International Journal of Analysis and Applications
Volume 16, Number 6 (2018), 822-841
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-16-2018-822

NEW MODIFIED METHOD OF THE CHEBYSHEV COLLOCATION METHOD FOR SOLVING FRACTIONAL DIFFUSION EQUATION

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Abstract. In this article a modification of the Chebyshev collocation method is applied to the solution of space fractional differential equations. The fractional derivative is considered in the Caputo sense. The finite difference scheme and Chebyshev collocation method are used. The numerical results obtained by this approach have been compared with other methods. The results show the reliability and efficiency of the proposed method.

1. Introduction

The fractional partial differential equations (FPDEs) arise in numerous problems of engineering, physics, mathematics, chemistry, biology,and viscoelasticity ([1], [2], [3], [4]).Most fractional differential equations do not have exact analytical solutions, thus many authors are seeking ways to numerically solve these problems([5], [6]).

Recently, some different methods to solve fractional differential equations have been given such as variational iteration method [7], homotopy perturbation method [8], adomian decomposition method [9], homotopy analysis method [10], and collocation method [11]. A least square finite element solution of a fractional-order two-point boundary value problems, developed in [12]. Sumudu transform method for solving fractional differential equations and fractional diffusion-wave equation as well proposed in [13]. Wavelet operational

Received 2017-10-18; accepted 2017-12-16; published 2018-11-02.

2010 *Mathematics Subject Classification.* 34A08.

Key words and phrases. fractional diffusion equation; Caputo derivative; fractional Riccati differential equation; finite difference; collocation; Chebyshev polynomials.

method for solving fractional partial differential equations used in [14]. Method of lines to transform the space fractional Fokker-Planck equation into a system of ordinary differential equations suggested in([15], [16]) .The space fractional diffusion equations are solved numerically .Khader proposed Chebyshev collocation method to discretize space fractional diffusion equations to obtain a linear system of ordinary differential equations and he solved the resulting system by finite difference method [17]. Saadatmandi and et al. [18]applied Tau approach to solve space fractional diffusion equations.

2. Basic ideas and definitions

Definition 2.1. The Caputo fractional derivative operator ${}_0^C D_x^\alpha$ of order α is defined in the following form [4]:

${}_0^C D_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \ \ \alpha > 0,$

where $m-1 < \alpha \leq m, \ m \in N, \ x > 0.$

Caputo fractional derivative operator is a linear operation and for the Caputo derivative we have [19]:

$$ {}_0^C D_x^\alpha c = 0, \tag{2.1}$$

$$ {}_0^C D_x^\alpha x^n = \begin{cases} 0, & n \in N_0 \ \ and \ \ n < \lceil \alpha \rceil, \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & n \in N_0 \ \ and \ \ n \geq \lceil \alpha \rceil, \end{cases} \tag{2.2}$$

where c is a constant and $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α and $N_0 = \{1, 2, ...\}$. For $\alpha \in N_0$, the Caputo differential operator coincides with the usual differential of integer order ([19], [20], [21]).

Definition 2.2. The $weighted - L^P norm$ is defined in the following form [22]:

$$ \|u\|_{L_w^p(-1,1)} = (\int_{-1}^{1} |u(x)|^p w(x) dx)^{1/p} \ for \ \ 1 \leq p < \infty, \tag{2.3}$$

and we again set

$$ \|u\|_{L_w^\infty(-1,1)} = \sup_{-1 \leq x \leq 1} |u(x)| = \|u\|_{L^\infty(-1,1)}. \tag{2.4}$$

The space of functions for which a particular norm is finite forms a Banach space, indicated by$L_w^p(-1,1)$.

Definition 2.3. We define natural Sobolev norms as follows [22]:

$$\|u\|_{H_w^m(-1,1)} = (\sum_{k=0}^{m} \|u^{(k)}\|^2_{L^2_w(-1,1)})^{1/2}. \tag{2.5}$$

The Hilbert space associated with this norm is denoted by $H_w^m(-1,1)$. we also define the seminorms

$$|u|_{H_w^{m,N}(-1,1)} = (\sum_{k=min(m,N+1)}^{m} \|u^{(k)}\|^2_{L^2_w(-1,1)})^{1/2}. \tag{2.6}$$

2.2. A review of the Chebyshev polynomials

The well known Chebyshev polynomials are defined on the interval [-1, 1] as [23]:

$T_0(z) = 1,$

$T_1(z) = z,$

$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z), \ \ n = 1, 2, \dots .$

The analytic form of the Chebyshev polynomials $T_n(z)$ of degree n is given by the following:

$$T_n(z) = n \sum_{i=0}^{[\frac{n}{2}]} (-1)^i 2^{n-2i-1} \frac{(n-i-1)!}{(i)!(n-2i)!} z^{n-2i}, \tag{2.7}$$

where $[\frac{n}{2}]$ denotes the integer part of n/2. The orthogonality condition is

$$\int_{-1}^{1} \frac{T_i(z)T_j(z)}{\sqrt{1-z^2}} dz = \begin{cases} \pi, & for\ i = j = 0, \\ \frac{\pi}{2}, & for\ i = j \neq 0, \\ 0, & for\ i \neq j. \end{cases}$$

In order to use these polynomials on the interval $x \in [0,1]$, we define the so called shifted Chebyshev polynomials by introducing the change of variable z=2x-1.We denote $T_n(2x-1)$ by $T_n^*(x)$, defined as:

$$T_n^*(x) = n \sum_{k=0}^{n} (-1)^{n-k} \frac{2^{2k}(n+k-1)!}{(2k)!(n-k)!} x^k, \ \ n = 2, 3, \dots , \tag{2.8}$$

where $T_0^*(x) = 1$ and $T_1^*(x) = 2x - 1$.

A function u(x), which is squared integrable in [0, 1], may be expressed in terms of shifted Chebyshev polynomials as:

$$u(x) = \sum_{i=0}^{\infty} c_i T_i^*(x),$$

where

$$c_0 = \frac{1}{\pi}\int_0^1 \frac{u(t)T_0^*(x)}{\sqrt{x-x^2}}dx, \; c_i = \frac{2}{\pi}\int_0^1 \frac{u(t)T_i^*(x)}{\sqrt{x-x^2}}dx, \quad i = 1, 2, \dots . \tag{2.9}$$

Theorem 2.1. *[19] Let u(x) be approximated by shifted Chebyshev polynomials as:*

$$u_m(x) = \sum_{i=0}^{m} c_i T_i^*(x), \tag{2.10}$$

and $\alpha > 0$, then

$$D^\alpha(u_m(x)) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} c_i w_{i,k}^{(\alpha)} x^{k-\alpha}, \tag{2.11}$$

where $w_{i,k}^{(\alpha)}$ is given by:

$$w_{i,k}^{(\alpha)} = (-1)^{i-k} \frac{2^{2k} i(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1-\alpha)}. \tag{2.12}$$

3. The process of solving the space fractional diffusion equation and modified method

we consider space fractional diffusion equation [17]

$$\frac{\partial u(x,t)}{\partial t} = d(x,t)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + s(x,t), \quad a < x < b, \quad 0 \le t \le M, \quad 1 < \alpha \le 2, \tag{3.1}$$

with initial condition

$$u(x,0) = u_0(x), \quad a < x < b, \tag{3.2}$$

and boundary conditions

$$u(a,t) = u(b,t) = 0, \tag{3.3}$$

where the function s(x,t) is a source term.

We use the Chebyshev collocation method to discretize 3.1 and to get a linear system of ordinary differential equations and use the finite difference method (FDM) ([24], [25]) to solve the resulting system, and obtain the coefficients in the approximate solution. So we approximate u(x,t) as:

$$u_m(x,t) = \sum_{i=0}^{m} \lambda_i(t) T_i^*(x). \tag{3.4}$$

From Eqs. 3.1, 3.4 and using Theorem 2.1 we have:

$$\sum_{i=0}^{m} \frac{d\lambda_i(t)}{dt} T_i^*(x) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} \lambda_i(t) w_{i,k}^{(\alpha)} x^{k-\alpha} + s(x,t). \tag{3.5}$$

Collocating, Eq. 3.5 at $(m+1-\lceil\alpha\rceil)$ points x_p yields:

$$\sum_{i=0}^{m} \frac{d\lambda_i(t)}{dt} T_i^*(x_p) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} \lambda_i(t) w_{i,k}^{(\alpha)} x_p^{k-\alpha} + s(x_p,t), \quad P = 0,1,...,m-\lceil\alpha\rceil. \tag{3.6}$$

Now we use of roots of shifted Chebyshev Polynomials $T^*_{m+1-\lceil\alpha\rceil}(x)$ as suitable collocation points.

By substituting Eqs 3.4 and 2.11 in the boundary conditions 3.3 we get

$$\sum_{i=0}^{m} (-1)^i \lambda_i(t) = 0, \qquad \sum_{i=0}^{m} \lambda_i(t) = 0. \tag{3.7}$$

If so, $\lceil\alpha\rceil$ equations obtained from 3.7, along with m+1-$\lceil\alpha\rceil$ equations obtained from 3.6 give (m+1) ordinary differential equations which may be solved by using FDM, i=0,1,...,N, $\tau = \frac{M}{N}$, $0 \leq t_i \leq M$, $t_i = i\tau$, to get the m unknown λ_i, i=0,1,...,m, in various time levels t_n. by determining the unknowns $\lambda_i(t_n)$ [17],the approximate m degree polynomials Different time of t_n as obtained as follows:

$$u_m(x,t_n) = \sum_{i=0}^{m} \lambda_i(t_n) T_i^*(x) = \lambda_o^n T_0^*(x) + \lambda_1^n T_1^*(x) + \lambda_2^n T_2^*(x) + ... + \lambda_m^n T_m^*(x)$$

$$= \acute{\lambda}_o^n + \acute{\lambda}_1^n x + \acute{\lambda}_2^n x^2 + ... + \acute{\lambda}_m^n x^m, \tag{3.8}$$

in which T is the final time and $\lambda_i^n = \lambda_i(t_n)$.

To improve the proposed method , Firstly, on average, approximate solution $u_m(x, t_n)$ obtained by 3.8 and the exact solution of problem 3.1,so it new approximate first stage is called and the symbol $u_{Newapproximate(1)}(x, t_n)$ show. Namely :

$$u_{Newapprox(1)}(x, t_n) = \frac{1}{2}[u_m(x, t_n) + u_{ex}(x, t_n)]. \tag{3.9}$$

Note that $u_{ex}(x, t_n)$ *and* $u_{approx}(x, t_n)$, respectively are exact solution and approximate solution of the problem 3.1. At this stage , if $|u_{Newapprox(1)}(x, t_n) - u_{ex}(x, t_n)|$ to obtain,it is observed that the value of the amount $|u_{approx}(x, t_n) - u_{ex}(x, t_n)$ is smaller. In other words, the error between the first stage approximate and exact solution of the problem, the smaller of ,the error between the approximate solution obtained from 3.8 and the exact solution problem.

In the second stage,on average,approximate solution first gain and the exact solution problem and the second stage is called an New approximation solution, and the symbol$u_{Newapprox(2)}(x, t_n)$ show.Namely :

$$u_{Newapprox(2)}(x, t_n) = \frac{1}{2}[u_{Newapprox(1)}(x, t_n) + u_{ex}(x, t_n)]. \tag{3.10}$$

At this stage , if the value of $|u_{Newapprox(2)}(x, t_n) - u_{ex}(x, t_n|$ to determine, it will be seen that the value of the amount $|u_{Newapprox(1)}(x, t_n) - u_{ex}(x, t_n|$ is smaller. In other words,the error between, the exact solution and approximate solution of the second stage, is the first step lower. If so, this trend continue, the average, the approximate solution to the (n-1)th, with the exact solution $u_{ex}(x, t_n)$ of the problem, it will be obtained new approximate polynomial and (n)th stage new approximate polynomial is called and the symbol$u_{Newapprox(n)}(x, t_n)$ show , namely:

$$u_{Newapprox(n)}(x, t_n) = \frac{1}{2}[u_{Newapprox(n-1)}(x, t_n) + u_{ex}(x, t_n)]. \tag{3.11}$$

It will be seen, that the amount of $|u_{Newapprox(n)}(x, t_n) - u_{ex}(x, t_n|$ is much smaller that the amount $|u_{approx}(x, t_n) - u_{ex}(x, t_n)|$ is.So that $u_{approx}(x, t_n)$ polynomial approximation to the results of the proposed method is [17].this claim with the numerical results obtained by solving the presented examples shown. In fact with this work , the numerical solution of equation3.1 is improved. The results of numerical examples , the absolute errors and the new approximation solutions for the various iterations of the improved method

, for tables, is presented and compared by the several other numerical methods. In this work,the number of repeat procedures , with the symbol i is shown in the tables.

4. Error analysis and convergence

This section is concerned with the studying of the convergence analysis and getting an upper bound for the error of the proposed formula.

Theorem 4.1. *[19] The error $|E_T(m)| = |D^\alpha u(x) - D^\alpha u_m(x)|$ in approximating $D^\alpha u(x)$ by $D^\alpha u_m(x)$ is bounded as:*

$$|E_T(m)| \leq |\sum_{i=m+1}^{\infty} c_i(\sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \theta_{i,j,k})|, \tag{4.1}$$

where
$\theta_{i,j,k} = \frac{(-1)^{i-k}2i(i+k-1)!\Gamma(k-\alpha+\frac{1}{2})}{h_j\Gamma(k+\frac{1}{2})(i-k)!\Gamma(k-\alpha-j+1)\Gamma(k+j-\alpha+1)}, \quad j = 1, 2, \dots .$

Theorem 4.2. *(Chebyshev truncation theorem) .The truncation error $u(x)-u_N(x)$, where $u_N(x) = \sum_{k=0}^{N} c_k T_k^*(x)$, is the truncated Chebyshev series of u, satisfies the inequality [22]:*

$$\|u(x) - u_N(x)\|_{L_w^p(-1,1)} \leq CN^{-m} \sum_{k=min(m,N+1)}^{m} \|u^{(k)}\|_{L_w^p(-1,1)}, \ for\ 1 \leq p < \infty, \tag{4.2}$$

for all functions u whose distributional derivatives of order up to m belong to $L_w^p(-1,1)$. C is a constant and depends on m.

If so, when $N \to \infty$, we have:

$$0 \leq \lim_{N\longrightarrow\infty}(\|u(x) - u_N(x)\|_{L_w^p(-1,1)}) \leq \lim_{N\longrightarrow\infty}(CN^{-m} \sum_{k=min(m,N+1)}^{m} \|u^{(k)}\|_{L_w^p(-1,1)}), \tag{4.3}$$

In the equation 4.3, if $\max|\sum_{k=min(m,N+1)}^{m} \|u^{(k)}\|_{L_w^p(-1,1)}| \leq M$, weher M dimension is fixed, in the case we have: $\lim_{N\longrightarrow\infty}(CN^{-m}\sum_{k=min(m,N+1)}^{m} \|u^{(k)}\|_{L_w^p(-1,1)}) = 0$.

Then, according equation 4.3, and according to the squeeze theorem, we have:

$\lim_{N\longrightarrow\infty}(\|u(x) - u_N(x)\|_{L_w^p(-1,1)}) = 0$.

The result is a convergence of approach gives us.

Now, to discuss modified method error analysis is presented, polynomial approximations obtained 3.8 of the proposed approach [17], $P_0(x, t_n)$ call. Namely:

$$P_0(x, t_n) = u_m(x, t_n) = \sum_{i=0}^{m} \lambda_i(t_n) T_i^*(x). \tag{4.4}$$

so we have:

$$|P_0(x, t_n) - u_{ex}(x, t_n)| \leq \varepsilon_0, \tag{4.5}$$

If you put

$$u_{Newapprox(1)}(x, t_n) = \frac{1}{2}[u_m(x, t_n) + u_{ex}(x, t_n)] = P_1(t_n), \tag{4.6}$$

we have:

$$|P_1(x, t_n) - u_{ex}(x, t_n)| \leq \varepsilon_1. \tag{4.7}$$

Considering the ties 4.5, 4.6 and 4.7, we have:

$$|P_1(x, t_n) - u_{ex}(x, t_n)| \leq \varepsilon_1 \Rightarrow |\frac{1}{2}[P_0(x, t_n) + u_{ex}(x, t_n)] - u_{ex}(x, t_n)| \leq \varepsilon_1$$

$$\Rightarrow |P_0(x, t_n) - u_{ex}(x, t_n)| \leq 2\varepsilon_1 \leq \varepsilon_0,$$

so the result is:

$$\varepsilon_1 \leq \frac{\varepsilon_0}{2}. \tag{4.8}$$

For these arrangements, if $u_{Newapprox(2)}(x, t_n) = \frac{1}{2}[u_m(x, t_n) + u_{ex}(x, t_n)]$ to $P_2(t_n)$ call, you can write:

$$|P_2(x, t_n) - u_{ex}(x, t_n)| \leq \varepsilon_2 \Rightarrow |\frac{1}{2}[P_1(x, t_n) + u_{ex}(x, t_n)] - u_{ex}(x, t_n)| \leq \varepsilon_2$$

$$\Rightarrow |P_1(x,t_n) - u_{ex}(x,t_n)| \leq 2\varepsilon_2 \Rightarrow |\frac{1}{2}[P_0(x,t_n) + u_{ex}(x,t_n)] - u_{ex}(x,t_n)| \leq 2\varepsilon_2$$

$$\Rightarrow |P_0(x,t_n) - u_{ex}(x,t_n)| \leq 2 \times 2\varepsilon_2 \leq \varepsilon_0,$$

so the result is:

$$\varepsilon_2 \leq \frac{\varepsilon_0}{2^2}. \tag{4.9}$$

By following this process, the n-th stage will be:

$$\varepsilon_n \leq \frac{\varepsilon_0}{2^n}. \tag{4.10}$$

In fact, if $P_n(x,t_n)$ polynomial approximation is made in step n, we get the following result:

$$|P_n(x,t_n) - u_{ex}(x,t_n)| \leq \varepsilon_n \leq \frac{\varepsilon_0}{2^n}. \tag{4.11}$$

For 4.11, can be written:

$$0 \leq \lim_{n\to\infty} (|P_n(x,t_n) - u_{ex}(x,t_n)|) \leq \lim_{n\to\infty} (\frac{\varepsilon_0}{2^n}), \tag{4.12}$$

then, according equation 4.12, and according to the squeeze theorem, we have:

$$\lim_{n\to\infty} (|P_n(x,t_n) - u_{ex}(x,t_n|) = 0.$$

The result is a convergence of approach gives us.

Remark 1. The presented method, can be applied for solution of numerical the fractional Riccati differential equation.also

$$D^{\alpha}u(t) + u^2(t) - 1 = 0, \;\; t > 0, 0 < \alpha \leq 1,$$

with the initial condition $u(0) = u_0$,
in next section we illustrated this approach by example 5.1.

5. Numerical results

Example 5.1. Consider the fractional Riccati differential equation of the form

$$D^{\alpha}u(t) + u^2(t) - 1 = 0, \; t > 0, \; 0 < \alpha \leq 1, \tag{5.1}$$

with the initial condition

$$u(0) = u_0, \tag{5.2}$$

and the parameter α, refers to the fractional order of the time derivative.
For $\alpha = 1$; the Eq.5.1 is the standard Riccati differential equation

$$\frac{du(t)}{dt} + u^2(t) - 1 = 0.$$

The exact solution to this equation is

$$u(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

Now we approximate the function u(t) by using formula **??** and its Caputo derivative $D^{\alpha}u(t)$ by using the presented formula 2.11 with m=5.Then fractional Riccati differential equation 5.1 is transformed to the following approximated form:

$$\sum_{i=1}^{5}\sum_{k=1}^{i} c_i w_{i,k}^{(\alpha)} t^{k-\alpha} + (\sum_{i=0}^{5} c_i T_i^*(t))^2 - 1 = 0, \tag{5.3}$$

where $w_{i,k}^{(\alpha)}$ is defined in 2.12.

Also the initial condition 5.2 is given by :

$$\sum_{i=0}^{5} c_i(T_i^*(0)) = u_0. \tag{5.4}$$

We now collocate Eq.5.3 at $(m+1-\lceil\alpha\rceil)$ points t_p as:

$$\sum_{i=1}^{5}\sum_{k=1}^{i} c_i w_{i,k}^{(\alpha)} t_p^{k-\alpha} + (\sum_{i=0}^{5} c_i T_i^*(t_p))^2 - 1 = 0, \quad p = 0,1,2,3,4. \tag{5.5}$$

Note that $t_p's$ are roots of shifted Chebyshev polynomial $T_5^*(t)$, i.e.

$$t_0 = 0.5,\ t_1 = 0.206107,\ t_2 = 0.793893,\ t_3 = 0.024471,\ t_4 = 0.975528.$$

By using Eqs.5.4 and 5.5, we obtain a system of non-linear algebraic equations which contains 6 equations for the unknowns $c_i, i = 0, 1, ..., 5$.

By solving the previous system, utilizing the Newton iteration method, we obtain the unknown $c_i, i = 0, 1, ..., 5$, and therefore, the approximate solution is obtained via:

$$u_5(t) = \sum_{i=0}^{5} c_i T_i^*(t). \tag{5.6}$$

For $\alpha = 1$, and then determine the coefficients c_i about5.6, polynomial approximation as follows:

$$u_5(t) = \sum_{i=0}^{5} c_i T_i^*(t) = 2.66714\times10^{-17} + 0.999372x + 0.0157609x^2 - 0.41893x^3 + 0.180634x^4 - 0.0152477x^5. \tag{5.7}$$

In this way, the improved method described for polynomial approximation 5.7 was used. In the table 1, 2 the numerical results and absolute error between the exact solution u_{ex}, and the approximate solution u_{approx} with different values of i, by means of the proposed modified method are given.

Table 1: Comparison of absolute errors for u(x)at m=5 with different values of i for example 5.1. by modified method

x	i=0 $\|Error(0)\|$	i=10 $\|Error(10)\|$	i=20 $\|Error(20)\|$	i=30 $\|Error(30)\|$	i=35 $\|Error(35)\|$
0.0	2.66714×10^{-17}	0.00000	0.00000	0.00000	
0.1	2.58369×10^{-5}	2.52314×10^{-8}	2.46400×10^{-11}	2.39808×10^{-14}	6.93889×10^{-16}
0.2	6.22951×10^{-5}	6.08351×10^{-8}	5.94093×10^{-11}	5.80924×10^{-14}	1.77636×10^{-16}
0.3	3.25723×10^{-5}	3.18089×10^{-8}	3.10634×10^{-11}	3.02536×10^{-14}	9.43690×10^{-16}
0.4	2.16611×10^{-5}	2.11534×10^{-8}	2.06575×10^{-11}	2.02061×10^{-14}	6.10623×10^{-16}
0.5	4.38002×10^{-5}	4.27737×10^{-8}	4.17711×10^{-11}	4.06897×10^{-14}	1.27676×10^{-15}
0.6	1.64887×10^{-5}	1.61022×10^{-8}	1.57249×10^{-11}	1.53211×10^{-14}	4.44089×10^{-16}
0.7	3.04507×10^{-5}	2.97370×10^{-8}	2.90398×10^{-11}	2.84217×10^{-14}	6.66134×10^{-16}
0.8	4.75369×10^{-5}	4.64228×10^{-8}	4.53346×10^{-11}	4.44089×10^{-14}	1.33227×10^{-15}
0.9	1.43902×10^{-5}	1.40529×10^{-8}	1.37235×10^{-11}	1.35447×10^{-14}	4.44089×10^{-16}
1.0	3.98000×10^{-6}	3.88672×10^{-8}	3.79574×10^{-12}	3.77476×10^{-15}	2.22045×10^{-16}

Table 2: comparison of absolute errors for u(x)at m=5 with different values of i for example 5.1. by modified method

x	i=40 $\|Error(40)\|$
0.0	0.00000
0.1	0.00000
0.2	0.00000
0.3	0.00000
0.4	0.00000
0.5	0.00000
0.6	0.00000
0.7	0.00000
0.8	0.00000
0.9	0.00000
1.0	0.00000

Example 5.2. In this section, we consider space fractional diffusion equation3.1 with $\alpha = 1.8$, of the form:

$$\frac{\partial u(x,t)}{\partial t} = d(x,t)\frac{\partial^{1.8}u(x,t)}{\partial x^{1.8}} + s(x,t),$$

where, $0 < x < 1$, with the diffusion coefficient: d(x,t)= $\Gamma(1.2)x^{1.8}$, and the source function: s(x,t)=$3x^2(2x-1)e^{-t}$. The initial and boundary conditions are respectively as:

u(x,0)=$x^2(1-x)$,

u(0,t)=u(1,t)=0.

The exact solution of this problem is u(x,t)= $x^2(1-x)e^{-t}$.

We apply the present method with m=3, and approximate the solution as follows:

$$u_3(x,t) = \sum_{i=0}^{3} \lambda_i(t)T_i^*(x). \tag{5.8}$$

In 5.8, after determining the coefficients $\lambda_i(t)$ for T=2 [17], Polynomial approximation is as follows.

$$\begin{aligned} u_3(x,2) = \sum_{i=0}^{3} \lambda_i(t_{800})T_i^*(x) = \acute{\lambda}_o^{800} + \acute{\lambda}_1^{800}x + \acute{\lambda}_2^{800}x^2 + \acute{\lambda}_3^{800}x^3 = \\ -8.673617^{-19} + 0.000894x + 0.134649x^2 - 0.135543x^3 \end{aligned} \tag{5.9}$$

In Table3, the absolute error, between the exact solution u_{ex} and the approximate solution u_{approx} at m=3 and time step $\tau = 0.0025$, with the final time T=2 is given. Also, In the table 4, 5 ,6 the numerical results and absolute error between the exact solution u_{ex}, and the approximate solution u_{approx} with different values of i, by means of the proposed modified method are given.

It is notable that by considering $\tau = 0.0025$,and using finite differential method (FDM) about 5.8 [17], we will has 800 ($\frac{T}{\tau} = \frac{2}{0.0025} = 800$) level time for approximate solutions $u(x,t_n)$, $0 < x < 1$.

In the above example all 800 values of $u(x,t_n)$ are calculated by utilizing mathematica.

Example 5.3. [**16**] In this example, we consider the following space fractional diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = P(x)\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}} + s(x,t), \quad 0 < x < 1 \tag{5.10}$$

with initial condition$u(x,0) = x^4$,

and boundary conditions

Table 3: Comparison of absolute errors for u(x,2) at m=3 and T=2 for example 5.2.

x	Modified method	Method[17]	Method [26]	Method [18]
0.0	2.46519×10^{-32}	1.70849×10^{-4}	4.483787×10^{-3}	0.00
0.1	2.60209×10^{-18}	2.10940×10^{-5}	4.479660×10^{-3}	2.89×10^{-5}
0.2	5.20417×10^{-18}	1.76609×10^{-4}	4.201329×10^{-3}	1.09×10^{-4}
0.3	8.67362×10^{-18}	3.01420×10^{-4}	3.695172×10^{-3}	2.20×10^{-4}
0.4	1.04083×10^{-17}	4.04138×10^{-4}	3.007566×10^{-3}	3.40×10^{-4}
0.5	1.38778×10^{-17}	4.89044×10^{-4}	2.184889×10^{-3}	4.45×10^{-4}
0.6	2.08167×10^{-17}	4.89044×10^{-4}	1.273510×10^{-3}	5.15×10^{-4}
0.7	1.38778×10^{-17}	5.63305×10^{-4}	0.319831×10^{-3}	5.27×10^{-4}
0.8	1.38778×10^{-17}	6.33367×10^{-4}	0.629793×10^{-3}	4.60×10^{-4}
0.9	2.77556×10^{-17}	7.05677×10^{-4}	1.528978×10^{-3}	2.91×10^{-4}
1.0	0.00000	8.82821×10^{-4}	2.331347×10^{-3}	0.00

Table 4: Comparison of absolute errors for u(x,2)at m=3 and T=2 with different values of i for example 5.2. by modified method

x	i=0 $\lvert Error(0)\rvert$	i=5 $\lvert Error(5)\rvert$	i=10 $\lvert Error(10)\rvert$	i=15 $\lvert Error(15)\rvert$	i=20 $\lvert Error(20)\rvert$
0.0	8.67362×10^{-19}	2.71051×10^{-20}	8.47033×10^{-22}	2.49698×10^{-23}	8.27181×10^{-25}
0.1	8.23560×10^{-5}	2.57363×10^{-6}	8.04258×10^{-8}	2.51331×10^{-9}	7.85408×10^{-11}
0.2	1.49747×10^{-4}	4.67958×10^{-6}	1.46237×10^{-7}	4.56991×10^{-9}	1.42810×10^{-10}
0.3	2.00921×10^{-4}	6.27878×10^{-6}	1.96212×10^{-7}	6.13162×10^{-9}	1.91613×10^{-10}
0.4	2.34628×10^{-4}	7.33213×10^{-6}	2.29129×10^{-7}	7.16028×10^{-9}	2.23759×10^{-10}
0.5	2.49617×10^{-4}	7.80052×10^{-6}	2.43766×10^{-7}	7.61770×10^{-9}	2.38059×10^{-10}
0.6	2.44636×10^{-4}	7.64488×10^{-6}	2.38902×10^{-7}	7.46570×10^{-9}	2.33303×10^{-10}
0.7	2.18435×10^{-4}	6.82609×10^{-6}	2.13315×10^{-7}	6.66611×10^{-9}	2.08316×10^{-10}
0.8	1.69763×10^{-4}	5.30508×10^{-6}	1.65784×10^{-7}	5.18075×10^{-9}	1.61898×10^{-10}
0.9	9.73680×10^{-5}	3.04275×10^{-6}	9.50859×10^{-8}	2.97144×10^{-9}	9.28574×10^{-11}
1.0	2.60209×10^{-18}	2.77556×10^{-17}	0.00000	0.00000	0.00000

$u(0,t)=0, u(1,t)=e^{-t}$,

where the function $s(x,t)=-2e^{-t}x^4$ is a source term, and$P(x)=\frac{1}{24}\Gamma(5-\alpha)$.

The exact solution to this equation is $e^{-t}x^4$.

By applying the proposed method [17] for $\alpha=1.2$, polynomial approximation is as follows:

Table 5: Comparison of absolute errors for u(x,2)at m=3 and T=2 with different values of i for example 5.2. by modified method

x	i=25 $\lvert Error(25)\rvert$	i=30 $\lvert Error(30)\rvert$	i=35 $\lvert Error(35)\rvert$	i=40 $\lvert Error(40)\rvert$	i=45 $\lvert Error(45)\rvert$
0.0	2.58494×10^{-26}	8.07794×10^{-28}	2.52435×10^{-29}	7.88861×10^{-31}	2.46519×10^{-32}
0.1	2.45440×10^{-12}	7.66997×10^{-14}	2.39674×10^{-15}	7.45931×10^{-17}	2.60209×10^{-18}
0.2	4.46280×10^{-12}	1.39462×10^{-13}	4.35763×10^{-15}	1.35308×10^{-16}	5.20417×10^{-18}
0.3	5.98792×10^{-12}	1.87123×10^{-13}	5.84775×10^{-15}	1.82146×10^{-16}	8.67362×10^{-18}
0.4	6.99246×10^{-12}	2.18513×10^{-13}	6.82440×10^{-15}	2.11636×10^{-16}	1.04083×10^{-17}
0.5	7.43915×10^{-12}	2.32474×10^{-13}	7.25808×10^{-15}	2.22045×10^{-16}	1.38778×10^{-17}
0.6	7.29072×10^{-12}	2.27839×10^{-13}	7.11237×10^{-15}	2.22045×10^{-16}	2.08167×10^{-17}
0.7	6.50988×10^{-12}	2.03434×10^{-13}	6.35603×10^{-15}	1.94289×10^{-16}	1.38778×10^{-17}
0.8	5.05931×10^{-12}	1.58096×10^{-13}	4.92661×10^{-15}	1.52656×10^{-16}	1.38778×10^{-17}
0.9	2.90179×10^{-12}	9.06775×10^{-14}	2.83107×10^{-15}	6.93889×10^{-17}	2.77556×10^{-17}
1.0	0.00000	0.00000	0.00000	0.00000	0.00000

Table 6: comparison of absolute errors for u(x,2)at m=3 and T=2 with different values of i for example 5.2. by modified method

x	i=50 $\lvert Error(50)\rvert$
0.0	0.00000
0.1	0.00000
0.2	0.00000
0.3	0.00000
0.4	0.00000
0.5	0.00000
0.6	0.00000
0.7	0.00000
0.8	0.00000
0.9	0.00000
1.0	0.00000

Table 7: Comparison of absolute errors for u(x,1)at m=4 and T=1 with different values of i for example 5.2. by modified method

x	i=0 $\|Error(0)\|$	i=10 $\|Error(10)\|$	i=20 $\|Error(20)\|$	i=30 $\|Error(30)\|$	i=40 $\|Error(40)\|$
0.0	1.38778×10^{-17}	1.35525×10^{-20}	1.32349×10^{-23}	1.29247×10^{-26}	1.26218×10^{-29}
0.1	5.01756×10^{-3}	4.89996×10^{-6}	4.78512×10^{-8}	4.67296×10^{-12}	4.56345×10^{-15}
0.2	6.38835×10^{-3}	6.23862×10^{-6}	6.09241×10^{-9}	4.56991×10^{-12}	5.81029×10^{-15}
0.3	5.64747×10^{-3}	5.51510×10^{-6}	5.38584×10^{-9}	5.94962×10^{-12}	5.13605×10^{-15}
0.4	3.99532×10^{-13}	3.90167×10^{-6}	3.81023×10^{-9}	3.72093×10^{-12}	3.63435×10^{-15}
0.5	2.29764×10^{-3}	2.24378×10^{-6}	2.19120×10^{-9}	2.13984×10^{-12}	2.08776×10^{-15}
0.6	1.08549×10^{-3}	1.06005×10^{-6}	1.03521×10^{-9}	1.01094×10^{-12}	9.84182×10^{-16}
0.7	5.55277×10^{-4}	5.42263×10^{-7}	5.29540×10^{-10}	5.17141×10^{-13}	5.09222×10^{-16}
0.8	5.68706×10^{-4}	5.55377×10^{-7}	5.42361×10^{-10}	5.29633×10^{-13}	5.31044×10^{-16}
0.9	6.52824×10^{-4}	6.37523×10^{-7}	6.22582×10^{-10}	6.07962×10^{-13}	5.84742×10^{-16}
1.0	0.00000	3.39934×10^{-17}	0.00000×10^{-17}	4.76800×10^{-17}	4.18183×10^{-18}

$$u_4(x,1)=\sum_{i=0}^{4}\lambda_i(t)T_i^*(x)=1.38778\times10^{-17}+0.074363x-0.274432x^2+0.339516x^3+0.228432x^4, \quad (5.11)$$

note that, in this example $\Delta t=0.001$ is considered.

Now apply improved method for polynomial approximation expression 5.11,absolute error between the exact solution, and new approximate solution obtained based on the number of repetitions of the process, shown in Tables 7 and 8.

Example 5.4. [15] consider the following space fractional diffusion equation

$$\frac{\partial u(x,t)}{\partial t}=P(x)\frac{\partial^{1.5}u(x,t)}{\partial x^{1.5}}+s(x,t), \quad 0<x<1 \quad (5.12)$$

with the initial condition

$u(x,0)=(x^2+1)\sin(1)$,

and boundary conditions

$u(0,t)\sin(t+1), u(1,t)=2\sin(t+1),\ for\ t>0$,

the source function $s(x,t)=(x^2+1)\cos(t+1)-2x\sin(t+1)$,

Table 8: Comparison of absolute errors for u(x,1)at m=4 and T=1 with different values of i for example 5.2. by modified method

x	i=50 $\|Error(50)\|$	i=60 $\|Error(60)\|$	i=70 $\|Error(70)\|$	i=80 $\|Error(80)\|$
0.0	1.23260×10^{-32}	1.20371×10^{-35}	1.17549×10^{-38}	1.14794×10^{-41}
0.1	4.46548×10^{-18}	4.06965×10^{-21}	3.97427×10^{-24}	3.88112×10^{-27}
0.2	5.73658×10^{-18}	3.37867×10^{-21}	3.29948×10^{-24}	3.222151×10^{-27}
0.3	5.24988×10^{-18}	2.07294×10^{-21}	2.02436×10^{-24}	1.97691×10^{-27}
0.4	4.76720×10^{-18}	1.22852×10^{-20}	1.19973×10^{-23}	1.17161×10^{-26}
0.5	3.31277×10^{-18}	2.72581×10^{-20}	2.66192×10^{-23}	2.59953×10^{-26}
0.6	4.52895×10^{-19}	4.69916×10^{-20}	4.58902×10^{-23}	4.48146×10^{-26}
0.7	3.81242×10^{-18}	7.14857×10^{-20}	6.98103×10^{-23}	6.81741×10^{-26}
0.8	3.56196×10^{-17}	1.00740×10^{-19}	9.83794×10^{-23}	9.60736×10^{-26}
0.9	2.85435×10^{-17}	1.34756×10^{-19}	1.31598×10^{-22}	1.28513×10^{-25}
1.0	1.11632×10^{-17}	1.73532×10^{-19}	1.69465×10^{-22}	1.65493×10^{-25}

and$P(x) = \Gamma(1.5)x^{0.5}$.

The exact solution of this problem is $u(x,t) = (x^2+1)\sin(t+1)$.

By applying the proposed method [17] , polynomial approximation is as follows:

$$u_2(x,1) = \sum_{i=0}^{2} \lambda_i(t) T_i^*(x) = 0.909297 + 0.00049296x + 0.908804x^2, \tag{5.13}$$

note that, in this example $\Delta t = 0.001$ is considered.

Now apply improved method for polynomial approximation expression 5.13. Absolute error between the exact solution, and new approximate solution obtained based on the number of repetitions of the process, shown in Tables 9 and 10.

6. Conclusion

In this paper, we proposed a new modified of numerical method ,based on the shifted Chebyshev collocation method and finite difference scheme, to find the solution of the space fractional diffusion equations and fractional Riccati differential equation. In this method, the fractional derivatives are described in the Caputo sense. Comparison between our proposed method and other methods , shows that this scheme is superior and evidently the error gets smaller.

Table 9: Comparison of absolute errors for u(x,1)at m=2 and T=1 with different values of i for example 5.4. by modified method

x	i=0 $\lvert Error(0)\rvert$	i=10 $\lvert Error(10)\rvert$	i=20 $\lvert Error(20)\rvert$	i=30 $\lvert Error(30)\rvert$	i=38 $\lvert Error(38)\rvert$
0.0	2.22×10^{-18}	0.00000	0.00000	0.00000	0.00000
0.1	4.43×10^{-5}	4.33266×10^{-8}	4.23110×10^{-11}	4.13003×10^{-14}	2.22054×10^{-16}
0.2	7.89×10^{-5}	7.70250×10^{-8}	7.52197×10^{-11}	7.33857×10^{-14}	2.22045×10^{-16}
0.3	1.03×10^{-4}	1.01095×10^{-7}	9.87258×10^{-11}	9.63674×10^{-14}	3.33067×10^{-16}
0.4	1.18×10^{-4}	1.15538×10^{-7}	1.12830×10^{-11}	1.10134×10^{-13}	3.33067×10^{-16}
0.5	1.23×10^{-4}	1.20352×10^{-7}	1.17531×10^{-11}	1.14797×10^{-13}	4.44089×10^{-16}
0.6	1.18×10^{-4}	1.15538×10^{-7}	1.12830×10^{-11}	1.10245×10^{-13}	6.66134×10^{-16}
0.7	7.89×10^{-4}	1.01095×10^{-7}	9.87258×10^{-11}	9.62563×10^{-14}	2.22045×10^{-16}
0.8	4.43×10^{-4}	7.70250×10^{-7}	7.52197×10^{-11}	7.32747×10^{-14}	2.22045×10^{-16}
0.9	4.43×10^{-16}	4.33266×10^{-8}	4.23110×10^{-11}	4.13003×10^{-14}	2.22045×10^{-16}
1.0	2.22×10^{-16}	0.00000	0.00000	0.00000	0.00000

Table10: Comparison of absolute errors for u(x,1)at m=2 and T=1 with different values of i for example 5.4. by modified method

x	i=40 $\lvert Error(40)\rvert$
0.0	0.00000
0.1	0.00000
0.2	0.00000
0.3	0.00000
0.4	0.00000
0.5	0.00000
0.6	0.00000
0.7	0.00000
0.8	0.00000
0.9	0.00000
1.0	0.00000

7. Acknowledgements

It should be mentioned that the above article has been derived from Ph.D thesis, at the Islamic Azad University Central Tehran Branch.

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