International Journal of Analysis and Applications
Volume 16, Number 4 (2018), 605-613
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-16-2018-605

HERMITE-HADAMARD TYPE INEQUALITIES FOR QUASI-CONVEX FUNCTIONS VIA KATUGAMPOLA FRACTIONAL INTEGRALS

ERHAN SET*, İLKER MUMCU

Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu, Turkey

* *Corresponding author:* erhanset@yahoo.com

Abstract. The paper deals with quasi-convex functions, Katugampola fractional integrals and Hermite-Hadamard type integral inequalities. The main idea of this paper is to present new Hermite-Hadamard type inequalities for quasi-convex functions using Katugampola fractional integrals, Hölder inequality and the identities in the literature.

1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda u + (1-\lambda) v) \leq \lambda f(u) + (1-\lambda) f(v)$$

holds for all $u, v \in I$ and $\lambda \in [0,1]$. This definition has been used in the following inequality that is called Hermite-Hadamard inequality:
Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This inequality has attracted many mathematicians. Especially, in the last three decades, numerous generalizations, variants, and extensions of this inequality have been presented (see, e.g., [1, 3, 13, 14, 20] and the references cited therein).

Received 2017-10-26; accepted 2018-01-03; published 2018-07-02.

2010 *Mathematics Subject Classification.* 26A33, 26D10, 33B20.

Key words and phrases. Hermite-Hadamard inequality; Riemann-Liouville fractional integrals; Katugampola fractional integrals.

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if

$$f(\lambda u + (1-\lambda) v) \leq \max\{f(x), f(y)\},$$

for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [4]).

Let $f \in L_1[a, b] := L(a, b)$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{R}^+$ are defined, respectively, by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt \quad (x > a)$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt \quad (x < b)$$

where Γ is the familiar Gamma function (see, e.g., [21, Section 1.1]). It is noted that $J^1_{a+}f(x)$ and $J^1_{b-}f(x)$ become the usual Riemann integrals.

In the case of $\alpha = 1$, the fractional integral reduces to classical integral.

For further results related to Hermite-Hadamard type inequalities involving fractional integrals on can see [7, 11–19].

The beta function $B(\alpha, \beta)$ is defined by (see, e.g., [21, Section 1.1] [10, p18])

$$B(\alpha, \beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt & (\Re(\alpha) > 0;\ \Re(\beta) > 0) \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \end{cases} \tag{1.2}$$

The hypergeometric function [6]:

$$_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \quad c > b > 0, \ \ |z| < 1.$$

A hypergeometric function can be written using Euler's hypergeometric transformations $(t \to 1-t)$ in equivalent form:

$$_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \tag{1.3}$$

Lemma 1.1. *[9] For $0 < \alpha \leq 1$ and $0 \leq a < b$, we have*

$$|a^{\alpha} - b^{\alpha}| \leq (b-a)^{\alpha}.$$

In [11], Sarıkaya et. al. proved a new version of Hermite-Hadamard's inequalities in Riemann-Liouville fractional integral form as follows:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)] \leq \frac{f(a)+f(b)}{2} \tag{1.4}$$

with $\alpha > 0$.

In [8], Özdemir et. al. gave following results for quasi-convex functions via Riemann-Liouville fractional integrals.

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$, be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a quasi-convex function $[a,b]$, then the following inequality for fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)] \leq \max\{f(a), f(b)\} \tag{1.5}$$

with $\alpha > 0$.

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is quasi-convex on $[a,b]$ with $\alpha > 0$, then the following inequality holds:*

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)]\right| \\ \leq \; & \frac{b-a}{\alpha+1}\left(1 - \frac{1}{2^{\alpha}}\right)\max\{f'(a), f'(b)\}. \end{aligned} \tag{1.6}$$

Theorem 1.4. *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on (a,b) such that $f' \in L_1[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$, and $p > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)]\right| \\ \leq \; & \frac{b-a}{2(\alpha p+1)^{1/p}}\left(1 - \frac{1}{2^{\alpha}}\right)\left(\max\{|f'(a)|^q, |f'(b)|^q\}\right)^{1/q}. \end{aligned} \tag{1.7}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0,1]$.

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on (a,b) such that $f' \in L_1[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$, and $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)]\right| \\ \leq \; & \frac{b-a}{\alpha+1}\left(1 - \frac{1}{2^{\alpha}}\right)\left(\max\{|f'(a)|^q, |f'(b)|^q\}\right)^{1/q}. \end{aligned} \tag{1.8}$$

with $\alpha \in [0,1]$.

Katugampola gave a new fractional integral that generalizes the Riemann-Liouville and the Hadamard fractional integrals into a single form.

Definition 1.1. *[5] Let $[a, b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order $(\alpha > 0)$ of $f \in X_c^p(a, b)$ are defined:*

$$^{\rho}\mathcal{I}_{a+}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)dt$$

and

$$^{\rho}\mathcal{I}_{b-}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t)dt$$

with $a < x < b$ and $\rho > 0$, if the integral exist.

Theorem 1.6. *[5] Let $\alpha > 0$ and $\rho > 0$. Then for $x > a$,*

1. $lim_{\rho \to 1} {}^{\rho}\mathcal{I}_{a+}^{\alpha} f(x) = J_{a+}^{\alpha} f(x)$,

2. $lim_{\rho \to 0^+} {}^{\rho}\mathcal{I}_{a+}^{\alpha} f(x) = H_{a+}^{\alpha} f(x)$.

Similar results also hold for right-sided operators.

In [2], Chen and Katugampola proved the following lemma:

Lemma 1.2. *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. Then the following equality holds if the fractional integrals exist:*

$$\begin{aligned} &\frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\alpha \rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \left[^{\rho}\mathcal{I}_{a+}^{\alpha} (f \circ g)(b) + {}^{\rho}\mathcal{I}_{b-}^{\alpha} (f \circ g)(a) \right] \\ = \; &\frac{b^\rho - a^\rho}{2} \int_0^1 [(1-t^\rho)^\alpha - t^{\rho\alpha}] t^{\rho-1} f'(t^\rho a^\rho + (1-t^\rho) b^\rho) dt \end{aligned} \tag{1.9}$$

where $g(x) = x^\rho$.

The main purpose of this paper is to establish Hermite-Hadamard's inequalities for quasi-convex functions via Katugampola fractional integral. We also obtain Hermite-Hadamard type inequalities of these classes functions.

2. Main Results

Theorem 2.1. *Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in X_c^p(a^\rho, b^\rho)$. If f is a quasi-convex function on $[a^\rho, b^\rho]$, then the following inequalities holds:*

$$\frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \left[^{\rho}\mathcal{I}_{a+}^{\alpha} (f \circ g)(b) + {}^{\rho}\mathcal{I}_{b-}^{\alpha} (f \circ g)(a) \right] \le max\{f(a^\rho), f(b^\rho)\} \tag{2.1}$$

where $g(x) = x^\rho$.

Proof. Since f is quasi-convex function on $[a^\rho, b^\rho]$, we get

$$f(t^\rho a^\rho + (1-t^\rho) b^\rho) \le max\{f(a^\rho), f(b^\rho)\}$$

and

$$f((1-t^\rho)a^\rho + t^\rho b^\rho) \leq max\{f(a^\rho), f(b^\rho)\}.$$

By adding these inequalities we have

$$\frac{1}{2}\left[f(t^\rho a^\rho + (1-t^\rho)b^\rho) + f((1-t^\rho)a^\rho + t^\rho b^\rho)\right] \leq max\{f(a^\rho), f(b^\rho)\}. \tag{2.2}$$

Multiplying both sides of (2.2) by $t^{\alpha\rho-1}$ and integrating the resulting inequality with respect to t over $[a^\rho, b^\rho]$, we obtain

$$\begin{aligned}
& \int_0^1 t^{\alpha\rho-1} f(t^\rho a^\rho + (1-t^\rho)b^\rho)dt + \int_0^1 t^{\alpha\rho-1} f((1-t^\rho)a^\rho + t^\rho b^\rho)dt \\
= & \int_a^b \left(\frac{b^\rho - x^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} f(x^\rho)\frac{x^{\rho-1}}{b^\rho - a^\rho}dx + \int_a^b \left(\frac{x^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} f(x^\rho)\frac{x^{\rho-1}}{b^\rho - a^\rho}dx \\
= & \frac{1}{(b^\rho - a^\rho)^\alpha}\int_a^b \frac{x^{\rho-1}}{(b^\rho - x^\rho)^{1-\alpha}} f(x^\rho)dx + \frac{1}{(b^\rho - a^\rho)^\alpha}\int_a^b \frac{x^{\rho-1}}{(x^\rho - a^\rho)^{1-\alpha}} f(x^\rho)dx \\
= & \frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho - a^\rho)^\alpha}\left[{}^\rho\mathcal{I}^\alpha_{a+}(f\circ g)(b) + {}^\rho\mathcal{I}^\alpha_{b-}(f\circ g)(a)\right] \\
\leq & \frac{2}{\rho\alpha} max\{f(a^\rho), f(b^\rho)\}
\end{aligned}$$

So we get desired result. The proof is completed. □

Remark 2.1. *In Theorem 2.1, taking limit $\rho \to 1$ we obtain inequality of (1.5).*

Theorem 2.2. *Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on $[a^\rho, b^\rho]$ with $0 \leq a < b$. If $|f'|$ is a quasi-convex function on $[a^\rho, b^\rho]$, then the following inequalities holds:*

$$\begin{aligned}
& \left|\frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[{}^\rho\mathcal{I}^\alpha_{a+}(f\circ g)(b) + {}^\rho\mathcal{I}^\alpha_{b-}(f\circ g)(a)\right]\right| \\
= & \frac{b^\rho - a^\rho}{\rho(\alpha+1)}\left(1 - \frac{1}{2^{\rho(\alpha+1)}}\right) max\{|f'(a^\rho)|\,|f'(b^\rho)|\}
\end{aligned} \tag{2.3}$$

where $g(x) = x^\rho$.

Proof. Using Lemma 1.2 and quasi-convex of $|f'|$ with modulus, we get

$$\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{I}^\alpha_{a+}(f\circ g)(b)+{}^\rho\mathcal{I}^\alpha_{b-}(f\circ g)(a)\right]\right|\\
\leq\ & \frac{b^\rho-a^\rho}{2}\int_0^1|(1-t^\rho)^\alpha-t^{\rho\alpha}|t^{\rho-1}|f'(t^\rho a^\rho+(1-t^\rho)b^\rho)|dt\\
\leq\ & \frac{b^\rho-a^\rho}{2}\int_0^1|(1-t^\rho)^\alpha-t^{\rho\alpha}|t^{\rho-1}max\{|f'(a^\rho)|\,|f'(b^\rho)|\}dt\\
=\ & \frac{b^\rho-a^\rho}{2}max\{|f'(a^\rho)|\,|f'(b^\rho)|\}\\
&\times\left\{\int_0^{1/2^{1/\rho}}[(1-t^\rho)^\alpha-t^{\rho\alpha}]t^{\rho-1}dt+\int_{1/2^{1/\rho}}^1[t^{\rho\alpha}+(1-t^\rho)^\alpha]t^{\rho-1}dt\right\}\\
=\ & \frac{b^\rho-a^\rho}{\rho(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)max\{|f'(a^\rho)|\,|f'(b^\rho)|\}
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^{1/2^{1/\rho}}[(1-t^\rho)^\alpha-t^{\rho\alpha}]t^{\rho-1}dt+\int_{1/2^{1/\rho}}^1[t^{\rho\alpha}+(1-t^\rho)^\alpha]t^{\rho-1}dt\\
=\ & \frac{1}{\rho}\left\{\int_0^{1/2}\left[(1-u)^\alpha-u^\alpha\right]du+\int_{1/2}^1\left[u^\alpha-(1-u)^\alpha\right]du\right\}\\
=\ & \frac{2}{\rho(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right). \qquad (2.4)
\end{aligned}$$

The proof is completed. □

Remark 2.2. *In Theorem 2.2, taking limit $\rho\to 1$ we obtain inequality of (1.6).*

Theorem 2.3. *Let $\alpha>0$ and $\rho>0$. Let $f:[a^\rho,b^\rho]\to\mathbb{R}$ be a differentiable mapping on $[a^\rho,b^\rho]$ with $0\leq a<b$. If $|f'|^q$ is a quasi-convex function on $[a^\rho,b^\rho]$ and $s>1$, then the following inequalities holds:*

$$\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{I}^\alpha_{a+}(f\circ g)(b)+{}^\rho\mathcal{I}^\alpha_{b-}(f\circ g)(a)\right]\right|\\
=\ & \frac{b^\rho-a^\rho}{2}\left(max\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\}\right)^{1/q}(K_1+K_2)^{1/s}
\end{aligned}$$

where

$$\begin{aligned}
K_1 &= \frac{1}{\rho 2^{s+\frac{1-s}{\rho}}}B(s+\frac{1-s}{\rho},\alpha s+1),\\
K_2 &= \frac{\alpha s+1}{2\rho}\,{}_2F_1\left(1-s+\frac{s-1}{\rho},1;\alpha s+2;\frac{1}{2}\right),
\end{aligned}$$

$\frac{1}{s}+\frac{1}{q}=1$ *and* $g(x)=x^\rho$.

Proof. From Lemma 1.1, Lemma 1.2, Hölder inequality and quasi-convex of $|f'|$ with proporties of modulus, we have

$$\begin{aligned}
& \left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{I}^\alpha_{a+}(f\circ g)(b) + {}^\rho\mathcal{I}^\alpha_{b-}(f\circ g)(a)\right]\right| \\
\leq\ & \frac{b^\rho-a^\rho}{2}\int_0^1 |(1-t^\rho)^\alpha - t^{\rho\alpha}| t^{\rho-1}|f'(t^\rho a^\rho+(1-t^\rho)b^\rho)|dt \\
\leq\ & \frac{b^\rho-a^\rho}{2}\left(\int_0^1 |(1-t^\rho)^\alpha - t^{\rho\alpha}|^s t^{s(\rho-1)}dt\right)^{1/s}\left(\int_0^1 |f'(t^\rho a^\rho+(1-t^\rho)b^\rho)|^q dt\right)^{1/q} \\
\leq\ & \frac{b^\rho-a^\rho}{2}\left(\int_0^1 |1-2t^\rho|^{\alpha s} t^{s(\rho-1)}dt\right)^{1/s}\left(max\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\}\right)^{1/q} \\
=\ & \frac{b^\rho-a^\rho}{2}\left(max\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\}\right)^{1/q} \\
& \times\left\{\int_0^{1/2^{1/\rho}}(1-2t^\rho)^{\alpha s}t^{s(\rho-1)}dt + \int_{1/2^{1/\rho}}^1 (2t^\rho-1)^{\alpha s}t^{s(\rho-1)}dt\right\}^{1/s} \\
=\ & \frac{b^\rho-a^\rho}{2}\left(max\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\}\right)^{1/q}(K_1+K_2)^{1/s} \qquad (2.5)
\end{aligned}$$

where

$$\begin{aligned}
K_1 = \int_0^{1/2^{1/\rho}}(1-2t^\rho)^{\alpha s}t^{s(\rho-1)}dt &= \frac{1}{\rho 2^{s+\frac{1-s}{\rho}}}\int_0^1 u^{s-1+\frac{1-s}{\rho}}(1-u)^{\alpha s}du \\
&= \frac{1}{\rho 2^{s+\frac{1-s}{\rho}}} B\left(s+\frac{1-s}{\rho}, \alpha s+1\right) \qquad (2.6) \\
K_2 = \int_{1/2^{1/\rho}}^1 (2t^\rho-1)^{\alpha s}t^{s(\rho-1)}dt &= \frac{1}{2^{s+\frac{1-s}{\rho}}\rho}\int_0^1 u^{\alpha s}(1+u)^{s-1+\frac{1-s}{\rho}}du \\
&= \frac{\alpha s+1}{2\rho}\,{}_2F_1\left(1-s+\frac{s-1}{\rho}, 1; \alpha s+2; \frac{1}{2}\right) \qquad (2.7)
\end{aligned}$$

So, if we use (2.6), (2.7) in (2.5), we obtain desired result. □

Remark 2.3. *In Theorem 2.3, taking limit $\rho \to 1$ we obtain inequality of (1.7).*

Theorem 2.4. *Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on $[a^\rho, b^\rho]$ with $0 \leq a < b$. If $|f'|^q$ is a quasi-convex function on $[a^\rho, b^\rho]$ and $q \geq 1$, then the following inequalities holds:*

$$\begin{aligned}
& \left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{I}^\alpha_{a+}(f\circ g)(b) + {}^\rho\mathcal{I}^\alpha_{b-}(f\circ g)(a)\right]\right| \\
\leq\ & \frac{b^\rho-a^\rho}{\rho(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left(max\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\}\right)^{1/q}
\end{aligned}$$

where $g(x) = x^\rho$.

Proof. From Lemma 1.2, quasi-convex of $|f'|$ and using power-mean inequality with proporties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{I}^\alpha_{a+}(f\circ g)(b)+{}^\rho\mathcal{I}^\alpha_{b-}(f\circ g)(a)\right]\right|\\
\leq\ & \frac{b^\rho-a^\rho}{2}\int_0^1|(1-t^\rho)^\alpha-t^{\rho\alpha}|t^{\rho-1}|f'(t^\rho a^\rho+(1-t^\rho)b^\rho)|dt\\
\leq\ & \frac{b^\rho-a^\rho}{2}\left(\int_0^1|(1-t^\rho)^\alpha-t^{\rho\alpha}|t^{\rho-1}dt\right)^{1-1/q}\\
&\times\left(\int_0^1|(1-t^\rho)^\alpha-t^{\rho\alpha}|t^{\rho-1}|f'(t^\rho a^\rho+(1-t^\rho)b^\rho)|^q dt\right)^{1/q}\\
\leq\ & \frac{b^\rho-a^\rho}{2}\left(\int_0^1|(1-t^\rho)^\alpha-t^{\rho\alpha}|t^{\rho-1}dt\right)^{1-1/q}\\
&\times(max\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\})^{1/q}\left(\int_0^1|(1-t^\rho)^\alpha-t^{\rho\alpha}|t^{\rho-1}dt\right)^{1/q}\\
=\ & \frac{b^\rho-a^\rho}{2}\left(\int_0^1|(1-t^\rho)^\alpha-t^{\rho\alpha}|t^{\rho-1}dt\right)(max\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\})^{1/q}
\end{aligned}$$

Using (2.4) we get desired result. □

Remark 2.4. *In Theorem 2.4, taking limit $\rho\to 1$ we obtain inequality of (1.8).*

3. Acknowledgement

This research is supported by Ordu University Scientific Research Projects Coordination Unit (BAP). Project Number: YKD-17224

References

[1] M.U. Awan, M.A. Noor, M.V. Mihai and K.I. Noor, Fractional Hermite-Hadamard inequalities for differentiables-Godunova-Levin functions, Filomat, 30(12) (2016), 3235C-3241.

[2] H. Chen, U.N. Katugampola, Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities for generalized fractional integrals, J. Math. Anal. Appl., 446, 1274-1291.

[3] S.S. Dragomir and C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[4] D.A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Ann. Univ. Craiova, Math. Comp. Sci. Ser., 34 (2007), 82-87.

[5] U.N. Katugampola, New approach to generalized fractional derivatives, Bull. Math. Anal. Appl., 6(4), (2014), 1-15.

[6] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, Amsterdam: Elsevier; (2006).

[7] M. A. Noor, K. I. Noor, and M. U. Awan, Generalized convexity and integral inequalities, Appl. Math. Inform. Sci., 9(1) (2015),233-243.

[8] M. E. Özdemir, Ç. Yıldız, The Hadamards inequality for quasi-convex functions via fractional integrals, Annals of the University of Craiova, Ann. Univ. Craiova, Math. Comp. Sci. Series, 40(2) (2013), 167-173.

[9] A.P. Prudnikov, Y.A. Brychkov, O.I. Marichev, Integral and series. In: Elementary Functions, vol. 1. Nauka, Moscow; (1981).

[10] E.D. Rainville, Special Functions, The Mcmillan Company, New York, 1960.

[11] M. Z. Sarıkaya, E. Set, H. Yaldız, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Math. Comput. Model., 57 (2013), 2403-2407.

[12] E. Set, M.Z. Sarikaya, M.E. Özdemir, H. Yildirim, The Hermite-Hadamards inequality for some convex functions via fractional integrals and related results, J. Appl. Math. Stat. Inf., 10(2) (2014), 69-83.

[13] E. Set, İ. İşcan, F. Zehir, On some new inequalities of Hermite-Hadamard type involving harmonically convex functions via fractional integrals, Konuralp J. Math., 3(1) (2015), 42-55.

[14] E. Set, A.O. Akdemir, İ. Mumcu, Hermite-Hadamard's Inequality and its Extensions for Conformable Fractional Integrals of Any Order $\alpha > 0$, ResearchGate, https://www.researchgate.net/publication/303382221.

[15] E. Set, B. Çelik, Fractional Hermite-Hadamard type inequalities for quasi-convex functions, Ordu Univ. J. Sci. Tech., 6(1) (2016), 137-149.

[16] E. Set, A.O. Akdemir, B. Çelik, On Generalization of Fejer Type Inequalities via fractional integral operator, ResearchGate, https://www.researchgate.net/publication/311452467.

[17] E. Set, A.O. Akdemir, İ. Mumcu, Ostrowski type inequalities involving special functions via conformable fractional integrals, J. Adv. Math. Stud., 10(3) (2017), 386-395.

[18] E. Set, M.A. Noor, M.U. Awan, A. Gözpinar, Generalized Hermite-Hadamard type inequalities involving fractional integral operators, J. Ineq. Appl., 2017(169) (2017), 1-10.

[19] E. Set,J. Choi, A. Gözpinar, Hermite-Hadamard type inequalities for the generalized k-fractional integral operators, J. Ineq. Appl., 2017 (2017), Art. ID 206.

[20] E. Set, M.E. Özdemir, M.Z. Sarıkaya, Inequalities of Hermite-Hadamard type for functions whose derivatives absolute values are m-convex, AIP Conf. Proc., 1309(1) (2010), 861-863.

[21] H.M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York, 2012.