1} |a_n|}{|B-A| - \sum_{n=2}^\infty \left\{ (B+1) [n, q]_q + \beta [n, q][n-1, q] - (A+1) \right\} \varphi_{n-1} |a_n|} \\ & < 1 \quad (\text{by (2.1)}). \end{aligned}$$

□

When  $A = 1 - 2\alpha$ ,  $B = -1$ ,  $\beta = 0$  with  $0 \leq \alpha < 1$ , then we have the following known result, proved by Kanas and Raducanu in [7].

**Corollary 2.1.** *A function  $f \in A$  and of the form (1.1) is in the class  $k-US_q(\lambda, 1 - 2\alpha, -1)$ , if it satisfies the condition*

$$\sum_{n=2}^\infty \{ (k+1) [n, q] - k - \alpha \} \varphi_{n-1} |a_n| \leq 1 - \alpha.$$

When  $q \rightarrow 1$ ,  $\beta = 0$ ,  $\lambda = 0$ , then we have the following known result, proved by Noor and Sarfraz [11].

**Corollary 2.2.** *A function  $f \in A$  and of the form (1.1) is in the class  $k-ST(A, B)$ , if it satisfies the condition*

$$\sum_{n=2}^\infty \{ 2(k+1)(n-1) + |n(B+1) - (A+1)| \} |a_n| \leq |B-A|.$$

When  $q \rightarrow 1, \lambda = 0, \beta = 0, A = 1 - 2\alpha, B = -1$  with  $0 \leq \alpha < 1$ , then we have the following known result, proved by Shams et-al. in [18].

**Corollary 2.3.** *A function  $f \in A$  and of the form (1.1) is in the class  $k - \mathcal{UST}(1 - 2\alpha, -1)$ , if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{n(k + 1) - (k + \alpha)\} |a_n| \leq 1 - \alpha,$$

where  $0 \leq \alpha < 1$  and  $k \geq 0$ .

When  $\lambda = 0, \beta = 0, A = 1 - 2\alpha, B = -1$  with  $0 \leq \alpha < 1$  and  $k = 0$ , then we have the following known result, proved by Selverman in [17].

**Corollary 2.4.** *A function  $f \in A$  and of the form (1.1) is in the class  $0 - \mathcal{UST}(1 - 2\alpha, -1)$ , if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{n - \alpha\} |a_n| \leq 1 - \alpha, \quad 0 \leq \alpha < 1.$$

**Theorem 2.2.** *If  $f(z) \in k - \mathcal{US}_q(\lambda, A, B, \beta)$  and is of the form (1.1). Then*

$$|a_n| \leq \prod_{j=0}^{n-2} \left( \frac{|L_1(k)(A - B) - 2[j, q]B|}{2[j + 1, q] \{q + \beta[j + 2, q]\} \varphi_{j+1}} \right), \quad n \geq 2, \tag{2.2}$$

where  $L_1(k)$  is defined by (1.3).

*Proof.* Let

$$\frac{z \partial_q R_q^\lambda f(z)}{R_q^\lambda f(z)} + \beta \frac{z^2 \partial_q^2 R_q^\lambda f(z)}{R_q^\lambda f(z)} = p(z). \tag{2.3}$$

Then

$$\begin{aligned} p(z) &< \frac{(A + 1)p_k(z) - (A - 1)}{(B + 1)p_k(z) - (B - 1)} \\ &= [(A + 1)p_k(z) - (A - 1)] [(B + 1)p_k(z) - (B - 1)]^{-1} \\ &= \frac{(A - 1)}{(B - 1)} \left[ 1 - \frac{(A + 1)}{(A - 1)} p_k(z) \right] \left[ 1 + \sum \left( \frac{(B + 1)}{(B - 1)} p_k(z) \right)^n \right] \\ &= \frac{(A - 1)}{(B - 1)} + \left( \frac{(A - 1)(B + 1)}{(B - 1)^2} - \frac{(A + 1)}{(B - 1)} \right) (p_k(z)) \\ &\quad + \left( \frac{(A - 1)(B + 1)^2}{(B - 1)^3} - \frac{(A + 1)(B + 1)}{(B - 1)^2} \right) (p_k(z))^2 + \dots \end{aligned}$$

By taking

$$p_k(z) = 1 + L_1(k)z + L_2(k)z^2 + \dots,$$

after some simplification, we obtain

$$p(z) < \sum_{n=1}^{\infty} \frac{-2(B + 1)^{n-1}}{(B - 1)^n} + \left\{ \sum_{n=1}^{\infty} \frac{-2n(A - B)(B + 1)^{n-1}}{(B - 1)^{n+1}} \right\} L_1(k) + \dots$$

Now we see that the series  $\sum_{n=1}^{\infty} \frac{-2(B+1)^{n-1}}{(B-1)^n}$  and  $\sum_{n=1}^{\infty} \frac{-2n(A-B)(B+1)^{n-1}}{(B-1)^{n+1}}$  are convergent and converge to 1 and  $\frac{A-B}{2}$  respectively. Therefore,

$$p(z) < 1 + \frac{A-B}{2} L_1(k)z + \dots$$

Now if  $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ , then by Lemma 1, we have

$$|c_n| \leq \frac{A-B}{2} L_1(k), \quad n \geq 1. \tag{2.4}$$

Now from (2.3), we have

$$z \partial_q R_q^\lambda f(z) + \beta z^2 \partial_q^2 R_q^\lambda f(z) = R_q^\lambda f(z) p(z),$$

which implies that

$$z + \sum_{n=2}^{\infty} \{[n, q] + \beta[n, q][n-1, q]\} \varphi_{n-1} a_n z^n = \left(1 + \sum_{n=1}^{\infty} c_n z^n\right) \left(z + \sum_{n=2}^{\infty} \varphi_{n-1} a_n z^n\right).$$

Equating coefficients of  $z^n$  on both sides, we have

$$[n-1, q] \{q + \beta[n, q]\} \varphi_{n-1} a_n = \sum_{j=1}^{n-1} \varphi_{j-1} a_j c_{n-j}, \quad a_1 = 1.$$

This implies that

$$|a_n| \leq \frac{1}{[n-1, q] \{q + \beta[n, q]\} \varphi_{n-1}} \sum_{j=1}^{n-1} \varphi_{j-1} |a_j| |c_{n-j}|, \quad a_1 = 1.$$

Using (2.4), we have

$$|a_n| \leq \frac{(A-B) |L_1(k)|}{2[n-1, q] \{q + \beta[n, q]\} \varphi_{n-1}} \sum_{j=1}^{n-1} \varphi_{j-1} |a_j|, \quad a_1 = 1. \tag{2.5}$$

Now we prove that

$$\frac{(A-B) |L_1(k)|}{2[n-1, q] \{q + \beta[n, q]\} \varphi_{n-1}} \sum_{j=1}^{n-1} \varphi_{j-1} |a_j| \leq \prod_{j=0}^{n-2} \left( \frac{|L_1(k)(A-B) - 2[j, q]B|}{2[j+1, q] \{q + \beta[j+2, q]\}} \right). \tag{2.6}$$

For this we use the induction method

For  $n = 2$ , from (2.5), we have

$$|a_2| \leq \frac{(A-B) |L_1(k)|}{2 \{q + \beta[2, q]\} \varphi_1}.$$

From (2.2), we have

$$|a_2| \leq \frac{(A-B) |L_1(k)|}{2 \{q + \beta[2, q]\} \varphi_1}.$$

For  $n = 3$  from (2.5), we have

$$\begin{aligned} |a_3| &\leq \frac{(A-B) |L_1(k)|}{2[2, q] \{q + \beta[3, q]\} \varphi_2} \{1 + \varphi_1 a_2\} \\ &\leq \frac{(A-B) |L_1(k)|}{2[2, q] \{q + \beta[3, q]\} \varphi_2} \left\{ 1 + \frac{(A-B) |L_1(k)|}{2 \{q + \beta[2, q]\}} \right\}. \end{aligned}$$

From (2.2), we have

$$\begin{aligned}
 |a_3| &\leq \frac{(A - B) |L_1(k)|}{2 \{q + \beta[2, q]\} \varphi_1} \left\{ \left( \frac{|(A - B)L_1(k) - 2B|}{2 [2, q] \{q + \beta[3, q]\} \varphi_2} \right) \right\} \\
 &\leq \frac{(A - B) |L_1(k)|}{2 \{q + \beta[2, q]\} \varphi_1} \left\{ \left( \frac{(A - B) |L_1(k)| + 2|B|}{2 [2, q] \{q + \beta[3, q]\} \varphi_2} \right) \right\} \\
 &\leq \frac{(A - B) |L_1(k)|}{2 [2, q] \{q + \beta[3, q]\} \varphi_2} \left\{ \frac{(A - B) |L_1(k)|}{2 \{q + \beta[2, q]\} \varphi_1} + \frac{1}{\{q + \beta[2, q]\} \varphi_1} \right\}.
 \end{aligned}$$

Let the hypothesis be true for  $n = m$ . From (2.4), we have

$$|a_m| \leq \frac{(A - B) |L_1(k)|}{2[m - 1, q] \{q + \beta[m, q]\} \varphi_{m-1}} \sum_{j=1}^{m-1} |a_j|, \quad a_1 = 1$$

From (2.2), we have

$$\begin{aligned}
 |a_m| &\leq \prod_{j=0}^{m-2} \left( \frac{|L_1(k)(A - B) - 2[j, q]B|}{2 [j + 1, q] \{q + \beta[j + 2, q]\} \varphi_{j+1}} \right), \quad n \geq 2 \\
 &\leq \prod_{j=0}^{m-2} \left( \frac{|L_1(k)| (A - B) + 2[j, q]}{2 [j + 1, q] \{q + \beta[j + 2, q]\} \varphi_{j+1}} \right), \quad n \geq 2.
 \end{aligned}$$

By the induction hypothesis, we have

$$\frac{(A - B) |L_1(k)|}{2[m - 1, q] \{q + \beta[m, q]\} \varphi_{m-1}} \sum_{j=1}^{m-1} \varphi_{j-1} |a_j| \leq \prod_{j=0}^{m-2} \left( \frac{|L_1(k)| (A - B) + 2[j, q]}{2 [j + 1, q] \{q + \beta[j + 2, q]\} \varphi_{j+1}} \right). \tag{2.7}$$

Multiplying both sides by (2.7)

$$\frac{(A - B) |L_1(k)| + 2[m - 1, q] \{q + \beta[m, q]\}}{2[m - 1, q] \{q + \beta[m, q]\} \varphi_{m-1}},$$

we have

$$\begin{aligned}
 &\prod_{j=0}^{m-2} \left( \frac{|L_1(k)| (A - B) + 2[j, q]}{2 [j + 1, q] \{q + \beta[j + 1, q]\} \varphi_{j+1}} \right) \\
 &\geq \left\{ \frac{(A - B) |L_1(k)| + 2[m - 1, q] \{q + \beta[m, q]\}}{2[m - 1, q] \{q + \beta[m, q]\} \varphi_{m-1}} \right\} \frac{(A - B) |L_1(k)|}{2[m - 1, q] \{q + \beta[m, q]\} \varphi_{m-1}} \sum_{j=1}^{m-1} \varphi_{j-1} |a_j| \\
 &= \frac{(A - B) |L_1(k)|}{2[m - 1, q] \{q + \beta[m, q]\} \varphi_{m-1}} \left[ \begin{aligned} &\left\{ \frac{(A - B) |L_1(k)| + 2[m - 1, q] \{q + \beta[m, q]\}}{2[m - 1, q] \{q + \beta[m, q]\} \varphi_{m-1}} \sum_{j=1}^{m-1} \varphi_{j-1} |a_j| \right\} \\ &+ \sum_{j=1}^{m-1} \varphi_{j-1} |a_j| \end{aligned} \right] \\
 &\geq \frac{(A - B) |L_1(k)|}{2[m - 1, q] \{q + \beta[m, q]\} \varphi_{m-1}} \left\{ |a_m| + \sum_{j=1}^{m-1} \varphi_{j-1} |a_j| \right\} \\
 &= \frac{(A - B) |L_1(k)|}{2[m - 1, q] \{q + \beta[m, q]\} \varphi_{m-1}} \sum_{j=1}^m \varphi_{j-1} |a_j|.
 \end{aligned}$$

That is,

$$\frac{(A - B) |L_1(k)|}{2[m - 1, q] \{q + \beta[m, q]\} \varphi_{m-1}} \sum_{j=1}^m \varphi_{j-1} |a_j| \leq \prod_{j=0}^{m-2} \left( \frac{|L_1(k)| (A - B) + 2[j, q]}{2[j + 1, q] \{q + \beta[j + 1, q]\} \varphi_{j+1}} \right).$$

which shows that inequality (2.7) is true for  $n = m + 1$ . Hence the required result. □

When  $q \rightarrow 1$ ,  $\lambda = 0$  and  $\beta = 0$ , then we have the following known result, proved by Noor and Sarfraz in [11].

**Corollary 2.5.** *A function  $f \in A$  and of the form (1.1) is in the class  $k - \mathcal{ST}[A, B]$ , if it satisfies the condition*

$$|a_n| \leq \prod_{j=0}^{n-2} \left( \frac{|L_1(k)(A - B) - 2jB|}{2(j + 1)} \right).$$

When  $\lambda = 0$ ,  $A = 1$ ,  $B = -1$  and  $\beta = 0$  then we have the following known result, proved by Kanas and Wisniowska in [6].

**Corollary 2.6.** *A function  $f \in A$  and of the form (1.1) is in the class  $k - \mathcal{UST}[A, B]$ , if it satisfies the condition*

$$|a_n| \leq \prod_{j=0}^{n-2} \left( \frac{|L_1(k) + j|}{(j + 1)} \right).$$

When  $\lambda = 0$ ,  $A = 1 - 2\alpha$ ,  $\beta = 0$ ,  $B = -1$  with  $0 \leq \alpha < 1$ , then we have the following known result, proved by Shams et al. in [18].

**Corollary 2.7.** *A function  $f \in A$  and of the form (1.1) is in the class  $\mathcal{SD}(k, \alpha)$ , if it satisfies the condition*

$$|a_n| \leq \prod_{j=0}^{n-2} \left( \frac{|L_1(k)(1 - \alpha) + j|}{(j + 1)} \right).$$

where  $0 \leq \alpha < 1$  and  $k \geq 0$ .

When  $\lambda = 0$ ,  $\beta = 0$ ,  $k = 0$ , then  $T_1(k) = 2$  and we get the following known result, proved in [4]

**Corollary 2.8.** *A function  $f \in A$  and of the form (1.1) is in the class  $\mathcal{S}^*[A, B]$ , if it satisfies the condition*

$$|a_n| \leq \prod_{j=0}^{n-2} \left( \frac{|(A - B) - jB|}{(j + 1)} \right), \quad -1 \leq B < A \leq 1.$$

When  $\lambda = 0$ ,  $\beta = 0$ ,  $A = 1 - 2\alpha$ ,  $B = -1$  with  $0 \leq \alpha < 1$  and  $k = 0$ , then we have the following known result, proved by Selverman in [17].

**Corollary 2.9.** *A function  $f \in A$  and of the form (1.1) is in the class  $\mathcal{S}^*(\alpha)$ , if it satisfies the condition*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} (j - 2\alpha)}{(n - 1)!}, \quad 0 \leq \alpha < 1.$$

**Theorem 2.3.** Let  $-1 \leq B < A \leq 1$  and  $0 \leq k < \infty$  be fixed and let  $f(z) \in k - \mathcal{US}_q(\lambda, A, B, \beta)$  and is of the form (1.1) Then for a complex number  $\mu$ .

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(A-B)L_1(k)}{2[2,q]\{q+[3,q]\beta\}\varphi_2} \left\{ \left[ 2 + \frac{2D(k)-(1+B)L_1(k)}{2} \left[ 2 + \frac{2D(k)-(1+B)L_1(k)}{2} \right. \right. \right. \\ \left. \left. \left. - \frac{(A-B)}{2\{q+\beta[2,q]\}} L_1(k) \left( 1 - \mu \frac{\varphi_2}{(\varphi_1)^2} \right) \right] \right\}, & (\mu > \delta_1), \\ \frac{(A-B)L_1(k)}{2[2,q]\{q+[3,q]\beta\}\varphi_2} & (\delta_1 \leq \mu \leq \delta_2), \\ \frac{(A-B)L_1(k)}{2[2,q]\{q+[3,q]\beta\}\varphi_2} \left[ \frac{2D(k)-(1+B)L_1(k)}{2} \right. \\ \left. \frac{(A-B)}{2\{q+\beta[2,q]\}} L_1(k) \left( 1 - \mu \frac{\varphi_2}{(\varphi_1)^2} \right) \right] & (\mu < \delta_2). \end{cases} \quad (2.8)$$

Where

$$\delta_1 = \frac{(\varphi_1)^2}{\varphi_2(A-B)L_1(k)} \left[ \begin{array}{c} (q + \beta[2, q]) \{2 + 2D(k) - (1 + B)L_1(k)\} \\ + (A - B)L_1(k) \end{array} \right], \quad (2.9)$$

$$\delta_2 = \frac{(\varphi_1)^2}{\varphi_2(A-B)L_1(k)} \left[ \begin{array}{c} (q + \beta[2, q]) \{2D(k) - (1 + B)L_1(k) - 2\} \\ + (A - B)L_1(k) \end{array} \right]. \quad (2.10)$$

and  $L_1(k)$ ,  $D(k)$  are defined in (1.3) and (1.4).

*Proof.* If  $f(z) \in k - \mathcal{US}_q(\lambda, A, B, \beta)$  then it follows that

$$\frac{z\partial_q R_q^\lambda f(z)}{R_q^\lambda f(z)} + \beta \frac{z^2 \partial_q^2 R_q^\lambda f(z)}{R_q^\lambda f(z)} \prec q_k(z) = 1 + \frac{A-B}{2} L_1(k)z + \frac{[2D(k) - (1+B)L_1(k)](A-B)}{4} L_1(k)z^2 + \dots \quad (2.11)$$

Now by the definition of subordination there exists a function  $w$  analytic in  $E$  with  $w(0) = 0$  and  $|w(z)| < 1$  such that

$$\frac{z\partial_q R_q^\lambda f(z)}{R_q^\lambda f(z)} + \beta \frac{z^2 \partial_q^2 R_q^\lambda f(z)}{R_q^\lambda f(z)} = 1 + \frac{A-B}{2} L_1(k)w(z) + \frac{[2D(k) - (1+B)L_1(k)](A-B)}{4} L_1(k)w^2(z) + \dots \quad (2.12)$$

Now from Lemma 3, equation (2.11) and equation (2.12), we have

$$a_2 = \frac{(A-B)L_1(k)}{2\{q+\beta[2,q]\}\varphi_1} c_1,$$

and

$$a_3 = \frac{(A-B)L_1(k)}{2[2,q]\{q+\beta[3,q]\}\varphi_2} \left\{ c_2 + \left\{ \frac{2D(k) - (1+B)L_1(k)}{2} + \frac{(A-B)}{2\{q+\beta[2,q]\}} L_1(k) \right\} c_1^2 \right\}.$$

Therefore

$$|a_3 - \mu a_2^2| = \frac{(A - B)L_1(k)}{2 [2, q] \{q + \beta[3, q]\} \varphi_2} \left| c_2 + \left\{ \frac{2D(k) - (1 + B)L_1(k)}{2} + \frac{(A - B)}{2 \{q + \beta[2, q]\}} L_1(k) \left( 1 - \mu \frac{\varphi_2}{(\varphi_1)^2} \right) \right\} c_1^2 \right|. \tag{2.13}$$

This gives

$$|a_3 - \mu a_2^2| = \frac{(A - B)L_1(k)}{2 [2, q] \{q + \beta[3, q]\} \varphi_2} \left| c_2 - c_1^2 + \left\{ 1 + \frac{2D(k) - (1 + B)L_1(k)}{2} + \frac{(A - B)}{2 \{q + \beta[2, q]\}} L_1(k) \left( 1 - \mu \frac{\varphi_2}{(\varphi_1)^2} \right) \right\} c_1^2 \right|. \tag{2.14}$$

Suppose that  $\mu > \delta_1$ , then using the estimate  $|c_2 - c_1^2| \leq 1$  from Lemma 3 and the well known estimate  $|c_1| \leq 1$  of the Schwarz lemma, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{(A - B)L_1(k)}{2 [2, q] \{q + \beta[3, q]\} \varphi_2} \left| \left\{ 2 + \frac{2D(k) - (1 + B)L_1(k)}{2} - \frac{(A - B)}{2 \{q + \beta[2, q]\}} L_1(k) \left( 1 - \mu \frac{\varphi_2}{(\varphi_1)^2} \right) \right\} \right|. \tag{2.15}$$

The inequality (2.15) is our required assertion (2.8) for  $\mu > \delta_1$ . On the other hand if  $\mu < \delta_2$ , then (2.13) gives

$$|a_3 - \mu a_2^2| \leq \frac{(A - B)L_1(k)}{2 [2, q] \{q + \beta[3, q]\} \varphi_2} \left[ |c_2| + \left\{ \frac{2D(k) - (1 + B)L_1(k)}{2} + \frac{(A - B)}{2 \{q + \beta[2, q]\}} L_1(k) \left( 1 - \mu \frac{\varphi_2}{(\varphi_1)^2} \right) \right\} |c_1|^2 \right].$$

Applying the estimates  $|c_2| \leq 1 - |c_1|^2$  of Lemma 3 and  $|c_1| \leq 1$ , we have

$$|a_3 - \mu a_2^2| \leq \frac{(A - B)L_1(k)}{2 [2, q] \{q + \beta[3, q]\} \varphi_2} \left[ \left\{ \frac{2D(k) - (1 + B)L_1(k)}{2} + \frac{(A - B)}{2 \{q + \beta[2, q]\}} L_1(k) \left( 1 - \mu \frac{\varphi_2}{(\varphi_1)^2} \right) \right\} \right].$$

This is the last inequality in (2.8). Finally if  $\delta_1 < \mu < \delta_2$ , then

$$\left| \frac{2D(k) - (1 + B)L_1(k)}{2} + \frac{(A - B)}{2 \{q + \beta[2, q]\}} L_1(k) \left( 1 - \mu \frac{\varphi_2}{(\varphi_1)^2} \right) \right| \leq 1.$$

Therefore (2.13), yields

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{(A - B)L_1(k)}{2 [2, q] \{q + \beta[3, q]\} \varphi_2} \{ |c_2| + |c_1|^2 \}, \\ &\leq \frac{(A - B)L_1(k)}{2 [2, q] \{q + \beta[3, q]\} \varphi_2} \{ 1 - |c_1|^2 + |c_1|^2 \}, \\ &\leq \frac{(A - B)L_1(k)}{2 [2, q] \{q + \beta[3, q]\} \varphi_2}. \end{aligned}$$

We get the middle inequality in (2.8). This completes the proof. □

**Theorem 2.4.** Let  $0 \leq k < \infty$ ,  $-1 \leq B < A \leq 1$ , be fixed and let  $f(z) \in k - \mathcal{US}_q(\lambda, A, B, \beta)$  and is of the form (1.1) Then for a complex number  $\mu$ .

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)L_1(k)}{2[2, q] \{q + [3, q]\beta\} \varphi_2} \max \{1, |2v - 1|\},$$

where  $v$  is given by (2.17).

*Proof.* From (2.13) we have

$$\begin{aligned} |a_3 - \mu a_2^2| &= \frac{(A-B)L_1(k)}{2[2, q] \{q + [3, q]\beta\} \varphi_2} \left| c_2 - \left\{ \frac{(1+B)L_1(k) - 2D(k)}{2} \right. \right. \\ &\quad \left. \left. - \frac{(A-B)}{2\{q + \beta[2, q]\}} L_1(k) \left( 1 - \mu \frac{\varphi_2}{(\varphi_1)^2} \right) \right\} c_1^2 \right|, \\ &= \frac{(A-B)L_1(k)}{2[2, q] \{q + [3, q]\beta\} \varphi_2} |c_2 - v c_1^2| \end{aligned} \quad (2.16)$$

where

$$v = \frac{(1+B)L_1(k) - 2D(k)}{2} - \frac{(A-B)}{2\{q + \beta[2, q]\}} L_1(k) \left( 1 - \mu \frac{\varphi_2}{(\varphi_1)^2} \right). \quad (2.17)$$

Applying the Lemma 2 on equation (2.16), we obtain the required result.  $\square$

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