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# STABILITY OF EULER-LAGRANGE-JENSEN'S $(a, b)$- SEXTIC FUNCTIONAL EQUATION IN MULTI-BANACH SPACES

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Abstract. In this paper, we prove the Hyers-Ulam Stability of Euler-Lagrange-Jensen's $(a, b)$-Sextic Functional Equation in Multi-Banach Spaces.

## 1. Introduction and Preliminaries

The theory of stability is an important branch of the qualitative theory of functional equations. The concept of stability for a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem of functional equation was raised by S.M. Ulam [17] about seventy seven years ago. Since then, this question has attracted the attention of many researchers. Note that the affirmtive solution to this question was given in the next year by D.H. Hyers [5] in 1941. In the year 1950, T. Aoki [1] generalized Hyers theorem for additive mappings. The result of Hyers was generalized independently by Th.M.Rassias [14] for linear mappings by considering an unbounded Cauchy difference. In 1994, a further generalization of Th.M. Rassias theorem was obtained by P.Gavruta [4]. Then, the stability problem of several functional equations has been extensively investigated by a number

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of authors, and there are many interesting results concerning this problem ( [3,6,7,9,11–13,15,16,18,19]). The Hyers-Ulam stability of functional equation is investigated and the investigation is following. Here, we establish the Hyers-Ulam Stability of Euler-Lagrange-Jensen's $(a,b)$- Sextic Functional Equation is of the form

$$\begin{aligned} &f(ax+by)+f(bx+ay)+(a-b)^6\left[f\left(\frac{ax-by}{a-b}\right)+f\left(\frac{bx-ay}{b-a}\right)\right]\\ &=64(ab)^2(a^2+b^2)\left[f(\frac{x+y}{2})+f(\frac{x-y}{2})\right]+2(a^2-b^2)(a^4-b^4)\left[f(x)+f(y)\right] \end{aligned} \tag{1.1}$$

where $a\neq b$ such that $k\in\mathbb{R},\ h=a+b\neq 0,\ \pm 1$ in Multi-Banach Spaces by using direct and fixed point method.

**Definition 1.1.** *[2] A Multi- norm on* $\left\{\mathcal{A}^k : k\in\mathbb{N}\right\}$ *is a sequence* $(\|.\|)=(\|.\|_k : k\in\mathbb{N})$ *such that* $\|.\|_k$ *is a norm on* $\mathcal{A}^k$ *for each* $k\in\mathbb{N}, \|x\|_1=\|x\|$ *for each* $x\in\mathcal{A}$, *and the following axioms are satisfied for each* $k\in\mathbb{N}$ *with* $k\geq 2:$

(1) $\left\|\left(x_{\sigma(1)},...,x_{\sigma(k)}\right)\right\|_k=\|(x_1...x_k)\|_k$, *for* $\sigma\in\Psi_k, x_1,...,x_k\in\mathcal{A}$;

(2) $\|(\alpha_1x_1,...,\alpha_kx_k)\|_k\leq(\max_{i\in\mathbb{N}_k}|\alpha_i|)\|(x_1...x_k)\|_k$
*for* $\alpha_1...\alpha_k\in\mathbb{C}, x_1,...,x_k\in\mathcal{A}$;

(3) $\|(x_1,...,x_{k-1},0)\|_k=\|(x_1,...,x_{k-1})\|_{k-1}$, *for* $x_1,...,x_{k-1}\in\mathcal{A}$;

(4) $\|(x_1,...,x_{k-1},x_{k-1})\|_k=\|(x_1,...,x_{k-1})\|_{k-1}$ *for* $x_1,...,x_{k-1}\in\mathcal{A}$.

*In this case, we say that* $\left((\mathcal{A}^k,\|.\|_k):k\in\mathbb{N}\right)$ *is a multi - normed space.*

*Suppose that* $\left((\mathcal{A}^k,\|.\|_k):k\in\mathbb{N}\right)$ *is a multi - normed space, and take* $k\in\mathbb{N}$. *We need the following two properties of multi - norms. They can be found in [2].*

$$(a)\,\|(x,...,x)\|_k=\|x\|\,,for\quad x\in\mathcal{A},$$

$$(b)\max_{i\in\mathbb{N}_k}\|x_i\|\leq\|(x_1,...,x_k)\|_k\leq\sum_{i=1}^{k}\|x_i\|\leq k\max_{i\in\mathbb{N}_k}\|x_i\|\,,\forall x_1,...,x_k\in\mathcal{A}.$$

*It follows from (b) that if* $(\mathcal{A},\|.\|)$ *is a Banach space, then* $(\mathcal{A}^k,\|.\|_k)$ *is a Banach space for each* $k\in\mathbb{N}$. *In this case,* $\left((\mathcal{A}^k,\|.\|_k):k\in\mathbb{N}\right)$ *is a multi - Banach space.*

## 2. Stability of Functional Equation (1.1) in Multi-Banach Spaces: Direct Method

**Theorem 2.1.** *Let* $X$ *be a linear space and* $((Y^n,\|.\|_n):n\in N)$ *be a multi-Banach Spaces. Let* $f:X\to Y$ *be a mapping satisfying* $f(0)=0$ *such that*

$$\sup_{k\in\mathbb{N}}\|(\mathcal{D}f(x_1,y_1),...,\mathcal{D}f(x_k,y_k))\|_k\leq\epsilon \tag{2.1}$$

$\forall x_1, ..., x_k, y_1, ..., y_k \in Y$. *Then there exists a unique sextic mapping* $S : X \to Y$ *such that*

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - S(x_1), ..., f(x_k) - S(x_k))\| \leq \frac{\epsilon}{h^6} \tag{2.2}$$

*Proof.* Letting $y_i = x_i$ where $i = 1, 2, ...k$ in (2.1), we arrive at

$$\sup_{k\in\mathbb{N}} \left\| \left( \frac{1}{h^6} f(hx_1) - f(x_1), ..., \frac{1}{h^6} f(hx_k) - f(x_k) \right) \right\| \leq \frac{\epsilon}{2h^6} \tag{2.3}$$

Now, Replacing $x_i$ by $2x_i$ where $i = 1, 2, .., k$ and dividing by 2 in above equation, we get

$$\sup_{k\in\mathbb{N}} \left\| \left( \frac{f(2hx_1)}{h^6} - f(x_1), ..., \frac{f(2hx_k)}{h^6} - f(x_k) \right) \right\| \leq \frac{\epsilon}{2^2h^6} + \frac{\epsilon}{2h^6} \tag{2.4}$$

By using induction for a positive integer $n$, we obtain

$$\sup_{k\in\mathbb{N}} \left\| \left( \frac{f(2^n hx_1)}{2^n h^6} - f(x_1), ..., \frac{f(2^n hx_k)}{2^n h^6} - f(x_k) \right) \right\| \leq \frac{1}{h^6} \sum_{i=0}^{n-1} \frac{\epsilon}{2^{i+1}} \leq \frac{1}{h^6} \sum_{i=0}^{\infty} \frac{\epsilon}{2^{i+1}} \tag{2.5}$$

Now, we have to show that the sequence $\left\{ \frac{f(2^n hx)}{2^n h^6} \right\}$ is a Cauchy sequence, by fixing $x \in X$ and replacing $x_1, ...x_k$ by $x, 2x, ..., 2^{k-1}x$ such that

$$\begin{aligned} &\sup_{k\in\mathbb{N}} \left\| \left( \frac{f(2^n hx)}{2^n h^6} - \frac{f(2^m x)}{2^m}, ..., \frac{f(2^{n+k-1} hx)}{2^{n+k-1} h^6} - \frac{f(2^{m+k-1} x)}{2^{m+k-1}} \right) \right\| \\ &\leq \sup_{k\in\mathbb{N}} \left\| \left( \frac{f(2^n hx)}{2^n h^6} - \frac{f(2^m x)}{2^m}, ..., \frac{1}{2^{k-1}} \left[ \frac{f(2^n (2^{k-1} hx))}{2^n h^6} - \frac{f(2^m (2^{k-1} x))}{2^m} \right] \right) \right\| \end{aligned}$$

Using the definition of Multi-norm, we arrive at

$$\begin{aligned} \sup_{k\in\mathbb{N}} &\left\| \left( \frac{f(2^n hx)}{2^n h^6} - \frac{f(2^m x)}{2^m}, ..., \frac{f(2^n (2^{k-1} hx))}{2^n h^6} - \frac{f(2^m (2^{k-1} x))}{2^m} \right) \right\| \\ &\leq \frac{1}{h^6} \sum_{i=m}^{n-1} \frac{\epsilon}{2^{i+1}}. \end{aligned} \tag{2.6}$$

Hence the above inequality (2.6), shows that $\left\{ \frac{f(2^n hx)}{2^n h^6} \right\}$ is a Cauchy sequence as $n \to \infty$. Since $Y$ is complete, then the sequence $\left\{ \frac{f(2^n hx)}{2^n h^6} \right\}$ converges to a fixed point $S(x) \in Y$ such that

$$S(x) = \lim_{n\to\infty} \frac{f(2^n hx)}{2^n h^6}.$$

Therefore, as $n \to \infty$, the inequality (2.5) implies the inequality (2.2). Obviously, one can find the uniqueness of the mapping $S : X \to Y$, using the definition of multi-norm. That is, we can prove $S = S'$. □

**Corollary 2.1.** *Let* $X$ *be a linear space and* $((Y^n, \|.\|_n) : n \in N)$ *be a multi-Banach space. Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *such that*

$$\sup_{k\in\mathbb{N}} \|Df(x_1, y_1, ..., x_k, y_k)\|_k \leq \phi(x_1, y_1, ..., x_k, y_k) \tag{2.7}$$

*for all $x_1,..,x_k,y_1,..,y_k \in X$. Then there exists a unique sextic mapping $S : X \to Y$ such that*

$$\sup_{k\in\mathbb{N}} \|f(x_1) - S(x_1), ..., f(x_k) - S(x_k)\|_k \leq \frac{1}{h^6}\sum_{i=1}^{\infty}\frac{1}{2^{i+1}}\phi\left(2^i x_1, x_1, ..., 2^i x_k, x_k\right) \tag{2.8}$$

*for all $x_1,..,x_k \in X$.*

*Proof.* Proof is similar to that of Theorem 2.1 by replacing the condition $\phi(x_1, y_1, ..., x_k, y_k)$ in place of $\epsilon$. □

**Corollary 2.2.** *Let $X$ be a linear space and $((Y^n, \|.\|_n) : n \in N)$ be a multi-Banach space. Let $0 < p < 6$ , $\theta \geq 0$ and $f : X \to Y$ be a mapping satisfying $f(0) = 0$ such that*

$$\sup_{k\in\mathbb{N}} \|Df(x_1, y_1, ..., x_k, y_k)\|_k \leq \theta\left(\|x_1\|^p + \|y_1\|^p, ..., \|x_k\|^p + \|y_k\|^p\right) \tag{2.9}$$

*for all $x_1,..,x_k,y_1,..,y_k \in X$. Then there exists a unique sextic mapping $S : X \to Y$ such that*

$$\sup_{k\in\mathbb{N}} \|f(x_1) - S(x_1), ..., f(x_k) - S(x_k)\|_k \leq \frac{\theta}{h^6(2^p - 1)}\left(\|x_1\|^p, ..., \|x_k\|^p\right) \tag{2.10}$$

*for all $x_1,..,x_k \in X$.*

*Proof.* Proof is similar to that of Theorem 2.1 by replacing the condition
$\theta\left(\|x_1\|^p + \|y_1\|^p, ..., \|x_k\|^p + \|y_k\|^p\right)$ in place of $\epsilon$. □

## 3. Stability of Functional Equation (1.1) in Multi-Banach Spaces: Fixed Point Method

**Theorem 3.1.** *Let $X$ be a linear space and $((Y^n, \|.\|_n) : n \in N)$ be a multi-Banach Spaces. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ such that*

$$\sup_{k\in\mathbb{N}} \|(\mathcal{D}f(x_1, y_1), ..., \mathcal{D}f(x_k, y_k))\|_k \leq \epsilon \tag{3.1}$$

*$\forall x_1, ..., x_k, y_1, ..., y_k \in Y$. Then there exists a unique sextic mapping $S : X \to Y$ such that*

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - S(x_1), ..., f(x_k) - S(x_k))\| \leq \frac{\epsilon}{2(h^6 - 1)} \tag{3.2}$$

*Proof.* Letting $y_i = x_i$ where $i = 1, 2, ...k$ in (2.1), we arrive at

$$\sup_{k\in\mathbb{N}} \left\|\left(\frac{1}{h^6}f(hx_1) - f(x_1), ..., \frac{1}{h^6}f(hx_k) - f(x_k)\right)\right\| \leq \frac{\epsilon}{2h^6} \tag{3.3}$$

Let $\Psi = \{l : X \to Y | l(0) = 0\}$ and introduce the generalized metric $d$ defined on $\Psi$ by

$$d(l, m) = \inf\left\{\Psi \in [0, \infty] | \sup_{k\in\mathbb{N}} \|l(x_1) - m(x_1), ..., l(x_k) - m(x_k)\|_k \leq \Psi \quad \forall \quad x_1, ..., x_k \in X\right\}$$

Then it is easy to show that $\Psi, d$ is a generalized complete metric space, See [8].

We define an operator $\mathcal{J} : \Psi \to \Psi$ by

$$\mathcal{J}l(x) = \frac{1}{h^6}l(hx) \qquad x \in X$$

We assert that $\mathcal{J}$ is a strictly contractive operator. Given $l, m \in \Psi$, let $\Psi \in [0,\infty]$ be an arbitary constant with $d(l,m) \leq \Psi$. From the definition if follows that

$$\sup_{k\in\mathbb{N}} \|l(x_1)-m(x_1),...,l(x_k)-m(x_k)\|_k \leq \Psi \quad x_1,...,x_k \in X.$$

Therefore, $\sup_{k\in\mathbb{N}} \|(\mathcal{J}l(x_1)-\mathcal{J}m(x_1),...,\mathcal{J}l(x_k)-\mathcal{J}m(x_k))\|_k \leq \frac{1}{h^6}\Psi$
$x_1,...,x_k \in X$. Hence,it holds that

$$d(\mathcal{J}l,\mathcal{J}m) \leq \frac{1}{h^6}\Psi d(\mathcal{J}l,\mathcal{J}m) \leq \frac{1}{h^6} d(l,m)$$

$\forall l, m \in \Psi$.
This Means that $\mathcal{J}$ is strictly contractive operator on $\Psi$ with the Lipschitz constant $L = \frac{1}{h^6}$.
By (3.3), we have $d(\mathcal{J}f,f) \leq \frac{\epsilon}{2h^6}$. Applying the Theorem 2.2 in [10], we deduce the existence of a fixed point of $\mathcal{J}$ that is the existence of mapping $S : X \to Y$ such that

$$S(hx) = h^6 S(x) \qquad \forall x \in X.$$

Moreover, we have $d\left(\mathcal{J}^n f, S\right) \to 0$, which implies

$$S(x) = \lim_{n\to\infty} \mathcal{J}^n f(x) = \lim_{n\to\infty} \frac{f(h^n x)}{h^{6n}}$$

for all $x \in X$.
Also, $d(f,S) \leq \frac{1}{1-\mathcal{L}} d(\mathcal{J}f,f)$ implies the inequality

$$\leq \frac{\epsilon}{2(h^6-1)}.$$

Doing $x_1 =,...,= x_k = h^n x$, and $y_1 =,...,= y_k = h^n y$ in (1.1) and dividing by $h^{6n}$. Now, applying the property (a) of multi-norms, we have

$$\begin{aligned}\|DS(x,y)\| &= \lim_{n\to\infty} \frac{1}{h^{6n}} \|Df\left(h^n x, h^n y\right)\| \\ &\leq \lim_{n\to\infty} \frac{1}{h^{6n}} = 0\end{aligned}$$

for all $x, y \in X$. The uniqueness of $S$ follows from the fact that $S$ is the unique fixed point of $\mathcal{J}$ with the property that there exists $\ell \in (0,\infty)$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1)-S(x_1),...,f(x_k)-S(x_k))\|_k \leq \ell$$

for all $x_1,...,x_k \in X$.
Hence the proof. $\square$

**Corollary 3.1.** *Let $X$ be a linear space and $((Y^n, \|.\|_n) : n \in N)$ be a multi-Banach space. Let $0 < p < 6$ , $\theta \geq 0$ and $f : X \to Y$ be a mapping satisfying $f(0) = 0$ such that*

$$\sup_{k\in\mathbb{N}} \|Df(x_1, y_1, ..., x_k, y_k)\|_k \leq \theta \left(\|x_1\|^p + \|y_1\|^p, ..., \|x_k\|^p + \|y_k\|^p\right) \tag{3.4}$$

*for all $x_1, .., x_k, y_1, .., y_k \in X$. Then there exists a unique sextic mapping $S : X \to Y$ such that*

$$\sup_{k\in\mathbb{N}} \|f(x_1) - S(x_1), ..., f(x_k) - S(x_k)\|_k \leq \frac{2\theta}{h^6 - 2h^p} \left(\|x_1\|^p, ..., \|x_k\|^p\right) \tag{3.5}$$

*for all $x_1, .., x_k \in X$.*

*Proof.* Proof is similar to that of Theorem 3.1 by replacing the condition $\theta \left(\|x_1\|^p + \|y_1\|^p, ..., \|x_k\|^p + \|y_k\|^p\right)$ in place of $\epsilon$. □

**Corollary 3.2.** *Let $X$ be a linear space and $((Y^n, \|.\|_n) : n \in N)$ be a multi-Banach space. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ such that*

$$\sup_{k\in\mathbb{N}} \|Df(x_1, y_1, ..., x_k, y_k)\|_k \leq \phi(x_1, y_1, ..., x_k, y_k) \tag{3.6}$$

*for all $x_1, .., x_k, y_1, .., y_k \in X$. Then there exists a unique sextic mapping $S : X \to Y$ such that*

$$\sup_{k\in\mathbb{N}} \|f(x_1) - S(x_1), ..., f(x_k) - S(x_k)\|_k \leq \frac{1}{2(h^6 - 1)}\phi(x_1, x_1, ..., x_k, x_k) \tag{3.7}$$

*for all $x_1, .., x_k \in X$.*

*Proof.* Proof is similar to that of Theorem 3.1 by replacing the condition $\phi(x_1, y_1, ..., x_k, y_k)$ in place of $\epsilon$. □

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