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DIFFERENTIAL SUBORDINATIONS FOR HIGHER-ORDER DERIVATIVES OF MULTIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH DZIOK-SRIVASTAVA OPERATOR

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Abstract. By making use of the principle of subordination, we introduce a new class for higher- order derivatives of multivalent analytic functions associated with Dziok-Srivastava operator. Also we obtain some results for this class.

1. Introduction

Let $R(p,m)$ denote the class of functions f of the form:

$$f(z) = z^p + \sum_{n=m}^{\infty} a_{n+p} z^{n+p}, \quad (p, m \in N = \{1, 2, \cdots\}; z \in U), \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$.

For two functions f and g analytic in U, we say that the function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z) (z \in U)$, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$ such that $f(z) = g(w(z)), (z \in U)$. In particular, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

If $f \in R(p,m)$ is given by (1.1) and $g \in R(p,m)$ given by

$$g(z) = z^p + \sum_{n=m}^{\infty} b_{n+p} z^{n+p}, \quad (p, m \in N = \{1, 2, \cdots\}; z \in U),$$

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then the Hadamard product (or convolution) $f * g$ of f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=m}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g * f)(z).$$

A function $f \in R(1, m)$ is said to be starlike of order α in U if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (0 \leq \alpha < 1; z \in U).$$

Denote the class of all starlike functions of order α in U by $S^*(\alpha)$.

A function $f \in R(1, m)$ is said to be prestarlike of order α in U if

$$\frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in S^*(\alpha), \quad (\alpha < 1).$$

Denote the class of all prestarlike functions of order α in U by $\Re(\alpha)$.

Clearly a function $f \in R(1, m)$ is in the class $\Re(0)$ if and only if f is convex univalent in U and $\Re(\frac{1}{2}) = S^*(\frac{1}{2})$.

For complex parameters $\alpha_i \in C, \beta_j \in C \setminus Z_0^-$, where $Z_0^- = \{0, -1, -2, \cdots\}$; $1 \leq i \leq l, 1 \leq j \leq k; l, k \in N_0 = N \cup \{0\}$, the generalized hypergeometric function ${}_lF_k(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_k; z)$ is defined by the following infinite series:

$${}_lF_k(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_k; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_k)_n} \frac{z^n}{n!},$$

$$(l \leq k+1; l, k \in N_0; z \in U),$$

where $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \text{for } n = 0 \\ x(x+1)\cdots(x+n-1) & \text{for } n \in N. \end{cases}$$

Corresponding to a function $h_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_k; z)$ defined by

$$h_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_k; z) = z^p {}_lF_k(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_k; z). \quad (1.2)$$

Dziok and Srivastava [2] introduced a linear operator

$$H_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_k) : R(p, 1) \longrightarrow R(p, 1),$$

defined in terms of the Hadamard product as

$$H_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_k) f(z) = h_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_k; z) * f(z).$$

If $f \in R(p, m)$ is given by (1.1), then we have

$$H_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_k) f(z) = z^p + \sum_{n=m}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_k)_n n!} a_{n+p} z^{n+p}. \quad (1.3)$$

In order to make the notation simple, we write

$$H_p^{l,k}(\alpha_1) = H_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_k).$$

We note from (1.3) that, we have

$$z\left(H_p^{l,k}(\alpha_1)f(z)\right)' = \alpha_1 H_p^{l,k}(\alpha_1+1)f(z) - (\alpha_1 - p)H_p^{l,k}(\alpha_1)f(z). \tag{1.4}$$

Differentiating (1.4), $(q-1)$ times, we get

$$z\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q)} = \alpha_1\left(H_p^{l,k}(\alpha_1+1)f(z)\right)^{(q-1)} - (\alpha_1 - p + q - 1)\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q-1)}. \tag{1.5}$$

We note that special cases of the Dziok-Srivastava operator $H_p^{l,k}(\alpha_1)$ include the Hohlov linear operator [3], the Carlson-Shafer operator [1], the Ruscheweyh derivative operator [8], the Srivastava-Owa fractional operator [7], and many others.

Let H be the class of functions h with $h(0) = 1$, which are analytic and convex univalent in U.

Definition 1.1. *A function $f \in R(p,m)$ is said to be in the class $E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$ if it satisfies the subordination condition:*

$$\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q)}}{z^{p-q}} \prec h(z), \tag{1.6}$$

where $\eta \in C, p, q \in N, p > q$ and $h \in H$.

By specializing the parameters $l, k, \alpha_i, \beta_j, \eta, p, q$ and m, we obtain the following subclasses of analytic functions studied by various authors:

1) For $l = 2$, $k = q = m = 1$, $\alpha_1 = \lambda + p(\lambda > -p), \alpha_2 = c$ and $\beta_1 = a$, the class $E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$ reduces to the class $\mathcal{B}_p^{\lambda}(a, c, \eta; h)$ which was studied by Liu [5].

2) For $l = 2$, $k = q = m = p = \alpha_2 = \beta_1 = 1$, $\alpha_1 = 2$ and $h(z) = \frac{1+az}{1+bz}$ $(-1 \le b < 1, a > b)$, the class $E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$ reduces to the class $H(\eta, a, b)$ which was studied by Yang [11].

3) For $l = 2$, $k = q = m = p = \eta = \alpha_2 = \beta_1 = 1$, $\alpha_1 = 2$ and $h(z) = \frac{1+z}{1-z}$, the class $E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$ reduces to the class which was studied by Singh and Singh [10].

4) For $l = 2$, $k = q = m = p = \alpha_2 = \beta_1 = 1$, $\alpha_1 = 2$ and $h(z) = 1 + Mz(M > 0)$, the class $E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$ reduces to the class $H_1(1, \alpha; 1 + Mz) = S(\alpha, M)$ which was studied by Zhou and Owa [12] and Liu [4] respectively.

In order to prove our main results, we need the following lemmas.

Lemma 1.1. *[6] Let g be analytic in U and let h be analytic and convex univalent in U with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu}zg'(z) \prec h(z), \tag{1.7}$$

where $Re(\mu) \geq 0$ *and* $\mu \neq 0$, *then*

$$g(z) \prec \breve{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t) dt \prec h(z)$$

and $\breve{h}$ *is the best dominant of (1.7).*

Lemma 1.2. *[9] Let* $\alpha < 1$ *,* $f \in S^*(\alpha)$ *and* $g \in R(\alpha)$*. Then, for any analytic function* F *in* U

$$\frac{g * (fF)}{g * f}(U) \subset \bar{co}\left(F(U)\right),$$

where $\bar{co}\left(F(U)\right)$ *denotes the closed convex hull of* $F(U)$*.*

2. Main Results

Theorem 2.1. *Let* $0 \leq \eta < \varepsilon$*. Then* $E_{p,q}^{l,k}(\varepsilon, \alpha_1, m; h) \subset E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$*.*

Proof. Let $0 \leq \eta < \varepsilon$ and $f \in E_{p,q}^{l,k}(\varepsilon, \alpha_1, m; h)$.

Suppose that

$$g(z) = \frac{(p-q+1)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1) f(z)\right)^{(q-1)}}{z^{p-q+1}}. \tag{2.1}$$

Then the function g is analytic in U with $g(0) = 1$.

Since $f \in E_{p,q}^{l,k}(\varepsilon, \alpha_1, m; h)$, then we have

$$\frac{(1-\varepsilon)(p-q+1)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1) f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\varepsilon(p-q)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1) f(z)\right)^{(q)}}{z^{p-q}} \prec h(z). \tag{2.2}$$

By taking the derivatives in the both sides of (2.1) with respect to z and using (2.2), we get

$$\frac{(1-\varepsilon)(p-q+1)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1) f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\varepsilon(p-q)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1) f(z)\right)^{(q)}}{z^{p-q}} = g(z) + \frac{\varepsilon}{p-q+1} z g'(z) \prec h(z).$$

Hence, an application of Lemma 1.1 with $\mu = \dfrac{p-q+1}{\varepsilon}$, yields

$$g(z) \prec h(z). \tag{2.3}$$

Nothing that $0 \leq \frac{\eta}{\varepsilon} < 1$ and that h is convex univalent in U, it follows from (2.1),(2.2) and (2.3) that

$$\frac{(1-\eta)(p-q+1)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1) f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1) f(z)\right)^{(q)}}{z^{p-q}}$$

$$= \frac{\eta}{\varepsilon} \left[\frac{(1-\varepsilon)(p-q+1)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1) f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\varepsilon(p-q)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1) f(z)\right)^{(q)}}{z^{p-q}} \right] + \left(1 - \frac{\eta}{\varepsilon}\right) g(z) \prec h(z).$$

Therefore, $f \in E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$ and the proof of Theorem 2.1 is completed. □

Theorem 2.2. *Let* $Re\left\{\alpha_1\right\} \geq 0$ *and* $\alpha_1 \neq 0$*. Then* $E_{p,q}^{l,k}(\eta, \alpha_1 + 1, m; h) \subset E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$*.*

Proof. Let $f \in E^{l,k}_{p,q}(\eta, \alpha_1 + 1, m; h)$ and suppose that

$$g(z) = \frac{(1-\eta)(p-q+1)!}{p!} \frac{\left(H^{l,k}_p(\alpha_1) f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!} \frac{\left(H^{l,k}_p(\alpha_1) f(z)\right)^{(q)}}{z^{p-q}}. \tag{2.4}$$

Then, from (1.5), (2.4) is equivalent to

$$g(z) = \left(1 - \frac{\eta\alpha_1}{p-q+1}\right) \frac{(p-q+1)!}{p!} \frac{\left(H^{l,k}_p(\alpha_1) f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta\alpha_1(p-q)!}{p!} \frac{\left(H^{l,k}_p(\alpha_1+1) f(z)\right)^{(q-1)}}{z^{p-q+1}}. \tag{2.5}$$

Differentiating both sides of (2.5) with respect to z and using (1.5), we have

$$\begin{aligned} g(z) + zg'(z) = &\left(1 - \frac{\eta(\alpha_1+p-q)}{p-q+1}\right) \frac{\alpha_1(p-q+1)!}{p!} \frac{\left(H^{l,k}_p(\alpha_1+1) f(z)\right)^{(q-1)}}{z^{p-q+1}} \\ &- \left(1 - \frac{\eta\alpha_1}{p-q+1}\right) \frac{(\alpha_1-1)(p-q+1)!}{p!} \frac{\left(H^{l,k}_p(\alpha_1) f(z)\right)^{(q-1)}}{z^{p-q+1}} \\ &+ \frac{\eta\alpha_1(p-q)!}{p!} \frac{\left(H^{l,k}_p(\alpha_1+1) f(z)\right)^{(q)}}{z^{p-q}}. \end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we get

$$\alpha_1 g(z) + zg'(z) = \frac{\alpha_1(1-\eta)(p-q+1)!}{p!} \frac{\left(H^{l,k}_p(\alpha_1+1) f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\alpha_1\eta(p-q)!}{p!} \frac{\left(H^{l,k}_p(\alpha_1+1) f(z)\right)^{(q)}}{z^{p-q}},$$

that is

$$g(z) + \frac{1}{\alpha_1} zg'(z) = \frac{(1-\eta)(p-q+1)!}{p!} \frac{\left(H^{l,k}_p(\alpha_1+1) f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!} \frac{\left(H^{l,k}_p(\alpha_1+1) f(z)\right)^{(q)}}{z^{p-q}}. \tag{2.7}$$

Since $f \in E^{l,k}_{p,q}(\eta, \alpha_1 + 1, m; h)$, then it follows from (2.7) that

$$g(z) + \frac{1}{\alpha_1} zg'(z) \prec h(z), \quad (Re\{\alpha_1\} \geq 0, \alpha_1 \neq 0).$$

Hence, an application of Lemma 1.1 with $\mu = \alpha_1$, yields $g(z) \prec h(z)$. By using (2.4), we obtain the following

$$\frac{(1-\eta)(p-q+1)!}{p!} \frac{\left(H^{l,k}_p(\alpha_1) f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!} \frac{\left(H^{l,k}_p(\alpha_1) f(z)\right)^{(q)}}{z^{p-q}} \prec h(z).$$

This shows that $f \in E^{l,k}_{p,q}(\eta, \alpha_1, m; h)$ and the proof of Theorem 2.2 is completed. □

Theorem 2.3. *Let $f \in R(p, 1)$ and*

$$Re\left\{\frac{\theta_p(a, b; z)}{z^p}\right\} > \frac{1}{2}, \tag{2.8}$$

where $\theta_p(a, b; z) = h_p(a, \alpha_2, \cdots, \alpha_k, 1; b, \alpha_2, \cdots, \alpha_k; z)$ is defined as in (1.2). Then

$$E^{l,k}_{p,q}(\eta, b, 1; h) \subset E^{l,k}_{p,q}(\eta, a, 1; h).$$

Proof. Let $f \in E^{l,k}_{p,q}(\eta, b, 1; h)$. Then, we have

$$\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H^{l,k}_p(a)f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!}\frac{\left(H^{l,k}_p(a)f(z)\right)^{(q)}}{z^{p-q}}$$

$$= \frac{(1-\eta)(p-q+1)!}{p!}\left(\frac{\theta_p(a,b;z)}{z^p}\right) * \left(\frac{\left(H^{l,k}_p(b)f(z)\right)^{(q-1)}}{z^{p-q+1}}\right)$$

$$+ \frac{\eta(p-q)!}{p!}\left(\frac{\theta_p(a,b;z)}{z^p}\right) * \left(\frac{\left(H^{l,k}_p(b)f(z)\right)^{(q)}}{z^{p-q}}\right) = \left(\frac{\theta_p(a,b;z)}{z^p}\right) * \psi(z), \tag{2.9}$$

where

$$\psi(z) = \frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H^{l,k}_p(b)f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!}\frac{\left(H^{l,k}_p(b)f(z)\right)^{(q)}}{z^{p-q}} \prec h(z). \tag{2.10}$$

From (2.8) note that the function $\frac{\theta_p(a,b;z)}{z^p}$ has the Herglotz representation

$$\frac{\theta_p(a,b;z)}{z^p} = \int_{|x|=1}\frac{d\mu(x)}{1-xz} \quad (z \in U), \tag{2.11}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since h is convex univalent in U, it follows from (2.9), (2.10) and (2.11) that

$$\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H^{l,k}_p(a)f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!}\frac{\left(H^{l,k}_p(a)f(z)\right)^{(q)}}{z^{p-q}} = \int_{|x|=1}\psi(xz)\, d\mu(x) \prec h(z).$$

This shows that $f \in E^{l,k}_{p,q}(\eta, a, 1; h)$ and the theorem is proved. □

Theorem 2.4. *Let $0 < a < b$ and $f \in R(p, 1)$. Then $E^{l,k}_{p,q}(\eta, b, 1; h) \subset E^{l,k}_{p,q}(\eta, a, 1; h)$.*

Proof. Define the function g by

$$g(z) = z + \sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n}z^{n+1}, \quad (0 < a < b; z \in U).$$

Then

$$\frac{\theta_p(a,b;z)}{z^{p-1}} = g(z) \in R(p,1), \tag{2.12}$$

where $\theta_p(a, b; z) = h_p(a, \alpha_2, \cdots, \alpha_k, 1; b, \alpha_2, \cdots, \alpha_k; z)$ is defined as in (1.2) and

$$\frac{z}{(1-z)^b} * g(z) = \frac{z}{(1-z)^a}. \tag{2.13}$$

By (2.13), we see that $\frac{z}{(1-z)^b} * g(z) \in S^*\left(1 - \frac{a}{2}\right) \subset S^*\left(1 - \frac{a}{2}\right)$.

For $0 < a < b$ which shows that

$$g(z) \in \Re\left(1 - \frac{b}{2}\right). \tag{2.14}$$

Let $f \in E^{l,k}_{p,q}(\eta, b, 1; h)$. Then from (2.9) (used in the proof of Theorem 2.3) and (2.12)), we can write

$$\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H^{l,k}_p(a)f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!}\frac{\left(H^{l,k}_p(a)f(z)\right)^{(q)}}{z^{p-q}} = \frac{g(z) * (z\psi(z))}{g(z) * z}, \tag{2.15}$$

where $\psi(z)$ is defined as in (2.10).

Since h is convex univalent in U, $\psi(z) \prec h(z)$ and $z \in S^*(1-\frac{b}{2})$, it follows from (2.14), (2.15) and Lemma 1.2 that

$$\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(a)f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!}\frac{\left(H_p^{l,k}(a)f(z)\right)^{(q)}}{z^{p-q}} \prec h(z).$$

Therefore, $f \in E_{p,q}^{l,k}(\eta, a, 1; h)$ and the proof is completed. □

Theorem 2.5. *Let $\eta > 0, \gamma > 0$ and $f \in E_{p,q}^{l,k}(\eta, \alpha_1, m; \gamma h + 1 - \gamma)$. If $\gamma \leq \gamma_0$, where*

$$\gamma_0 = \frac{1}{2}\left(1 - \frac{(p-q+1)}{\eta}\int_0^1 \frac{u^{\frac{p-q+1}{\eta}-1}}{1+u}du\right)^{-1}, \tag{2.16}$$

then $f \in E_{p,q}^{l,k}(0, \alpha_1, m; h)$. The bound γ_0 is the sharp when $h(z) = \frac{1}{1-z}$.

Proof. Suppose that

$$g(z) = \frac{(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q-1)}}{z^{p-q+1}}. \tag{2.17}$$

Let $f \in E_{p,q}^{l,k}(\eta, \alpha_1, m; \gamma h + 1 - \gamma)$ with $\eta > 0$ and $\gamma > 0$. Then, we have

$$g(z) + \frac{\eta}{p-q+1}zg'(z) = \frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q)}}{z^{p-q}}$$
$$\prec \gamma h(z) + 1 - \gamma.$$

By using Lemma 1.1, we have

$$g(z) \prec \frac{\gamma(p-q+1)}{\eta}z^{-\frac{(p-q+1)}{\eta}}\int_0^z t^{\frac{p-q+1}{\eta}-1}h(t)dt + 1 - \gamma = (h*\phi)(z), \tag{2.18}$$

where

$$\phi(z) = \frac{\gamma(p-q+1)}{\eta}z^{-\frac{(p-q+1)}{\eta}}\int_0^z \frac{t^{\frac{p-q+1}{\eta}-1}}{1-t}dt + 1 - \gamma. \tag{2.19}$$

If $0 < \gamma \leq \gamma_0$, where $\gamma_0 > 1$ is given by (2.16), then it follows from (2.19) that

$$Re\left(\phi(z)\right) = \frac{\gamma(p-q+1)}{\eta}\int_0^1 u^{\frac{p-q+1}{\eta}-1}Re\left(\frac{1}{1-uz}\right)du + 1 - \gamma$$
$$> \frac{\gamma(p-q+1)}{\eta}\int_0^1 \frac{u^{\frac{p-q+1}{\eta}-1}}{1+u}du + 1 - \gamma \geq \frac{1}{2}.$$

Now, by using the Herglotz representation for $\phi(z)$, from (2.17) and (2.18), we get

$$\frac{(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q-1)}}{z^{p-q+1}} \prec (h*\phi)(z) \prec h(z).$$

Since h is convex univalent in U, then $f \in E_{p,q}^{l,k}(0, \alpha_1, m; h)$.

For $h(z) = \frac{1}{1-z}$ and $f \in R(p,m)$ defined by

$$\frac{(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q-1)}}{z^{p-q+1}} = \frac{\gamma(p-q+1)}{\eta}z^{-\frac{(p-q+1)}{\eta}}\int_0^z \frac{t^{\frac{p-q+1}{\eta}-1}}{1-t}dt + 1 - \gamma,$$

we have

$$\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q-1)}}{z^{p-q+1}}+\frac{\eta(p-q)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q)}}{z^{p-q}}=\gamma h(z)+1-\gamma.$$

Thus, $f\in E_{p,q}^{l,k}(\eta,\alpha_1,m;\gamma h+1-\gamma)$. Also, for $\gamma>\gamma_0$, we have

$$Re\left\{\frac{(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q-1)}}{z^{p-q+1}}\right\}\longrightarrow\frac{\gamma(p-q+1)}{\eta}\int_0^1\frac{u^{\frac{p-q+1}{\eta}-1}}{1+u}du+1-\gamma<\frac{1}{2},\quad(z\to-1),$$

which implies that $f\notin E_{p,q}^{l,k}(0,\alpha_1,m;h)$. Therefore the bound γ_0 cannot be increased when $h(z)=\frac{1}{1-z}$. This completes the proof of the theorem. □

Theorem 2.6. *Let $f\in E_{p,q}^{l,k}(\eta,\alpha_1,m;h)$ be defined as in (1.1). Then the function I defined by*

$$I(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt,\ (Re(c)>-p),\tag{2.20}$$

is also in the class $E_{p,q}^{l,k}(\eta,\alpha_1,m;h)$.

Proof. Let $f\in E_{p,q}^{l,k}(\eta,\alpha_1,m;h)$ be defined as in (1.1). Then, we have

$$\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q-1)}}{z^{p-q+1}}+\frac{\eta(p-q)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q)}}{z^{p-q}}\prec h(z).\tag{2.21}$$

For $f\in R(p,m)$ and $Re(c)>-p$, we find from (2.20) that $I\in R(p,m)$ and

$$f(z)=\frac{cI(z)+zI'(z)}{c+p}.\tag{2.22}$$

Define the function J by

$$J(z)=\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q-1)}}{z^{p-q+1}}+\frac{\eta(p-q)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q)}}{z^{p-q}}.\tag{2.23}$$

Differentiating both sides of (2.23) with respect to z and using (2.21) and (2.22), we have

$$\begin{aligned}&\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q-1)}}{z^{p-q+1}}+\frac{\eta(p-q)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q)}}{z^{p-q}}\\&=\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)\left(\frac{cI(z)+zI'(z)}{c+p}\right)\right)^{(q-1)}}{z^{p-q+1}}+\frac{\eta(p-q)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)\left(\frac{cI(z)+zI'(z)}{c+p}\right)\right)^{(q)}}{z^{p-q}}\\&=\frac{c}{c+p}\left(\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q-1)}}{z^{p-q+1}}+\frac{\eta(p-q)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q)}}{z^{p-q}}\right)\\&+\frac{1}{c+p}\left(\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)\left(zI'(z)\right)\right)^{(q-1)}}{z^{p-q+1}}+\frac{\eta(p-q)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)\left(zI'(z)\right)\right)^{(q)}}{z^{p-q}}\right)\\&=\frac{c}{c+p}J(z)+\frac{1}{c+p}\left(zJ'(z)+pJ(z)\right)=J(z)+\frac{1}{c+p}zJ'(z)\prec h(z).\end{aligned}$$

Hence, an application of Lemma 1.1 with $\mu=c+p$, yields $J(z)\prec h(z)$. By using (2.23), we get

$$\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q-1)}}{z^{p-q+1}}+\frac{\eta(p-q)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q)}}{z^{p-q}}\prec h(z),$$

which implies that $I \in E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$. □

Theorem 2.7. *Let $f \in R(p, m)$ and I be defined as in (2.20). If*

$$\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q-1)}}{z^{p-q+1}} \prec h(z),\ (\eta > 0), \tag{2.24}$$

then $I \in E_{p,q}^{l,k}(0, \alpha_1, m; h)$.

Proof. Suppose that

$$J(z) = \frac{(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q-1)}}{z^{p-q+1}}. \tag{2.25}$$

Then the function J is analytic in U with $J(0) = 1$. Differentiating both sides of (2.25) with respect to z, we have

$$zJ'(z) = \frac{(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q)}}{z^{p-q}} - (p-q+1)J(z). \tag{2.26}$$

Making use of (2.22), (2.24), (2.25) and (2.26), we deduce that

$$\begin{aligned}
&\frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(q-1)}}{z^{p-q+1}} \\
&= \frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)\left(\frac{cI(z)+zI'(z)}{c+p}\right)\right)^{(q-1)}}{z^{p-q+1}} \\
&= \frac{(1-\eta)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q-1)}}{z^{p-q+1}} \\
&+ \frac{\eta}{c+p}\left[\frac{(c+q-1)(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q)}}{z^{p-q}}\right] \\
&= J(z) + \frac{\eta}{c+p}zJ'(z) \prec h(z).
\end{aligned}$$

Hence, an application of Lemma 1.1 with $\mu = \frac{c+p}{\eta}$, yields $J(z) \prec h(z)$. By using (2.25), we get

$$\frac{(p-q+1)!}{p!}\frac{\left(H_p^{l,k}(\alpha_1)I(z)\right)^{(q-1)}}{z^{p-q+1}} \prec h(z),$$

which implies that $I \in E_{p,q}^{l,k}(0, \alpha_1, m; h)$. □

Theorem 2.8. *Let $f \in E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$, $g \in R(p, m)$ and*

$$Re\left\{\frac{g(z)}{z^p}\right\} > \frac{1}{2}. \tag{2.27}$$

Then $f * g \in E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$.

Proof. Let $f \in E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$ and $g \in R(p, m)$. Then, we have

$$\begin{aligned}
&\frac{(1-\eta)(p-q+1)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1)\,(f*g)\,(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1)\,(f*g)\,(z)\right)^{(q)}}{z^{p-q}} \\
&= \frac{(1-\eta)(p-q+1)!}{p!} \left(\frac{g(z)}{z^p}\right) * \left(\frac{\left(H_p^{l,k}(\alpha_1) f(z)\right)^{(q-1)}}{z^{p-q+1}}\right) \\
&+ \frac{\eta(p-q)!}{p!} \left(\frac{g(z)}{z^p}\right) * \left(\frac{\left(H_p^{l,k}(\alpha_1) f(z)\right)^{(q)}}{z^{p-q}}\right) = \left(\frac{g(z)}{z^p}\right) * \varphi(z),
\end{aligned} \tag{2.28}$$

where

$$\varphi(z) = \frac{(1-\eta)(p-q+1)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1) f(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1) f(z)\right)^{(q)}}{z^{p-q}} \prec h(z). \tag{2.29}$$

From (2.27) note that the function $\frac{g(z)}{z^p}$ has the Herglotz representation

$$\frac{g(z)}{z^p} = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in U), \tag{2.30}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since h is convex univalent in U, it follows from (2.28), (2.29) and (2.30) that

$$\begin{aligned}
\frac{(1-\eta)(p-q+1)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1)\,(f*g)\,(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1)\,(f*g)\,(z)\right)^{(q)}}{z^{p-q}} &= \int_{|x|=1} \psi(xz)\, d\mu(x) \\
&\prec h(z).
\end{aligned}$$

This shows that $f * g \in E_{p,q}^{l,k}(\eta, \alpha_{,1}, m; h)$. $\square$

Theorem 2.9. *Let $f \in E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$, $g \in R(p, m)$ and $z^{1-p} g(z) \in \Re(\alpha)$, $(\alpha < 1)$. Then $f * g \in E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$.*

Proof. For $f \in E_{p,q}^{l,k}(\eta, \alpha_1, m; h)$ and $g \in R(p, m)$, from (2.28) (used in the proof of Theorem 2.8), we can write

$$\begin{aligned}
&\frac{(1-\eta)(p-q+1)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1)\,(f*g)\,(z)\right)^{(q-1)}}{z^{p-q+1}} + \frac{\eta(p-q)!}{p!} \frac{\left(H_p^{l,k}(\alpha_1)\,(f*g)\,(z)\right)^{(q)}}{z^{p-q}} \\
&= \frac{\left(z^{1-p} g(z)\right) * (z\varphi(z))}{(z^{1-p} g(z)) * z}, \quad (z \in U),
\end{aligned} \tag{2.31}$$

where $\varphi(z)$ is defined as in (2.29). Since h is convex univalent in U, $\psi(z) \prec h(z)$, $z^{1-p} g(z) \in \Re(\alpha)$ and $z \in S^*(\alpha)$,$(\alpha < 1)$, it follows from (2.31) and Lemma 1.2, we get the result. $\square$

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