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SOME PROPERTIES OF SPECIAL MAGNETIC CURVES

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Abstract. In the theory of curves, a magnetic field generates a magnetic flow whose trajectories are curves called magnetic curves. This paper aims at studying some properties for these curves which corresponding to the Killing magnetic fields in the 3-dimensional Euclidean space. We investigate the trajectories of the magnetic fields called T-magnetic and e-magnetic curves, also we give some characterizations of these curves. In addition, we determine all magnetic curves for new spherical images of a spherical curve and finally, we defray some examples to confirm our main results.

1. Introduction

The magnetic curves on a Riemannian manifold (M, g) are trajectories of charged particles moving on M under the action of a magnetic field F. Each trajectory γ may be found by solving the Lorentz equation $\nabla_{\gamma'}\gamma' = \phi(\gamma')$, where ϕ is the Lorentz force corresponding to F and ∇ is the Levi Civita connection of g. In particular, the trajectories of (charged) free particles moving without the action of a magnetic field are geodesics, which satisfy $\nabla_{\gamma'}\gamma' = 0$. Here "free" means *subject to no forces other than gravity* (see [1]).
The magnetic trajectories of the uniform magnetic fields obtained by multiplying the volume form on (M, g) by scalars were determined on some 2-dimensional spaces, as $S^2(c)$, $H^2(-c)$, E^2, $c > 0$ (see [2,3]). In such an ambient, the Killing vector fields define the Killing magnetic fields. In a three-dimensional space, when a

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charged particle moves along a regular curve the tangent, normal, and binormal vectors describe kinematic and geometric properties of this particle. These vectors and the time dimension affect the trajectory of the charged particle during the motion in a magnetic field.

The study of magnetic curves was extended to other ambient spaces, such as complex space forms [4,5], Sasakian 3-manifold [6,7], and so on. Very recent results of classification for the Killing magnetic trajectories on two special 3-dimensional manifolds, namely E^3 and $S^2 \times R$, were obtained in [8,9], respectively. Barros and Romero proved in [10] that if (M, g) has constant curvature, then the magnetic curves corresponding to a Killing magnetic field are centerlines of Kirchhoff elastic rods. Moreover, if the space is also simply connected, they proved, using a variational approach to characterize Killing magnetic flows, that a Killing magnetic flow is equivalent to the Kirchhoff elastic rod variational model. In this paper, we investigate effects of magnetic fields on charged particle trajectories by variational approach to magnetic flow associated with the Killing magnetic field on a three-dimensional Riemannian manifold M.

2. The basic concepts

In this section we give some facts, notations and basic meanings which we are needed through this study (see [11–13]). The sphere of radius $r > 0$ and with center in the origin in the space E^3 is defined by

$$S^2 = \{x = (x_1, x_2, x_3) \in E^3 : \langle x, x \rangle = r^2\}.$$

Let $\gamma : I \to S^2 \in E^3$ be a spherical curve. We say that γ is parameterized by its arc length if it satisfies $\|\gamma'(s)\| = 1$. Throughout this paper, we denote the parameter s of γ as the arc length parameter. Let us denote $T(s) = \gamma'(s)$, and we call $T(s)$ a unit tangent vector of γ at s. We remark that $\langle x \wedge y, z \rangle = \det(x, y, z)$. Hence, $x \wedge y$ is orthogonal to x, y. We now set a vector $e(s) = \gamma(s) \wedge T(s)$ such that

$$\begin{aligned}
\langle e(s), e(s) \rangle &= \langle \gamma(s) \wedge T(s), \gamma(s) \wedge T(s) \rangle = \langle \gamma(s), \gamma(s) \rangle \ \langle T(s), T(s) \rangle = 1, \\
T(s) \wedge e(s) &= \langle T(s), T(s) \rangle \gamma(s) + \langle T(s), \gamma(s) \rangle T(s) = \gamma(s),
\end{aligned}$$

and

$$\gamma(s) \wedge e(s) = -T(s).$$

Therefore, we have an orthonormal frame $\{\gamma(s), T(s), e(s)\}$ along γ. Its Frenet–Serret formula is given as follows:

$$\begin{aligned}
\gamma'(s) &= T(s), \\
T'(s) &= -\gamma(s) + \kappa_g(s) e(s), \\
e'(s) &= -\kappa_g(s) T(s),
\end{aligned}$$

where κ_g is the geodesic curvature of the curve γ in S^2 and given by $\kappa_g = \det(\gamma(s), T(s), T'(s))$. For any $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in R^3$, the scalar product and the vector product of x and y respectively defined:

$$g(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

$$x \wedge y = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where $\wedge$ is the cross-product of S^2 and (i, j, k) is the canonical basis of R^3. If $x \neq 0$, the norm $\|x\|$ is defined as

$$\|x\| = \sqrt{g(x, x)}.$$

Now, for our study it is important to consider the following [6, 14]:

Definition 2.1. *A magnetic field on a Riemannian manifold (M, g) is defined as a closed 2-form F on M, related to a skew-symmetric $(1,1)-$tensor field ϕ, called the Lorentz force of F, by:*

$$g(\phi(X), Y) = F(X, Y), \qquad \forall\ X, Y \in \chi(M).$$

For a trivial magnetic field, $F = 0$, i.e. in the case of vanishing Lorentz force, the magnetic curves are given by the trajectories of the charged particles moving freely, only under the influence of gravity. The Lorentz equation becomes $\nabla_{\gamma'}\gamma' = 0$ and its solutions are the geodesics. The magnetic trajectories of F are curves γ on M which satisfy the Lorentz equation

$$\nabla_{\gamma'}\gamma' = \phi(\gamma').$$

Let V be a Killing vector field on M, then the *Lorentz force* can be written as

$$\phi(X) = V \times X. \tag{2.1}$$

In this case, the *Lorentz force equation* can be given as

$$\nabla_{\gamma'}\gamma' = V \times \gamma'.$$

Proposition 2.1. *Let $\gamma : I \to S^2 \subset M^3$ be a curve in a three-dimensional oriented Riemannian Manifold (M^3, g) and V be a vector field along the curve γ. One can take a variation of γ in the direction of V, say, a map $\Gamma : I \times (-\epsilon, \epsilon) \to M^3$ which satisfies $\Gamma(s, 0) = \gamma(s)$, $\left(\frac{\partial \Gamma}{\partial s}(s, t)\right) = V(s)$.*

In this setting, we have the following functions:

1. *the speed function $v(s, t) = \left\| \frac{\partial \Gamma}{\partial s}(s, t) \right\|$,*

2. the curvature function $\kappa_g(s,t)$ of $\gamma_t(s)$. The variations of those functions at $t=0$,

$$V(v)=\left(\frac{\partial v}{\partial t}(s,t)\right)\Big|_{t=0}=g(\nabla_T V,T)v,$$

$$V(\kappa_g)=\left(\frac{\partial \kappa_g}{\partial t}(s,t)\right)\Big|_{t=0}=g(\nabla_T^2 V,e)-2\kappa_g\ g(\nabla_T V,T)+g(\Omega(V,T)T,e),$$

where Ω is curvature tensor of M^3.

Proposition 2.2. *Let $V(s)$ be the restriction to $\gamma(s)$ of a Killing vector field ,say V of M^3. Then*

$$V(v)=V(\kappa_g)=0.$$

3. Some Magnetic Curves in 3D Oriented Riemannian Manifolds

3.1. $T-$**Magnetic Curve.** In this section, we give some characterizations for T-magnetic curves in a Riemannian manifolds.

Definition 3.1. *Let $\gamma: I\to S^2\subset M^3$ be a spherical curve in 3D oriented Riemannian space, (M^3,g) and F be a magnetic field on M. We call the curve γ is a T-magnetic curve if T satisfy the Lorentz force equation,*

$$\nabla_{_T}T=\phi(T)=V\times T.$$

Proposition 3.1. *Let γ be a T-magnetic curve in 3D oriented Riemannian space (M^3,g) with the Frenet apparatus $\{\gamma,T,e,\kappa_g\}$. Then, we have the sphere Frenet–Serret formula:*

$$\begin{bmatrix}\gamma'\\ T'\\ e'\end{bmatrix}=\begin{bmatrix}0&1&0\\-1&0&\kappa_g\\0&-\kappa_g&0\end{bmatrix}\begin{bmatrix}\gamma\\ T\\ e\end{bmatrix},$$

and then the Lorentz force in the Frenet frame can be written as

$$\begin{bmatrix}\phi(\gamma)\\ \phi(T)\\ \phi(e)\end{bmatrix}=\begin{bmatrix}0&1&\delta\\-1&0&\kappa_g\\-\delta&-\kappa_g&0\end{bmatrix}\begin{bmatrix}\gamma\\ T\\ e\end{bmatrix}.$$

where δ is a certain function defined by $\delta=g(\phi(\gamma),e)$.

Proof. Let γ be a unit speed T-magnetic curve in 3D oriented Riemannian space (M^3,g) with the Frenet apparatus $\{\gamma,T,e,\kappa_g\}$. From the definition of a magnetic curve, one can write

$$\phi(T)=-\gamma+\kappa_g e.$$

Since $\phi(\gamma)\in span\{\gamma,T,e\}$, we have

$$\phi(\gamma)=\lambda_1\gamma+\lambda_2 T+\lambda_3 e.$$

By using the following equalities:

$$\begin{aligned} \lambda_1 &= g(\phi(\gamma), \gamma) = 0, \\ \lambda_2 &= g(\phi(\gamma), T) = -g(\phi(T), \gamma) = 1, \\ \lambda_3 &= g(\phi(\gamma), e) = \delta, \end{aligned}$$

we get

$$\phi(\gamma) = T + \delta e.$$

Similarly, we can easily obtain that

$$\phi(e) = -\delta\gamma - \kappa_g T,$$

hence, this completes the proof. □

Proposition 3.2. *Let γ be a spherical curve in 3D oriented Riemannian space (M^3, g). The curve γ is a T-magnetic trajectory of a magnetic field V if and only if the vector field V along γ can be written as*

$$V = \kappa_g \gamma - \delta T + e. \tag{3.1}$$

Proof. Let γ be a unit speed T-magnetic trajectory of a magnetic field V. Using Proposition 3.1 and Eq. (1.1), we can easily see that

$$V = \kappa_g \gamma - \delta T + e.$$

Conversely, we assume that Eq.(3.1) holds. Then we get $V \times T = \phi(T)$, and so the curve γ is a T-magnetic trajectory of the magnetic vector field V. □

Theorem 3.1. ***(Main result)*** *Let γ be a spherical T-magnetic curve and V be a Killing vector field on a simply connected space form $(M^3(C), g)$. If the curve γ is one of the T-magnetic trajectories of $(M^3(C), g, V)$, then its curvature hold the following equations:*

$$\begin{aligned} \delta &= const., \\ \delta\kappa_g' &= C, \end{aligned}$$

where C is the curvature of the Riemannian space M^3.

Proof. Let V be a magnetic field in a Riemannian manifold M^3. Then V satisfies Eq.(3.1). Differentiating this equation with respect to s, we have

$$\nabla_T V = (\kappa_g' + \delta)\gamma - \delta' T - \delta\kappa_g e, \tag{3.2}$$

Now, if V is Killing, Proposition 2.2 implies that $V(v) = 0$ and so, from Eq.(3.2), one obtains that $\nabla_T V$ has no tangential component, i.e.,

$$\delta = const.$$

and differentiation of Eq.(3.2) and using the sphere Frenet–Serret formulas, gives us

$$\nabla_T^2 V = (\kappa_g' + \delta)'\gamma + (\kappa_g' + \delta + \delta\kappa_g^2)T - (\delta\kappa_g)'e, \tag{3.3}$$

Eqs.(3.2), (3.3) together with $V(\kappa_g) = 0$ in Proposition 2.2 lead to

$$-(\delta\kappa_g)' + g(\Omega(V,T)T, e) = 0.$$

In particular, if M^3 has constant curvature C, then $g(\Omega(V,T)T, e) = Cg(V,e) = C$ and so,

$$C - \delta\kappa_g' = 0. \tag{3.4}$$

Thus, the theorem is proved. □

3.2. $e-$**Magnetic Curve.**

Definition 3.2. *Let $\gamma : I \to S^2 \subset M^3$ be a curve in 3D oriented Riemannian space, (M^3, g) and F be a magnetic field on M. We call the curve γ is a e-magnetic curve if e satisfy the Lorentz force equation:*

$$\nabla_T e = \phi(e) = V \times e.$$

Proposition 3.3. *Let γ be a e-magnetic curve in 3D oriented Riemannian space (M^3, g) with the Frenet apparatus $\{\gamma, T, e, \kappa_g\}$. Then, we have the sphere Frenet–Serret formula:*

$$\begin{bmatrix} \gamma' \\ T' \\ e' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_g \\ 0 & -\kappa_g & 0 \end{bmatrix} \begin{bmatrix} \gamma \\ T \\ e \end{bmatrix},$$

and then the Lorentz force in the Frenet frame can be written as

$$\begin{bmatrix} \phi(\gamma) \\ \phi(T) \\ \phi(e) \end{bmatrix} = \begin{bmatrix} 0 & \rho & 0 \\ -\rho & 0 & \kappa_g \\ 0 & -\kappa_g & 0 \end{bmatrix} \begin{bmatrix} \gamma \\ T \\ e \end{bmatrix},$$

where ρ is a certain function defined by $\rho = g(\phi(\gamma), T)$.

Proof. Let γ be a unit speed e-magnetic curve in 3D oriented Riemannian space (M^3, g), with the Frenet apparatus $\{\gamma, T, e, \kappa_g\}$. Since we have

$$\phi(\gamma) = \mu_1\gamma + \mu_2 T + \mu_3 e,$$

then using the following equalities:

$$\begin{aligned} \mu_1 &= g(\phi(\gamma), \gamma) = 0, \\ \mu_2 &= g(\phi(\gamma), T) = -g(\phi(T), \gamma) = \Omega_2, \\ \mu_3 &= g(\phi(\gamma), e) = -g(\phi(e), \gamma) = 0. \end{aligned}$$

One can get

$$\phi(\gamma) = \rho T.$$

Similarly, we can easily calculate that

$$\begin{aligned}\phi(T) &= -\rho\gamma + \kappa_g e,\\ \phi(e) &= -\kappa_g T,\end{aligned}$$

therefore, it completes the proof. □

Proposition 3.4. *Let γ be a spherical curve in 3D oriented Riemannian space (M^3, g). The curve γ is then the e-magnetic trajectory of a magnetic field V if and only if the magnetic vector field V can be written along the curve γ as*

$$V = \kappa_g\gamma + \rho e. \tag{3.5}$$

Proof. Let γ be a unit speed e-magnetic trajectory of a magnetic field V. Using proposition 3.3 and Eq.(1.1), we can easily see that

$$V = \kappa_g\gamma + \rho e.$$

Conversely, we assume that Eq.(3.5) holds. Then we get $V \times e = \phi(e)$, and so the curve γ is an e-magnetic trajectory of the magnetic vector field V. This completes the proof. □

Theorem 3.2. ***(Main result)*** *Let V be a Killing vector field on a simply connected space form $(M^3(C), g)$. Then, the unit speed e-magnetic trajectories of $(M^3(C), g, V)$ are curves with curvature satisfying*

$$\begin{aligned}\kappa_g &= 0,\\ \rho &= 1,\\ C &= 1,\end{aligned}$$

where C is the curvature of the Riemannian space M^3.

Proof. Let V be a magnetic field in a Riemannian manifold M^3. Then V satisfies Eq.(3.5). Differentiating Eq.(3.5) with respect to s, we have

$$\nabla_T V = \kappa_g'\gamma + (\kappa_g - \rho\kappa_g)T + \rho' e, \tag{3.6}$$

and differentiation of Eq.(3.6) and using the sphere Frenet–Serret formulas, gives us

$$\nabla_T^2 V = (\kappa_g'' + \rho\kappa_g - \kappa_g)\gamma + (\kappa_g' + (\kappa_g - \rho\kappa_g)' - \rho'\kappa_g)T + (\rho'' + \kappa_g^2 - \rho\kappa_g^2)e, \tag{3.7}$$

Now, if V is Killing, Proposition 2.2 implies that $V(v)=0$ and so, from Eq.(3.6), one obtains that $\nabla_T V$ has no tangential component, i.e.,

$$\kappa_g - \rho\kappa_g = 0,$$

If Eq.(3.6) and Eq.(3.7) are then considered with $V(\kappa_g)=0$ in Proposition 2.2, we obtain

$$\Omega_2'' + \kappa_g^2 - \rho\kappa_g^2 + g(\Omega(V,T)T,e) = 0.$$

In particular, if M^3 has constant curvature C, then $g(\Omega(V,T)T,e) = Cg(V,e) = C\rho$ and so,

$$\rho'' + \kappa_g^2 - \rho\kappa_g^2 + C\rho = 0,$$

Which gives the required result. □

4. Magnetic curves of spherical indicatrices

In this section we introduce a new representation of spherical indicatrices of magnetic curves we start as follows: (see [15–17])

Definition 4.1. *Let γ be a curve in the sphere S^2 with Frenet vectors γ, T and e. The unit tangent vectors along the curve $\gamma(s)$ generate a curve $\gamma_t = T$ on the sphere of radius 1 about the origin. The curve γ_t is called the tangent indicatrix of the curve γ. If $\gamma = \gamma(s)$ is a natural representations of the curve γ, then $\gamma_t(s) = T(s)$ will be a representation of γ_t. Similarly, one can consider the principal normal indicatrix $\gamma_e = e(s)$.*

Lemma 4.1. *Let γ_t be the tangent indicatrix of γ and $\{\gamma_t, T_t, e_t\}$ be its frenet frame, then we have Frenet formula:*

$$\begin{bmatrix} \gamma_t'(s_t) \\ T_t'(s_t) \\ e_t'(s_t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_{g_t}(s_t) \\ 0 & -\kappa_{g_t}(s_t) & 0 \end{bmatrix} \begin{bmatrix} \gamma_t(s_t) \\ T_t(s_t) \\ e_t(s_t) \end{bmatrix},$$

where

$$\begin{cases} \gamma_t = T, \\ T_t = \frac{-1}{\sqrt{1+\kappa_g^2}}\gamma + \frac{\kappa_g}{\sqrt{1+\kappa_g^2}}e, \\ e_t = \frac{\kappa_g}{\sqrt{1+\kappa_g^2}}\gamma + \frac{1}{\sqrt{1+\kappa_g^2}}e, \end{cases}$$

and

$$s_t = \int \sqrt{1+\kappa_g^2}ds, \qquad \kappa_{g_t} = \frac{\kappa_g'}{\sqrt{1+\kappa_g^2}},$$

where s_t is the natural representation of γ_t and κ_{g_t} its the geodesic curvature.

Lemma 4.2. *Consider $\gamma_e = e$ is the the principal normal indicatrix of γ and $\{\gamma_e, T_e, e_e\}$ is the Frenet vectors of it, then we have the Frenet equations:*

$$\begin{bmatrix} \gamma_e'(s_e) \\ T_e'(s_e) \\ e_e'(s_e) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_{g_e}(s_e) \\ 0 & -\kappa_{g_e}(s_e) & 0 \end{bmatrix} \begin{bmatrix} \gamma_e(s_e) \\ T_e(s_e) \\ e_e(s_e) \end{bmatrix},$$

where

$$\begin{cases} \gamma_e = e, \\ T_e = -T, \\ e_e = \gamma, \end{cases}$$

and

$$s_e = \int \kappa_g \, ds, \qquad \kappa_{g_e} = \frac{1}{\kappa_g}.$$

The parameter s_e is the natural representation of γ_e and κ_{g_e} is the geodesic curvature of that one.

Through what has been studied previously about magnetic curves it is easy to market the following:

4.1. **T_t-Magnetic Curve.**

Definition 4.2. *Let $\gamma_t : I \to S^2 \subset M^3$ be a curve in 3D oriented Riemannian space, (M^3, g) and F be a magnetic field on M. We call the curve γ_t is a T_t-magnetic curve if T_t satisfy the Lorentz force equation,*

$$\nabla_{T_t} T_t = \phi(T_t) = V_t \times T_t.$$

Proposition 4.1. *Let γ_t be a T_t-magnetic curve in 3D oriented Riemannian space (M^3, g) with the Frenet apparatus $\{\gamma_t, T_t, e_t\}$. Then, the Lorentz force in the Frenet frame can be written as*

$$\begin{bmatrix} \phi(\gamma_t) \\ \phi(T_t) \\ \phi(e_t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \zeta \\ -1 & 0 & \frac{\kappa_g'}{\sqrt{1+\kappa_g^2}} \\ -\zeta & -\frac{\kappa_g'}{\sqrt{1+\kappa_g^2}} & 0 \end{bmatrix} \begin{bmatrix} \gamma_t \\ T_t \\ e_t \end{bmatrix}.$$

where ζ is a certain function defined by $\zeta = g(\phi(\gamma_t), e_t)$.

Proposition 4.2. *Let γ_t be the tangent indicatrix of γ and $\{\gamma_t, T_t, e_t\}$ be its frenet frame in 3D oriented Riemannian space (M^3, g). The curve γ_t is a T_t-magnetic trajectory of a magnetic field V_t if and only if the vector field V_t along γ_t can be written as*

$$V_t = \frac{\kappa_g + \zeta}{\sqrt{1+\kappa_g^2}}\gamma + \frac{\kappa_g'}{\sqrt{1+\kappa_g^2}}T + \frac{1-\zeta\kappa_g}{\sqrt{1+\kappa_g^2}}e.$$

Theorem 4.1. *Let γ_t be a T_t-magnetic curve and V_t be its Killing vector field on a simply connected space form $(M^3(K), g)$. If the curve γ is one of the T_t-magnetic trajectories of $(M^3(K), g, V_t)$, then its curvature hold the following equations:*

$$\kappa_g''(1+\kappa_g^2) - \kappa_g(\kappa_g')^2 = \frac{K}{\zeta},\ \zeta = const.,$$

where K is the curvature of the Riemannian space M^3.

4.2. e_t**-Magnetic Curve.**

Definition 4.3. *Let $\gamma_t : I \to S^2 \subset M^3$ be a curve in 3D oriented Riemannian space, (M^3, g) and F be a magnetic field on M. We call the curve γ_t is a e_t-magnetic curve if e_t satisfy the Lorentz force equation:*

$$\nabla_{T_t} e_t = \phi(e_t) = V_t \times e_t.$$

Proposition 4.3. *Let γ_t be a e_t-magnetic curve in 3D oriented Riemannian space (M^3, g) with the Frenet apparatus $\{\gamma_t, T_t, e_t\}$. Then, the Lorentz force in the Frenet frame can be written as*

$$\begin{bmatrix} \phi(\gamma_t) \\ \phi(T_t) \\ \phi(e_t) \end{bmatrix} = \begin{bmatrix} 0 & \varpi & 0 \\ -\varpi & 0 & \frac{\kappa_g'}{\sqrt{1+\kappa_g^2}} \\ 0 & -\frac{\kappa_g'}{\sqrt{1+\kappa_g^2}} & 0 \end{bmatrix} \begin{bmatrix} \gamma_t \\ T_t \\ e_t \end{bmatrix}.$$

where ϖ is a certain function defined by $\varpi = g(\phi(\gamma_t), T_t)$.

Proposition 4.4. *Let γ_t be the tangent indicatrix of γ in three-dimensional oriented Riemannian space (M^3, g). The curve γ_t is a e_t-magnetic trajectory of a magnetic field V_t if and only if the vector field V_t along γ_t can be written as*

$$V_t = \frac{\kappa_g \varpi}{\sqrt{1+\kappa_g^2}}\gamma + \frac{\kappa_g'}{\sqrt{1+\kappa_g^2}}T + \frac{\varpi}{\sqrt{1+\kappa_g^2}}e.$$

Theorem 4.2. *Let γ_t be the e_t-magnetic curve and V_t be a Killing vector field on a simply connected space form $(M^3(K), g)$. If the curve γ_t is one of the e_t-magnetic trajectories of $(M^3(K), g, V_t)$, then the following equations hold:*

$$\begin{aligned} \kappa_g &= const., \\ \varpi &= 1, \\ K &= 0, \end{aligned}$$

where K is the curvature of the Riemannian space M^3.

5. Examples

In this section, we give some examples on $T-magnetic$ and $e-magnetic$ curves and calculate their spherical indicatrices to illustrate our main results and draw their pictures by using Mathematica computer program.

Example 5.1. *Let us consider the following spherical curve $\alpha : I \to S^2$, (see Fig. 1)*

$$\alpha(s) = \left(\frac{-4}{5} \sin\left[\frac{5}{4}s\right], \frac{4}{5} \cos\left[\frac{5}{4}s\right], \frac{3}{5} \right), \tag{5.1}$$

Differentiating (5.1), we get the tangent T vector as follows:

$$T = \alpha'(s) = \left(-\cos\left[\frac{5}{4}s\right], -\sin\left[\frac{5}{4}s\right], 0 \right).$$

Then we obtain the principal normal vector of α:

$$e(s) = \left(\frac{3}{5} \sin\left[\frac{5}{4}s\right], \frac{-3}{5} \cos\left[\frac{5}{4}s\right], \frac{4}{5} \right),$$

Also, the geodesic curvature of α given by:

$$\kappa_g = \frac{3}{4}.$$

We have that $\kappa_g = const.$, so α is a T-magnetic curve and it is also a e_t-magnetic curve. From the above calculations, the spherical image of the tangent indicatrix of α is (see Fig. 2(A))

$$\alpha_t = T = \left(-\cos\left[\frac{5}{4}s\right], -\sin\left[\frac{5}{4}s\right], 0 \right).$$

Its natural representation and the geodesic curvature are

$$s_t = \frac{5}{4}s, \quad \kappa_{g_t} = 0.$$

Also, the spherical image of the principal normal indicatrix of α is (see Fig. 2(B))

$$\alpha_e = e = \left(\frac{3}{5} \sin\left[\frac{5}{4}s\right], \frac{-3}{5} \cos\left[\frac{5}{4}s\right], \frac{4}{5} \right),$$

It has the natural representation and the geodesic curvature as follows:

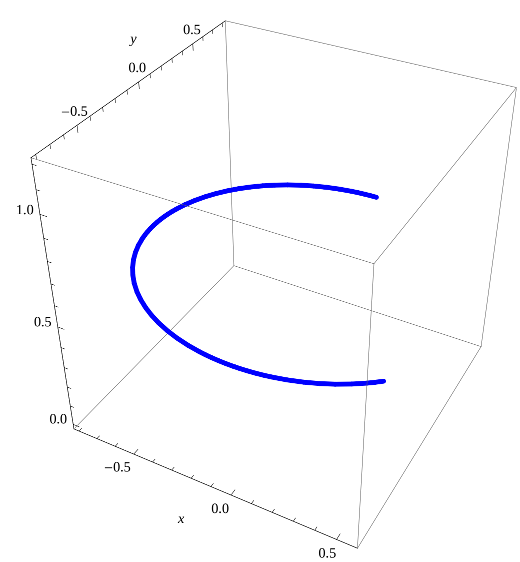
$$s_e = \frac{3}{4}s, \quad \kappa_{g_e} = \frac{4}{3}.$$

Example 5.2. *Let β be a T-magnetic curve in S^2 given by (see Fig. 3)*

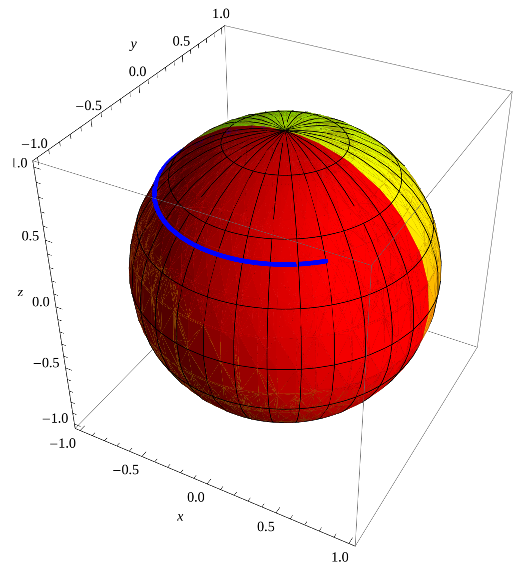
$$\beta(s) = \left(\frac{1}{\sqrt{2}} \cos\left[\sqrt{2}s\right], \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \sin\left[\sqrt{2}s\right] \right).$$

By differentiating, we get

$$\beta' = T = \left(-\sin\left[\sqrt{2}s\right], 0, \cos\left[\sqrt{2}s\right] \right),$$

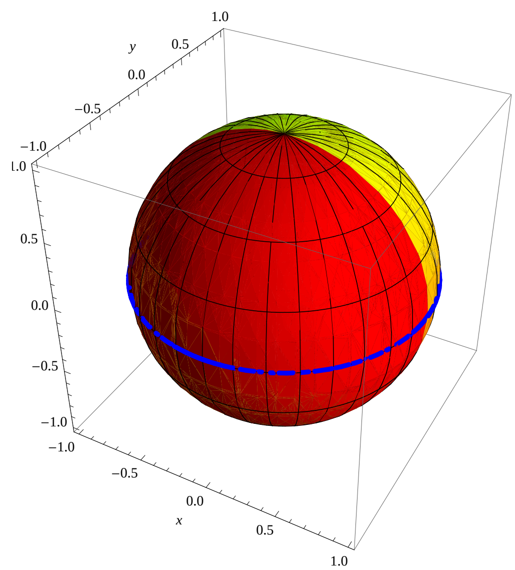


(A)

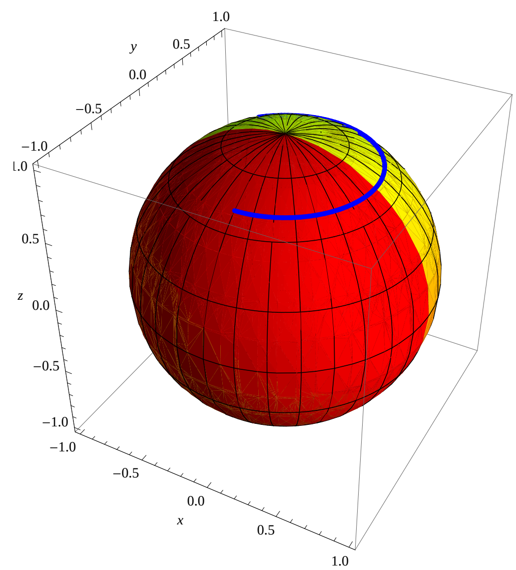


(B)

Figure 1



(A)



(B)

Figure 2

which implies that $\langle \beta', \beta' \rangle = \langle T, T \rangle = 1$*. From which, the principal normal vector of the* T*-magnetic curve is given as follows:*

$$e = \left(\frac{1}{\sqrt{2}} \cos\left[\sqrt{2}s\right], \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \sin\left[\sqrt{2}s\right] \right).$$

Besides, the geodesic curvature and the certain function of β *are respectively,*

$$\kappa_g = -1, \ \delta = 0.$$

Which mean that κ_g is constant so β is also a e_t-magnetic curve. Moreover, the spherical image of the tangent indicatrix of the T-magnetic curve β is (see Fig. 4(A))

$$\beta_t = T = \left(-\sin\left[\sqrt{2}s\right], 0, \cos\left[\sqrt{2}s\right]\right).$$

From this curve the natural representation and the geodesic curvature are respectively,

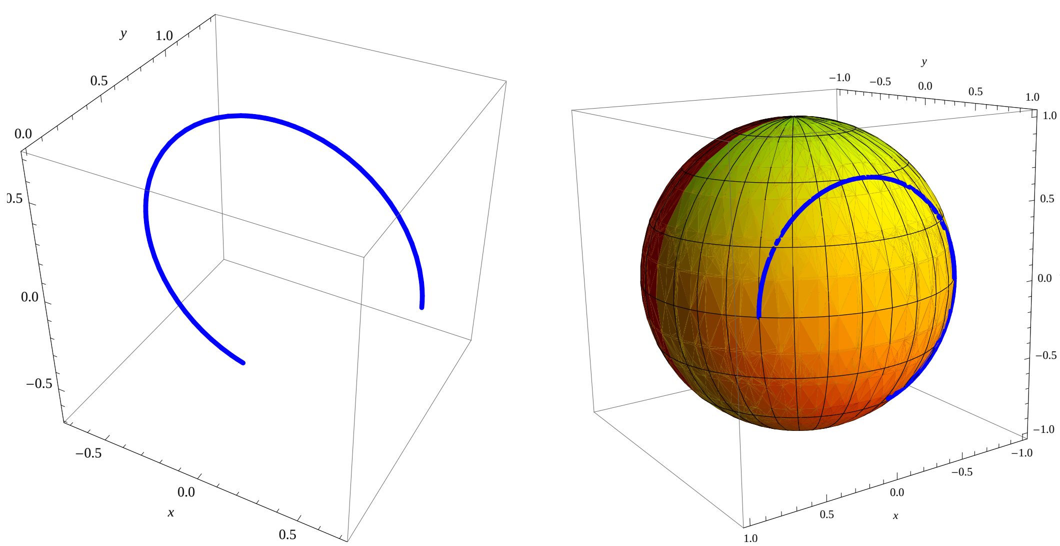
$$s_t = \sqrt{2}s, \quad \kappa_{g_t} = 0.$$

Also, the spherical image of the principal normal indicatrix of the T-magnetic curve β is (see Fig. 4(B))

$$\beta_e = e = \left(\frac{1}{\sqrt{2}}\cos\left[\sqrt{2}s\right], \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin\left[\sqrt{2}s\right]\right),$$

which has the natural representation and the geodesic curvature as

$$s_e = -s, \quad \kappa_{g_e} = -1.$$



(A) (B)

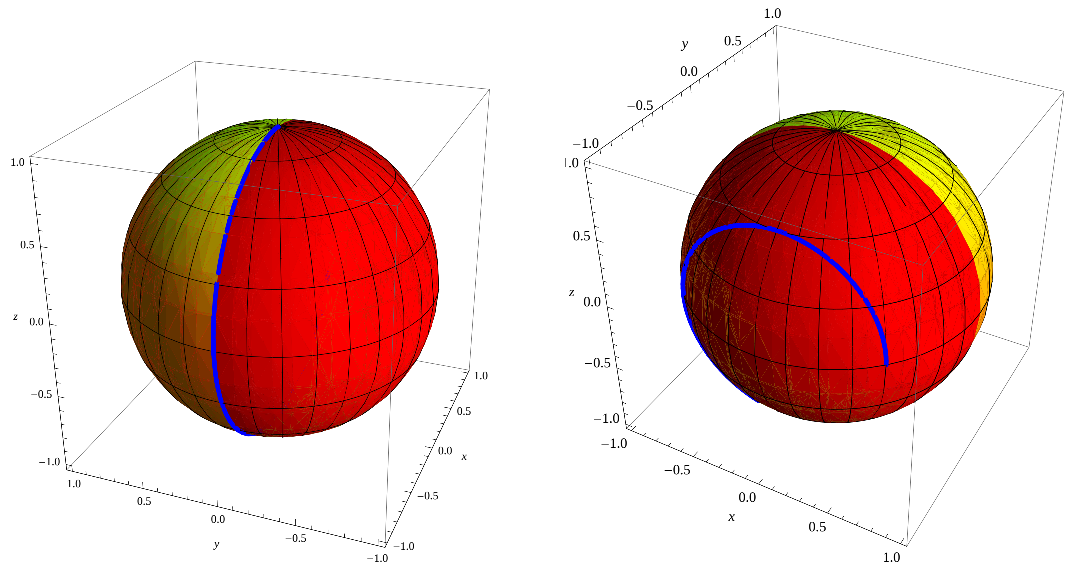
FIGURE 3

Example 5.3. *We consider a e-magnetic curve γ in S^2 is defined by (see Fig. 5)*

$$\gamma(s) = (\cos[s], \sin[s], 0).$$

By differentiating this vector we have

$$\phi(\gamma) = T = (-\sin[s], \cos[s], 0),$$



(A) (B)

Figure 4

which implies that $\langle \phi(\gamma), \phi(\gamma) \rangle = \langle T, T \rangle = 1$. *From which, the principal normal vector of the e-magnetic curve is given as follows:*

$$e = (0, 0, 1).$$

Besides, the geodesic curvature

$$\kappa_g = 0,$$

which means that γ *is a e-magnetic curve and the certain function of* γ *is* $\rho = 1$.

Moreover, the spherical image of the tangent indicatrix and the principal normal indicatrix of the e-magnetic curve are respectively, (see Fig. 6)

$$\gamma_t = T = (-\sin[s], \cos[s], 0),$$

$$\gamma_e = e = (0, 0, 1),$$

For γ_t *the natural representation and the geodesic curvature are respectively,*

$$s_t = s, \quad \kappa_{g_t} = 0.$$

From the above we can see that the geodesic curvature κ_g *is constant which mean that the curve is also* e_t*-magnetic curve.*

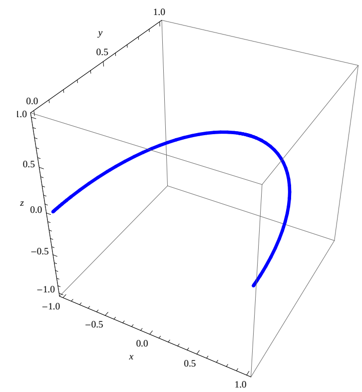
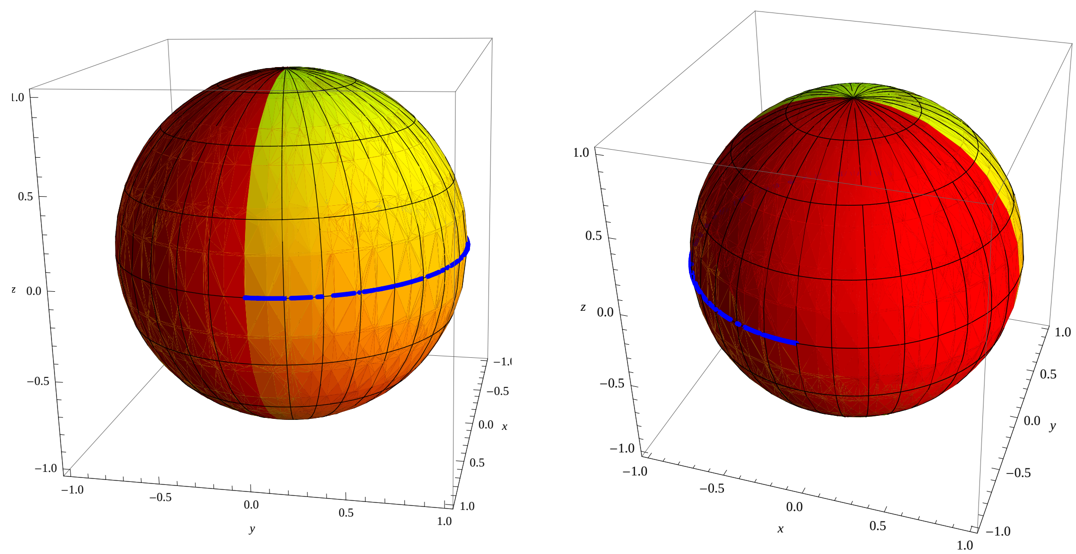


Figure 5



(A) (B)

Figure 6

Remark: For the principal normal indicatrix γ_e of the spherical curve γ, similar procedure as we have done for the tangent indicatrix of γ to get magnetic curves of γ_e.

6. Conclusion

In summary, we examine the conditions of spherical curve γ to be T-magnetic curve or e-magnetic curve and give some characterizations of these curves. Furthermore, for this curve, we investigate spherical images of the tangent indicatrix and binormal indicatrix. Finally as an application for this work, two examples have been given and plotted to confirm our main results.

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