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GLOBAL UNIQUENESS RESULT FOR FUNCTIONAL DIFFERENTIAL EQUATIONS DRIVEN BY A WIENER PROCESS AND FRACTIONAL BROWNIAN MOTION

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Abstract. We prove a global existence and uniqueness result for the solution of a mixed stochastic functional differential equation driven by a Wiener process and fractional Brownian motion with Hurst index $H > 1/2$. We also study the dependence of the solution on the initial condition.

1. Introduction

Fractional Brownian motion (fBm) with a Hurst parameter $H \in (0,1)$ is defined formally as a continuous centered Gaussian process $B_t^H = \{B_t^H, t \geq 0\}$ with the covariance

$$R_H = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{1}$$

For $H > 1/2$ it exhibits a property of long-range dependence, which makes it a popular model for long-range dependence in natural sciences, financial mathematics etc. For this reason, equations driven by fractional Brownian motion have been an object of intensive study during the last decade.

From (1) we deduce that $\mathbb{E}(|B_t^H - B_s^H|^2) = |t-s|^{2H}$ and, as a consequence, the trajectories of B^H are almost surely locally α-Hölder continuous for all $\alpha \in (0, H)$. Since B^H is not a semimartingale if $H \neq 1/2$ (see [7]), we cannot use the classical Itô theory to construct a stochastic calculus with respect to the fBm. Over the last years some new techniques have been developed in order to define stochastic integrals with respect to fBm. Essentially two different types of integrals can be defined:

One possibility is Skorokhod, or divergence integral introduced in the fractional Brownian setting in [2]. However this definition is not very practical: it is based on Wick rather than usual products, and unlike Brownian case, in the fractional Brownian case this makes difference when integrating non-anticipating functions because of dependence of increments. This makes this definition worthless for most applications (most notably, those in financial mathematics). Moreover, it is impossible to solve stochastic differential equations with such integral except the cases of additive or multiplicative noise; the latter case was considered in [10].

Another approach is a pathwise integral, defined first in [13] for fBm with

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$H > 1/2$ as a Young integral. The papers [6, 11] were the first to prove existence and uniqueness of stochastic differential equations involving such integrals. Later the pathwise approach was extended with the help of Lyons?rough path theory to the case of arbitrary H in [1] where also unique solvability of equations with $H > 1/4$ was proved.

Very recently, the stochastic differential equations driven simultaneously by a fractional Brownian motion and standard Brownian motion have been studied by several authors. In [5] Guerra and Nualart have proved an existence and uniqueness theorem for solutions of multidimensional, time dependent, stochastic differential equations driven by a multidimensional fractional Brownian motion with Hurst parameter $H > 1/2$ and a multidimensional standard Brownian motion using a techniques of the classical fractional calculus and the classical Itô stochastic calculus. Their (existence) result is based on the Yamada-Watanabe theorem. In [8] the existence and uniqueness of solutions is proved by Mishura and Shevchenko for differential equations driven by a fractional Brownian motion with parameter $H > 1/2$ and a Wiener process in one dimensional case, under mild regularity assumptions on the coefficients. For the same equation, with nonhomogeneous coefficients and random initial condition, the convergence in Besov space of the solutions depending on a parameter has been studied in [9] by Mishura and Posashkova.

In this paper we focus on the following mixed stochastic functional differential equation involving Wiener process and fractional Brownian motion, with nonconstant delay

(2)

$$\begin{aligned} x(t) &= \phi(0) + \int_0^t b(s, x_s)ds + \int_0^t \sigma_W(s, x_s)dW(s) + \int_0^t \sigma_H(s, x_s)dB^H(s), \quad t \geq 0 \\ x_0 &= \phi \in \mathcal{C}_r, \end{aligned}$$

where $B^H = \{B^H(t); t \in [0,T]\}$ is a fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$, $W = \{W(t); t \in [0,T]\}$ is a Wiener process and $\mathcal{C}_r$ is the space of all continuous functions f from $[-r, 0]$ to $\mathbb{R}$ endowed by the uniform norm $\|\cdot\|$. Here, $x_t \in \mathcal{C}_r$ denote the function defined by $x_t(u) = x(t+u)$, $\forall u \in [-r, 0]$ and the coefficients $b, \sigma_W, \sigma_H : [0,T] \times \mathcal{C}_r \to \mathbb{R}$ are appropriate functions. The stochastic integral w.r.t. Wiener process in (2) is the standard Itô integral, and the integral w.r.t. fBm is pathwise generalized Lebesgue-Stieltjes, or Young integral.

Our goal in this paper is to prove the existence and uniqueness of the solution for equation (2). Then we will study the dependance of the solution on the initial condition. We first prove our results for deterministic equations and we will easily apply them pathwise to the Wiener processus and fractional Brownian motion.

The paper is organized as follows. In Section 2, we state the problem and list our assumptions on the coefficients of Eq. (2). Section 3, contains some basic facts about extended Stieltjes integrals. In Section 4, we derive some precise estimates for the integrals involved in Eq. (2). Section 5 is devoted to obtain the existence, uniqueness and dependence on the initial data for the solution of the deterministic equations. In Section 6, we apply the results of the previous sections to stochastic equations driven by both Wiener process and fractional Brownian motion and we give the proofs of our main theorems.

2. Main result

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \in [0,T]), \mathbb{P})$ be a complete probability space with a filtration satisfying the standard conditions. Denote by $\{W(t), \mathcal{F}_t, t \in [0,T]\}$ the standard Wiener process adapted to this filtration. Suppose that $B = \{B(t); t \in [0,T]\}$ is an $\mathcal{F}_t$-fBm with Hurst index $H \in (\frac{1}{2}, 1)$. Consider the mixed stochastic functional differential equation (2) and let us consider the following assumptions on the coefficients.

($\mathbf{H}b$) The function $b(t,y)$ is continuous. Moreover, it is Lipschitz continuous in the variable y and has linear growth in the same variable, uniformly in t, that is, there exist constants L_1 and L_2 such that

$$\begin{aligned} |b(t,y) - b(t,z)| &\leq L_1 \|y - z\|, \\ |b(t,y)| &\leq L_2(1 + \|y\|), \end{aligned}$$

for all $y, z \in \mathcal{C}_r$ and $t \in [0,T]$.

($\mathbf{H}\sigma_W$) The function $\sigma_W(t,y)$ is continuous. Moreover, it is Lipschitz continuous in y and has linear growth in the same variable, uniformly in t, that is, there exist constants L_3 and L_4 such that

$$\begin{aligned} |\sigma_W(t,y) - \sigma_W(t,z)| &\leq L_3 \|y - z\|, \\ |\sigma_W(t,y)| &\leq L_4(1 + \|y\|), \end{aligned}$$

for all $y, z \in \mathcal{C}_r$ and $t \in [0,T]$.

($\mathbf{H}\sigma_H$) The function $\sigma_H(t,y)$ is continuous and Fréchet differentiable in the variable y. Moreover, there exist constants L_5, L_6 and L_7 such that

$$\begin{aligned} |\nabla_y \sigma_H(t,y)|_{\mathcal{L}(\mathcal{C}_r, \mathrm{R})} &\leq L_5, \\ |\nabla_y \sigma_H(t,y) - \nabla_y \sigma_H(t,z)|_{\mathcal{L}(\mathcal{C}_r, \mathrm{R})} &\leq L_6 \|y - z\|, \\ |\sigma_H(t,y) - \sigma_H(s,y)| + |\nabla_y \sigma_H(t,y) - \nabla_y \sigma_H(s,y)|_{\mathcal{L}(\mathcal{C}_r, \mathrm{R})} &\leq L_7 |t - s|, \end{aligned}$$

for all $y, z \in \mathcal{C}_r$ and $t \in [0,T]$.
Note that ($\mathbf{H}\sigma_H$) implies the linear growth property, i. e., there exists a constant L such that

$$|\sigma_H(t,y)| \leq L(1 + \|y\|),$$

for all $y \in \mathcal{C}_r$ and $t \in [0,T]$.

Let us define for $\lambda \in (0,1]$ the space $\mathcal{C}^\lambda$ of λ-Hölder continuous functions $f : [0,T] \to \mathbb{R}$, equipped with the norm

$$\|f\|_\lambda := \|f\|_\infty + \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t-s)^\lambda} < \infty,$$

where $\|f\|_\infty := \sup_{t \in [0,T]} |f(t)|$.

Our main results are the following theorems on the uniqueness, existence and dependence of the solution of Eq. (2) on the initial condition.

Theorem 2.1. *Let the assumptions* ($\mathbf{H}b$), ($\mathbf{H}\sigma_W$) *and* ($\mathbf{H}\sigma_H$) *be satisfied, and C be a generic constant which depends on the constants L_i, $1 \leq i \leq 7$.*

(1) *If $1 - H < \alpha < H$ and ϕ is a stochastic process whose trajectories belong to the space $\mathcal{C}^{1-\alpha}([-r,0])$ $\mathbb{P}$-a.s, then there exists a unique solution x of mixed equation* (2) *with paths in $\mathcal{C}^{1-\alpha}([-r,0])$ $\mathbb{P}$-a.s.*

(2) *If in addition $\alpha + H > \frac{3}{2}$, C is independent of ω and the process ϕ satisfies $\mathbb{E}\|\phi\|_{1-\alpha}^p < \infty$ for $p \geq 1$, then the solution x satisfies $\mathbb{E}\|x\|_{1-\alpha}^p < \infty$ for $p \geq 1$.*

Theorem 2.2. *Let the assumptions $(\mathbf{H}b)$, $(\mathbf{H}\sigma_W)$ and $(\mathbf{H}\sigma_H)$ be satisfied, $\phi, \phi^n \in \mathcal{C}^{1-\alpha}([-r, 0])$ and C be a generic constant which depends on the constants L_i, $1 \leq i \leq 7$. Let x be a solution of the mixed equation (2) and x^n the solution of the same equation with ϕ^n in place of ϕ. We assume that $1 - H < \alpha < H$.*

(1) *If $\lim_n \|\phi^n - \phi\|_{\mathbf{1}-\alpha} = 0$, a.s., then we have, for $\mathbf{P}$-almost all $\omega \in \Omega$, $\lim_n \|x^n(\omega, .) - x(\omega, .)\|_{\mathbf{1}-\alpha} = 0$.*

(2) *If in addition $\alpha + H > \frac{3}{2}$, C is independent of ω and ϕ, ϕ^n are deterministic functions, then $\lim_n \mathbb{E}\|x^n - x\|_{1-\alpha}^p = 0$ for $p \geq 1$.*

Remark 2.3. *We note that the regularity and absolute continuity results for the above mixed equation in d-dimensional case, but without delay, was studied in [5] by Guerra and Nualart. For the equations driven only by fBm, and the constant delay situation, we refer the reader to [4].*

3. Generalized Stieltjes integral

Let $\alpha \in (0, \frac{1}{2})$. For any measurable function $f : [0, T] \to \mathbb{R}$ we introduce the following notation

$$\|f(t)\|_\alpha := |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds. \tag{3}$$

Denote by $W^{\alpha,\infty}$ the space of measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$\|f(t)\|_{\alpha,\infty} := \sup_{t \in [0,T]} \|f(t)\|_\alpha < \infty. \tag{4}$$

A equivalent norm can be defined by

$$\|f\|_{\alpha,\mu} = \sup_{t \in [0,T]} e^{-\mu t} \left(|f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds \right); \qquad \mu \geq 0. \tag{5}$$

Note that for any ϵ, $(0 < \epsilon < \alpha)$, we have the inclusions

$\mathcal{C}^{\alpha+\epsilon}([0, T]; \mathbb{R}) \subset W^{\alpha,\infty}([0, T]; \mathbb{R}) \subset \mathcal{C}^{\alpha-\epsilon}([0, T]; \mathbb{R})$ (for more details, see [7]).

In particular, both the fractional Brownian motion B^H, with $H > \frac{1}{2}$, and the standard Brownian motion W, have their trajectories in $W^{\alpha,\infty}$. We refer the reader to [7, 5] for further details on this topics.

We denote by $W_T^{1-\alpha,\infty}([0, T]; \mathbb{R})$ the space of continuous functions $g : [0, T] \to \mathbb{R}$ such that

$$\|g\|_{1-\alpha,\infty,T} := \sup_{0<s<t<T} \left(\frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y) - g(s)|}{(y-s)^{2-\alpha}} dy \right) < \infty.$$

Clearly, for all $\epsilon > 0$ we have

$$\mathcal{C}^{1-\alpha+\epsilon}([0, T]; \mathbb{R}) \subset W_T^{1-\alpha,\infty}([0, T]; \mathbb{R}) \subset \mathcal{C}^{1-\alpha}([0, T]; \mathbb{R}).$$

Denoting

$$\Lambda_\alpha(g; [0, T]) = \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} |(D_{t^-}^{1-\alpha} g_{t^-})(s)|,$$

where $\Gamma(\alpha)=\int_0^\infty r^{\alpha-1}e^{-r}dr$ is the Euler function and

$$(D_{t^-}^{1-\alpha}g_{t-})(s)=\frac{e^{i\pi(1-\alpha)}}{\Gamma(\alpha)}\left(\frac{g(s)-g(t)}{(t-s)^{1-\alpha}}+(1-\alpha)\int_s^t\frac{g(s)-g(y)}{(y-s)^{2-\alpha}}dy\right)\mathbf{1}_{(0,t)}(s).$$

We also define the space $W^{\alpha,1}([0,T];\mathbb{R})$ of measurable functions f on $[0,T]$ such that

$$\|f\|_{\alpha,1;[0,T]}=\int_0^T\left[\frac{|f(t)|}{t^\alpha}+\int_0^t\frac{|f(t)-f(y)|}{(t-y)^{\alpha+1}}dy\right]dt<\infty.$$

We have $W^{\alpha,\infty}([0,T];\mathbb{R})\subset W^{\alpha,1}([0,T];\mathbb{R})$ and $\|f\|_{\alpha,1;[0,T]}\le\left(T+\frac{T^{1-\alpha}}{1-\alpha}\right)\|f\|_{\alpha,\infty;[0,T]}$.

In [13], Zähle introduced the generalized Stieltjes integral

$$\int_0^T f(t)dg(t)=(-1)^\alpha\int_0^T(D_{0+}^\alpha f)(t)(D_{T-}^{1-\alpha}g_{T-})(t)dt, \tag{6}$$

defined in terms of the fractional derivative operators

$$(D_{0+}^\alpha f)(t)=\frac{1}{\Gamma(1-\alpha)}\left(\frac{f(t)}{t^\alpha}+\alpha\int_0^t\frac{f(t)-f(y)}{(t-y)^{\alpha+1}}dy\right)\mathbf{1}_{(0,T)}(t),$$

and

$$(D_{T-}^{1-\alpha}g_{T-})(t)=\frac{e^{-i\pi\alpha}}{\Gamma(\alpha)}\left(\frac{g_{T-}(t)}{(T-t)^{1-\alpha}}+(1-\alpha)\int_t^T\frac{g_{T-}(t)-g_{T-}(y)}{(y-t)^{2-\alpha}}dy\right)\mathbf{1}_{(0,T)}(t).$$

The following proposition is the estimate of the generalized Stieltjes integral.

Proposition 3.1 ([7]). *Fix $0<\alpha<\frac{1}{2}$. Given two functions $g\in W_T^{1-\alpha,\infty}(0,T)$ and $f\in W^{\alpha,1}(0,T)$ we set*

$$G_s^t(f)=\int_s^t f_r dg_r.$$

Then for all $r<t\le T$ we have

$$\begin{aligned}\left|\int_s^t f_r dg_r\right| &\le \sup_{s\le r<\tau\le t}|(D_{\tau-}^{1-\alpha}g_{\tau-})(r)|\int_s^t|(D_{s+}^\alpha f)(\tau)|d\tau\\ &\le \Lambda_\alpha(g;[s,t])\|f\|_{\alpha,1;[0,T]}\\ &\le c_{\alpha,T}\Lambda_\alpha(g;[s,t])\|f\|_{\alpha,\infty},\end{aligned} \tag{7}$$

$$c_{\alpha,T}=\left(T+\frac{T^{1-\alpha}}{1-\alpha}\right).$$

4. A priori estimates

We will first deduce useful estimates for the integrals involved in Equation (2).

Let $\lambda\in(\frac{1}{2},1)$ be fixed and $\alpha\in(1-\lambda,\lambda)$. For $h\in\mathcal{C}^\alpha(0,T,\mathbb{R})$, $g\in\mathcal{C}^\lambda(0,T,\mathbb{R})$ and $x\in\mathcal{C}^{1-\alpha}([-r,T])$, we denote

$$F_t^b(x):=\int_0^t b(s,x_s)ds,\quad G_t^{\sigma_h}(x):=\int_0^t\sigma_h(s,x_s)dh_s\quad\text{and}\quad G_t^{\sigma_g}(x):=\int_0^t\sigma_g(s,x_s)dg_s.$$

Proposition 4.1. *Let the assumptions $(\mathbf{H}b)$, $(\mathbf{H}\sigma_W)$ and $(\mathbf{H}\sigma_H)$ be satisfied for the coefficients b, σ_h and σ_g respectively, $h\in\mathcal{C}^\alpha(0,T,\mathbb{R})$, $g\in\mathcal{C}^\lambda(0,T,\mathbb{R})$ and $x\in\mathcal{C}^{1-\alpha}([-r,T])$. Then*

(1) $F_t^b(x)\in\mathcal{C}^{1-\alpha}(0,T,\mathbb{R})$.

(2) $G_t^{\sigma_h}(x) \in \mathcal{C}^{1-\alpha}(0,T,\mathbb{R})$.
(3) $G_t^{\sigma_g}(x) \in \mathcal{C}^{1-\alpha}(0,T,\mathbb{R})$.

Proof.

(1) It is easy to see that $F_t^b(x) \in \mathcal{C}^1(0,T,\mathbb{R})$ and for $0 \le s \le t \le T$

$$\begin{aligned}|F_t^b(x) - F_s^b(x)| &= \left|\int_0^t b(u,x_u)du - \int_0^s b(u,x_u)du\right| \\ &= \left|\int_s^t b(u,x_u)du\right| \\ &\le C(1+\|x\|)(t-s) \\ &\le C(T+T^\alpha)(1+\|x\|)\end{aligned}$$

where C is defined as in above theorems.

Hence,

$$\|F_t^b(x) - F_s^b(x)\|_{1-\alpha} \le C_1(1+\|x\|_{1-\alpha}),$$

where C_1 constants depending only on C, α and T.

(2) It follows from the assumption $(\mathbf{H}\sigma_W)$ and the Garsia-Rodemich-Rumsey inequality (see, Theorem 2.1.3 of [12]) that for any $\alpha \in (1-\lambda, \lambda)$ and any $t \in [0,T]$ there exists a continuous random variable $\zeta(\alpha,t,h)$ in t with finite moments of any order such that

$$\begin{aligned}|G_t^{\sigma_h}(x) - G_s^{\sigma_h}(x)| &= \left|\int_s^t \sigma_h(u,x_u)dh_u\right| \\ &\le C(1+\|x\|)\zeta(\alpha,t,h)(t-s)^{\frac{1}{2}-\alpha} \\ &\le CT^{\frac{1}{2}-\alpha}\zeta(1+\|x\|),\end{aligned}$$

where ζ is defined as $\zeta(\alpha,t,h) = C_\alpha\left(\int_0^t\int_0^t \frac{|h_\tau - h_\theta|^{2/\alpha}}{|\tau-\theta|^{1/\alpha}}d\tau d\theta\right)^{\alpha/2}$, C_α constants depending only on α.

Hence,

$$\|G_t^{\sigma_h}(x) - G_s^{\sigma_h}(x)\|_{1-\alpha} \le C_2\zeta(1+\|x\|_{1-\alpha}),$$

C_2 depend only on C, α and T.

(3) Let $0 \le s < t \le T$. Using the proposition 3.1 we have for any $\alpha \in (1-\lambda,\lambda)$ and $\lambda \in (\frac{1}{2}, 1)$

(8)
$$\begin{aligned}|G_t^{\sigma_g}(x) - G_s^{\sigma_g}(x)| &= \left|\int_s^t \sigma_g(u,x_u)dg_u\right| \\ &\le \Lambda_\alpha(g)\int_s^t\left(\frac{|\sigma_g(\theta,x_\theta)|}{(\theta-s)^\alpha} + \alpha\int_s^\theta \frac{|\sigma_g(\theta,x_\theta) - \sigma_g(v,x_v)|}{(\theta-v)^{\alpha+1}}dv\right)d\theta.\end{aligned}$$

Hence, by condition $(\mathbf{H}\sigma_H)$,

$$\int_s^t \frac{|\sigma_g(\theta, x_\theta)|}{(\theta - s)^\alpha} d\theta \le CT^\alpha \frac{(t-s)^{1-\alpha}}{1-\alpha}(1 + \|x\|_{1-\alpha}), \tag{9}$$

and

$$\int_s^t \int_s^\theta \frac{|\sigma_g(\theta, x_\theta) - \sigma_g(v, x_v)|}{(\theta - v)^{\alpha+1}} dv d\theta \tag{10}$$

$$\begin{aligned}
&\le \int_s^t \int_s^\theta \frac{|\sigma_g(\theta, x_\theta) - \sigma_g(\theta, x_v)| + |\sigma_g(\theta, x_v) - \sigma_g(v, x_v)|}{(\theta - v)^{\alpha+1}} dv d\theta \\
&\le \int_s^t \int_s^\theta \frac{C(\theta - v)^{1-\alpha}(\|x\|_{1-\alpha} + (\theta - v)^\alpha)}{(\theta - v)^{\alpha+1}} dv d\theta \\
&\le C(t-s)^{1-\alpha} \frac{T^\alpha}{1-\alpha}\left[\frac{\|x\|_{1-\alpha}}{\alpha} + T\right].
\end{aligned}$$

Then the above estimates lead to the third assertion.

□

Consider the following equivalent norm in the space $\mathcal{C}^{1-\alpha}$ defined for any $\nu \ge 0$ by

$$\|x\|_{1-\alpha,\nu} = \sup_t e^{-\nu t}|x(t)| + \sup_{s<t} e^{-\nu t} \frac{|x(t) - x(s)|}{(t-s)^{1-\alpha}}.$$

We now present some estimates in order to be able to use the Banach fixed point theorem.

Proposition 4.2. *Let the assumptions* $(\mathbf{H}b)$, $(\mathbf{H}\sigma_W)$ *and* $(\mathbf{H}\sigma_H)$ *be satisfied for the coefficients* b, σ_h *and* σ_g *respectively,* $h \in \mathcal{C}^\alpha(0, T, \mathbf{R})$, $g \in \mathcal{C}^\lambda(0, T, \mathbf{R})$ *and* $x \in \mathcal{C}^{1-\alpha}([-r, T])$. *Then there exist* c_i^*, $1 \le i \le 3$, *such that*

(1) $\|F^b(x)\|_{1-\alpha,\nu} \le c_1^*(\nu)\Big(1 + \|x\|_{1-\alpha,\nu}\Big)$
(2) $\|G^{\sigma_h}(x)\|_{1-\alpha,\nu} \le c_2^*(\nu)\Big(1 + \|x\|_{1-\alpha,\nu}\Big)$
(3) $\|G^{\sigma_g}(x)\|_{1-\alpha,\nu} \le c_3^*(\nu)\Big(1 + \|x\|_{1-\alpha,\nu}\Big)$

where $c_i^*(\nu) \to 0$ *as* $\nu \to \infty$.

Proof.

(1) We remark that

$$\begin{aligned}
e^{-\nu t}(t-s)^{\alpha-1}|F_t^b(x) - F_s^b(x)| &\le e^{-\nu t}(t-s)^{\alpha-1}\int_s^t |b(u, x_u)| du \\
&\le e^{-\nu t}(t-s)^{\alpha-1}\int_s^t C(1 + \|x\|_u) du \\
&\le C \sup_t e^{-\nu t}(t-s)^{\alpha-1}\int_s^t (1 + \|x\|_u) du \\
&\le C \sup_t \int_s^t e^{-\nu(t-u)} \frac{1 + e^{-\nu u}\|x_u\|}{(t-u)^{1-\alpha}} du \\
&\le C\nu^{\alpha-1}\frac{T^{2\alpha-1}}{2\alpha-1}\Big(\mathbf{1} + \|x\|_{\nu_{1-\alpha,\nu}}\Big),
\end{aligned}$$

and therefore

$$\begin{aligned}\|F^b(x)\|_{1-\alpha,\nu} &= \sup_t e^{-\nu t}|F_t^b(x)| + \sup_t e^{-\nu t}(t-s)^{\alpha-1}|F_t^b(x) - F_s^b(x)| \\ &\leq \frac{C}{\nu^{1-\alpha}(2\alpha-1)}(T^{1-\alpha} + T^{2\alpha-1})\Big(\mathbf{1} + \|x\|_{\nu_{1-\alpha,\nu}}\Big).\end{aligned}$$

Then the above inequalities yield that there exist $c_1^*(\nu)$ such that

$$\|F^b(x)\|_{1-\alpha,\nu} \leq c_1^*(\nu)\Big(1 + \|x\|_{1-\alpha,\nu}\Big),$$

and $c_1^*(\nu) \to 0$ as $\nu \to \infty$.

(2) We have

$$\begin{aligned}|G_t^{\sigma_h}(x) - G_s^{\sigma_h}(x)|e^{-\nu t}(t-s)^{\alpha-1} &\leq e^{-\nu t}(t-s)^{\alpha-1}\int_s^t |\sigma_h(u, x_u)|dh_u \\ &\leq e^{-\nu t}(t-s)^{\alpha-1}CT^{\frac{1}{2}-\alpha}\zeta(\alpha, t, h)\int_s^t (1 + \|x_u\|)du \\ &\leq CT^{\frac{1}{2}-\alpha}\zeta(\alpha, t, h)\sup_t \int_s^t e^{-\nu(t-u)}\frac{1 + e^{-\nu u}\|x_u\|}{(t-u)^{1-\alpha}}du,\end{aligned}$$

and therefore

$$\|G^{\sigma_h}(x)\|_{1-\alpha,\nu} \leq c_2^*(\nu)\Big(1 + \|x\|_{1-\alpha,\nu}\Big),$$

where $c_2^*(\nu) = C\dfrac{T^{\alpha-1}}{(2\alpha-1)\nu^{1-\alpha}}\zeta(\alpha, t, h)$, and $c_2^*(\nu) \to 0$ as $\nu \to \infty$.

(3) Using proposition 3.1 we have for $s, t \in [0, T]$ with $s < t$

(11)
$$|G_t^{\sigma_g}(x) - G_s^{\sigma_g}(x)|e^{-\nu t}(t-s)^{\alpha-1} \leq$$
$$e^{-\nu t}(t-s)^{\alpha-1}\Lambda_\alpha(g)\int_s^t \left(\frac{|\sigma_g(\theta, x_\theta)|}{(\theta-s)^\alpha} + \alpha\int_s^\theta \frac{|\sigma_g(\theta, x_\theta) - \sigma_g(v, x_v)|}{(\theta-v)^{\alpha+1}}dv\right)d\theta.$$

By the assumption $(\mathbf{H}\sigma_H)$ we have,

(12)
$$\begin{aligned}e^{-\nu t}(t-s)^{\alpha-1}\int_s^t \frac{|\sigma_g(\theta, x_\theta)|}{(\theta-s)^\alpha}d\theta &\leq e^{-\nu t}(t-s)^{\alpha-1}\int_s^t C\frac{1 + \|x_\theta\|}{(\theta-s)^\alpha}d\theta \\ &\leq CT^{\alpha-1}\sup_t \int_s^t e^{-\nu(t-\theta)}\frac{1 + e^{-\nu\theta}\|x_\theta\|}{(\theta-s)^\alpha}d\theta \\ &\leq \frac{C}{T^\alpha(1-2\alpha)\nu^\alpha}\Big(1 + \|x\|_{1-\alpha,\nu}\Big),\end{aligned}$$

and

(13)
$$\begin{aligned}&e^{-\nu t}(t-s)^{\alpha-1}\int_s^t\int_s^\theta \frac{|\sigma_g(\theta, x_\theta) - \sigma_g(v, x_v)|}{(\theta-v)^{\alpha+1}}dvd\theta \\ \leq\ & e^{-\nu t}(t-s)^{\alpha-1}\int_s^t\int_s^\theta C\frac{(\theta-v)^{1-\alpha}(\|x_\theta - x_v\| + 1)}{(\theta-v)^{\alpha+1}}dvd\theta \\ \leq\ & CT^{\alpha-1}\sup_t \int_s^t\int_s^\theta e^{-\nu(\theta-v)}\frac{1 + e^{-\nu v}\|x_\theta - x_v\|}{(\theta-v)^{2\alpha}}dvd\theta \\ \leq\ & C\frac{T^{2\alpha-1}}{(2\alpha-1)\nu^{2\alpha-1}}\Big(1 + \|x\|_{1-\alpha,\nu}\Big).\end{aligned}$$

Thanks to (11)-(12) and (13) one easily remarks that there exist $c_3^*(\nu)$ such that

$$\|G^{\sigma_g}(x)\|_{1-\alpha,\nu} \leq c_3^*(\nu)\Big(1 + \|x\|_{1-\alpha,\nu}\Big),$$

$c_3^*(\nu) \to 0$ as $\nu \to \infty$.

□

Proposition 4.3. *Let the assumptions* ($\mathbf{H}b$), ($\mathbf{H}\sigma_W$) *and* ($\mathbf{H}\sigma_H$) *be satisfied for the coefficients* b, σ_h *and* σ_g *respectively,* $h \in \mathcal{C}^\alpha(0, T, \mathbb{R})$, $g \in \mathcal{C}^\lambda(0, T, \mathbb{R})$ *and* $x, y \in \mathcal{C}^{1-\alpha}([-r, T])$. *Then there exist* c_i^*, $1 \leq i \leq 3$, *such that*

(1) $\|F^b(x) - F^b(y)\|_{1-\alpha,\nu} \leq c_1^*(\nu)||x - y||_{1-\alpha,\nu}$
(2) $\|G^{\sigma_h}(x) - G^{\sigma_h}(y)\|_{1-\alpha,\nu} \leq c_2^*(\nu)\|x - y\|_{1-\alpha,\nu}$
(3) $\|G^{\sigma_g}(x) - G^{\sigma_g}(y)\|_{1-\alpha,\nu} \leq c_3^*(\nu)(1 + \|x\|_{1-\alpha} + \|y\|_{1-\alpha})\|x - y\|_{1-\alpha,\nu}$

where $c_i^*(\nu) \to 0$ *as* $\nu \to \infty$ *for* $1 \leq i \leq 3$.

Proof.

(1) Using the Lipschitz property of b, we see that for $0 \leq s < t \leq T$

$$e^{-\nu t}(t-s)^{\alpha-1}\left|\Big(F_t^b(x)-F_t^b(y)\Big)-\Big(F_s^b(x)-F_s^b(y)\Big)\right| \leq C\nu^{\alpha-1}(2\alpha+1)T^{2\alpha+1}\|x-y\|_{1-\alpha,\nu}$$

which imply the first claim.

(2) Let $s, t \in [0, T]$ with $s < t$, we have by the Lipschitz property of σ_h

$$\begin{aligned}
&\leq e^{-\nu t}(t-s)^{\alpha-1}\left|\Big(G_t^{\sigma_h}(x) - G_t^{\sigma_h}(y)\Big) - \Big(G_s^{\sigma_h}(x) - G_s^{\sigma_h}(x)\Big)\right| \\
&\leq e^{-\nu t}(t-s)^{\alpha-1}\int_s^t |\sigma_h(u, x_u) - \sigma_h(u, y_u)|du \\
&\leq CT^{\frac{1}{2}-\alpha}\zeta(\alpha, t, h)\sup_t \int_s^t e^{-\nu u}\|x_u - y_u\|e^{-\nu(t-u)}(t-u)^{\alpha-1}du.
\end{aligned}$$

Hence,

$$\|G^{\sigma_h}(x) - G^{\sigma_h}(y)\|_{1-\alpha,\nu} \leq C\frac{T^{\alpha-1}}{\alpha(\alpha-1)\nu^{\alpha-1}}\zeta(\alpha, t, h)\|x - y\|_{1-\alpha,\nu},$$

which give the result of the second assertion.

(3) We have for $s, t \in [0, T]$ with $s < t$

$$\left|\Big(G_t^{\sigma_g}(x) - G_t^{\sigma_g}(y)\Big) - \Big(G_s^{\sigma_g}(x) - G_s^{\sigma_g}(y)\Big)\right|e^{-\nu t}(t-s)^{\alpha-1} \leq e^{-\nu t}(t-s)^{\alpha-1}\Lambda_\alpha(g)\times$$
$$\int_s^t\left(\frac{|\sigma_g(\theta, x_\theta) - \sigma_g(\theta, y_\theta)|}{(\theta-s)^\alpha} + \alpha\int_s^\theta \frac{|\sigma_g(\theta, x_\theta) - \sigma_g(\theta, y_\theta) - \sigma_g(v, x_v) + \sigma_g(v, y_v)|}{(\theta-v)^{\alpha+1}}dv\right)d\theta.$$

By the Lipschitz property of σ_g we obtain

(14)
$$e^{-\nu t}(t-s)^{\alpha-1}\int_s^t \frac{|\sigma_g(\theta, x_\theta) - \sigma_g(\theta, y_\theta)|}{(\theta-s)^\alpha}d\theta \quad \leq \quad C\frac{T^{1-\alpha}}{\alpha(1-\alpha)\nu^\alpha}\|x - y|\|_{1-\alpha,\nu}.$$

Remark that for all $u, v \in [0,T]$

$$\begin{aligned}
& \Big|\Big(\sigma_g(\theta,x_\theta)-\sigma_g(\theta,y_\theta)\Big)-\Big(\sigma_g(v,x_v)-\sigma_g(v,y_v)\Big)\Big| \\
\leq & \Big|\int_0^1 \nabla\sigma_g(v, vx_v+y_v-vy_v)\big((x_\theta-y_\theta)-(x_v-y_v)\big)dv\Big| \\
+ & \Big|\int_0^1 \Big[\nabla\sigma_g(\theta, vx_\theta+y_\theta-vy_\theta)-\nabla\sigma_g(v, vx_v+y_v-vy_v)\Big](x_\theta-y_\theta)dv\Big| \\
\leq & C\Big[\|(x_\theta-y_\theta)-(x_v-y_v)\|+\|(x_\theta-y_\theta\|\Big(\|x_\theta-x_v\|+\|y_\theta-y_v\|+(\theta-v)\Big)\Big],
\end{aligned}$$

and therefore

$$\begin{aligned}
& e^{-\nu t}(t-s)^{\alpha-1}\int_s^t\int_s^\theta \frac{|(\sigma_g(\theta,x_\theta)-\sigma_g(\theta,y_\theta))-(\sigma_g(v,x_v)-\sigma_g(v,y_v))|}{(\theta-v)^{\alpha+1}}dvd\theta \\
\leq & e^{-\nu t}(t-s)^{\alpha-1}C\int_s^t\int_s^\theta \frac{\|(x_\theta-y_\theta)-(x_v-y_v)\|}{(\theta-v)^{\alpha+1}}dvd\theta \\
+ & e^{-\nu t}(t-s)^{\alpha-1}C\int_s^t\int_s^\theta \frac{\|x_\theta-y_\theta\|}{(\theta-v)^{\alpha}}dvd\theta \\
+ & e^{-\nu t}(t-s)^{\alpha-1}C\int_s^t\int_s^\theta \frac{\|x_\theta-y_\theta\|(\|x_\theta-x_v\|+||y_\theta-y_v\|)}{(\theta-v)^{\alpha+1}}dvd\theta \\
\leq & (t-s)^{\alpha-1}C\int_s^t\int_s^\theta \frac{e^{-\nu(t-\theta)}\|x-y\|_{1-\alpha,\nu}}{(\theta-v)^{2\alpha}}dvd\theta \\
+ & (t-s)^{\alpha-1}C\int_s^t\int_s^\theta \frac{e^{-\nu(t-\theta)}\|x-y\|_{1-\alpha,\nu}}{(\theta-v)^{\alpha}}dvd\theta \\
+ & e^{-\nu t}(t-s)^{\alpha-1}C\int_s^t\int_s^\theta \frac{\|x_\theta-y_\theta\|(\|x\|_{1-\alpha}+\|y\|_{1-\alpha})}{(\theta-v)^{2\alpha}}dvd\theta.
\end{aligned}$$

Thus

(15)
$$\begin{aligned}
& e^{-\nu t}(t-s)^{\alpha-1}\int_s^t\int_s^\theta \frac{|(\sigma_g(\theta,x_\theta)-\sigma_g(\theta,y_\theta))-(\sigma_g(v,x_v)-\sigma_g(v,y_v))|}{(\theta-v)^{\alpha+1}}dvd\theta \\
\leq & \frac{CT^{2\alpha-1}}{\nu^{1-\alpha}(1-2\alpha)}\|x-y\|_{1-\alpha,\nu}+\frac{CT^{\alpha-1}}{\nu(1-\alpha)}\|x-y\|_{1-\alpha,\nu} \\
+ & \frac{CT^{2\alpha-1}}{\nu^{1-\alpha}(1-\alpha)}\|x-y\|_{1-\alpha,\nu}\Big(\|x\|_{1-\alpha}+\|y\|_{1-\alpha}\Big)
\end{aligned}$$

From (14) and (15) we can derive the estimate:

$$\sup_{s<t}\Big|\Big(G_t^{\sigma_g}(x)-G_t^{\sigma_g}(y)\Big)-\Big(G_s^{\sigma_g}(x)-G_s^{\sigma_g}(y)\Big)\Big|e^{-\nu t}(t-s)^{\alpha-1}$$
$$\leq c_3^*(\nu)(1+\|x\|_{1-\alpha}+\|y\|_{1-\alpha})\|x-y\|_{1-\alpha,\nu},$$

$c_3^*(\nu)\to 0$ as $\nu\to\infty$, which implies the third claim.

□

Consider the equation on $\mathbb{R}$

$$\psi(x)(t)=x(0)+\int_0^t b(s,x_s)ds+\int_0^t \sigma_h(s,x_s)dh(s)+\int_0^t \sigma_g(s,x_s)dg(s), \quad t\geq 0,$$

where the function ψ is defined from $\mathcal{C}^{1-\alpha}([-r,T])$ into $\mathcal{C}^{1-\alpha}([-r,T])$ by $\psi(x)(t)=x(t)$, $t\in[-r,0]$.

Proposition 4.2 and Proposition 4.3 combined together give the following result.

Proposition 4.4. *Let the assumptions* $(\mathbf{H}b)$, $(\mathbf{H}\sigma_W)$ *and* $(\mathbf{H}\sigma_H)$ *be satisfied for the coefficients* b, σ_h *and* σ_g *respectively,* $h \in \mathcal{C}^\alpha(0,T,\mathbb{R})$, $g \in \mathcal{C}^\lambda(0,T,\mathbb{R})$ *and* $x, y \in \mathcal{C}^{1-\alpha}([-r,T])$. *Then there exist* $\tilde{c}_i$, $i = 1,2$ *such that*

(1) $\|\psi(x)\|_{1-\alpha,\nu} \le \|x_0\|_{1-\alpha,\nu} + \tilde{c}_1(\nu)\Big(1 + \|x\|_{1-\alpha,\nu}\Big)$,

(2) $\|\psi(x) - \psi(y)\|_{1-\alpha,\nu} \le \|x_0 - y_0\|_{1-\alpha,\nu} + \tilde{c}_1(\nu)\Big(1 + \|x\|_{1-\alpha} + \|y\|_{1-\alpha}\Big)\|x - y\|_{1-\alpha,\nu}$,

$\tilde{c}_i(\nu) \to 0$, $i = 1,2$, *as* $\nu \to \infty$.

5. Deterministic functional equation

In this section we fix the parameters λ and α such that $\frac{1}{2} < \lambda < 1$, $1 - \lambda < \alpha < \lambda$.
Consider the deterministic functional equation

$$
\begin{aligned}
x(t) &= \varphi(0) + \int_0^t b(s,x_s)ds + \int_0^t \sigma_h(s,x_s)dh(s) + \int_0^t \sigma_g(s,x_s)dg(s), \\
x_0 &= \varphi \in \mathcal{C}_r,
\end{aligned}
\tag{16}
$$

for $g \in \mathcal{C}^\lambda(0,T\mathbb{R})$, $h \in \mathcal{C}^\alpha(0,T,\mathbb{R})$ and $t \ge 0$.
The following theorem is the main result of this section.

Theorem 5.1. *Let the assumptions* $(\mathbf{H}b)$, $(\mathbf{H}\sigma_W)$ *and* $(\mathbf{H}\sigma_H)$ *be satisfied for the coefficients* b, σ_h *and* σ_g *respectively,* $\varphi \in \mathcal{C}^{1-\alpha}([-r,0])$. *Then Eq.* (16) *has unique solution* x. *Moreover the solution is* $(1-\alpha)$*-Hölder continuous on* $[-r,T]$.

Proof.

Existence. We shall prove the existence of the solution by a fixed point argument. We first define $\mathcal{C}^{1-\alpha}([-r,T],\varphi)$ as the space of all $x \in \mathcal{C}^{1-\alpha}([-r,T])$ such that $x = \varphi$ on $[-r,0]$. Let Γ be an operator defined from $\mathcal{C}^{1-\alpha}([-r,T],\varphi)$ into itself by $\Gamma(x)(t) = \varphi(t)$ for $t \in [-r,0]$ and

$$\Gamma(x)(t) = \varphi(0) + \int_0^t b(s,x_s)ds + \int_0^t \sigma_h(s,x_s)dh(s) + \int_0^t \sigma_g(s,x_s)dg(s), \quad t \ge 0.$$

From Proposition 4.4. we remark that

$$\|\Gamma(x)\|_{1-\alpha,\nu} \le \|\varphi\|_{1-\alpha,\nu} + \tilde{c}(\nu)\Big(1 + \|x\|_{1-\alpha,\nu}\Big),$$

where $\tilde{c}(\nu) \to 0$ as $\nu \to \infty$.
Let $\nu = \nu_0$ be sufficiently large such that $\tilde{c}(\nu_0) \le \frac{1}{2}$. If $||x||_{1-\alpha,\nu_0} \le 2(1 + ||\varphi||_{1-\alpha,\nu_0})$, then $\|\Gamma(x)\|_{1-\alpha,\nu_0} \le 2(1 + \|\varphi\|_{1-\alpha,\nu_0})$ and hence $\Gamma(B_{\nu_0}) \subset B_{\nu_0}$ where

$$B_{\nu_0} = \Big\{x \in \mathcal{C}^{1-\alpha}([-r,T],\varphi) : \|x\|_{1-\alpha,\nu_0} \le 2(1 + \|\varphi\|_{1-\alpha,\nu_0})\Big\}.$$

As consequence, Γ maps B_{ν_0} into itself.
We now show that there exists $\nu > \nu_0$ such that the operator Γ is a contraction on B_{ν_0} under the norm $\|\cdot\|_{1-\alpha,\nu}$. Using Proposition 4.4., we have for all $x, y \in \mathcal{C}^{1-\alpha}([-r,T],\varphi)$

$$\|\Gamma(x) - \Gamma(y)\|_{1-\alpha,\nu} \le \tilde{c}(\nu)(1 + \|x\|_{1-\alpha} + \|y\|_{1-\alpha})\|x - y\|_{1-\alpha,\nu}.$$

If $l_0 = \sup_{x \in B_{\nu_0}} \|x\|_{1-\alpha}$, then for all $x, y \in B_{\nu_0}$ we have

$$\|\Gamma(x) - \Gamma(y)\|_{1-\alpha,\nu} \leq \tilde{c}(\nu)(1 + 2l_0)\|x - y\|_{1-\alpha,\nu}.$$

Let $\nu > \nu_0$ be sufficiently large such that $\tilde{c}(\nu)(1+2l_0) < 1/2$. Then for all $x, y \in B_{\nu_0}$ we have

$$\|\Gamma(x) - \Gamma(y)\|_{1-\alpha,\nu} \leq \frac{1}{2}\|x - y\|_{1-\alpha,\nu}.$$

Consequently, the operator Γ is a contraction on the closed subset B_{ν_0} of the complete metric space $\mathcal{C}^{1-\alpha}([-r, T])$ which implies that it has a unique fixed point x in B_{ν_0}. So from the definition of Γ it follows that x is a solution of Eq. (16) in $\mathcal{C}^{1-\alpha}([-r, T])$.

Uniqueness. Assume that x, y are two solutions of (16) in the space $\mathcal{C}^{1-\alpha}([-r, T])$ and using Proposition 4.4., with ν sufficiently large, we get

$$||x - y||_\nu \leq \frac{1}{2}||x - y||_{1-\alpha,\nu},$$

and, therefore, $x = y$.

$\square$

Theorem 5.2. *Let the assumptions* (**H**b)*,* (**H**σ_W) *and* (**H**σ_H) *be satisfied for the coefficients* b*,* σ_h *and* σ_g *respectively,* $\varphi \in \mathcal{C}^{1-\alpha}([-r, 0])$*. Then the solution* x *of Eq. (16) satisfies*

$$\|x\|_{1-\alpha} \leq \hat{c}_1\Big(1 + \|\varphi\|_{1-\alpha}\Big) \exp\Big(\hat{c}_2 \Lambda_\alpha^{1/\alpha}(g)\Big),$$

where $\hat{c}_1, \hat{c}_2$ *are constants depending only on* α, T *and* C.

Proof. Set

$$J(t) = \sup_{s \in [-r,t]} |x(s)| + \sup_{-r \leq s < u \leq t} \frac{|x(u) - x(s)|}{(u - s)^{1-\alpha}}, \quad t \geq 0.$$

We have for $0 \leq s < u \leq t$,

$$\frac{|x(u) - x(s)|}{(u - s)^{1-\alpha}} \leq (u-s)^{\alpha-1}\Big(\Big|\int_s^u b(v, x_v)dv\Big| + \Big|\int_s^u \sigma_h(v, x_v)dh\Big| + \Big|\int_s^u \sigma_g(v, x_v)dg\Big|\Big).$$

By assumption (**H**b) we have

$$(u - s)^{\alpha-1}\Big(\Big|\int_s^u b(v, x_v)dv\Big|\Big) \leq C\int_0^t (t - v)^{\alpha-1}(1 + J(v))dv. \tag{17}$$

By assumption (**H**σ_W) we get

(18)
$$\begin{aligned}
(u - s)^{\alpha-1}\Big(\Big|\int_s^u \sigma_h(v, x_v)dh\Big|\Big) &\leq CT^{\frac{1}{2}-\alpha}\zeta(\alpha, t, h)\int_s^u \frac{1 + \|x_v\|}{(u - v)^{1-\alpha}}dv \\
&\leq CT^{\frac{1}{2}-\alpha}\zeta(\alpha, t, h)\int_0^t (t - v)^{\alpha-1}(1 + J(v))dv.
\end{aligned}$$

Using Proposition 3.1., we have for $1-\lambda<\alpha<\lambda$ and $\frac{1}{2}<\lambda<1$

$$\begin{aligned} & (u-s)^{\alpha-1}\left(\left|\int_s^u \sigma_g(v,x_v)dg\right|\right) \\ \leq\ & (u-s)^{\alpha-1}\Lambda_\alpha(g)\int_s^u\left(\frac{|\sigma_g(v,x_v)|}{(v-s)^\alpha}+\alpha\int_s^v\frac{|\sigma_g(v,x_v)-\sigma_g(\tau,x_\tau)|}{(v-\tau)^{\alpha+1}}d\tau\right)dv. \end{aligned}$$

By the Lipschitz property of σ_g, we have

$$\begin{aligned} & (u-s)^{\alpha-1}\Lambda_\alpha(g)\int_s^u\frac{|\sigma_g(v,x_v)|}{(v-s)^\alpha}dv \\ \leq\ & (u-s)^{\alpha-1}\Lambda_\alpha(g)\int_s^u\frac{|\sigma_g(v,x_v)-\sigma_g(v,x_s)|+|\sigma_g(v,x_s)|}{(v-s)^\alpha}dv \\ \leq\ & \Lambda_\alpha(g)C\frac{T^{2\alpha}}{1-\alpha}+\Lambda_\alpha(g)C(T^{1-2\alpha}+2\alpha)\int_0^t(t-s)^{\alpha-1}J(v)dv, \end{aligned} \tag{19}$$

and

$$\begin{aligned} & (u-s)^{\alpha-1}\Lambda_\alpha(g)\int_s^u\int_s^v\frac{|\sigma_g(v,x_v)-\sigma_g(\tau,x_\tau)|}{(v-\tau)^{\alpha+1}}d\tau dv \\ \leq\ & (u-s)^{\alpha-1}\Lambda_\alpha(g)\int_s^u\int_s^v C\frac{(v-\tau)^{1-\alpha}(1+\|x_v-x_\tau\|)}{(v-\tau)^{\alpha+1}}d\tau dv \\ \leq\ & C\frac{T^{2\alpha-1}}{(2\alpha-1)}\Lambda_\alpha(g)\int_0^t(t-v)^{\alpha-1}J(v)dv. \end{aligned} \tag{20}$$

Inequalities (17), (18), (19) and (20) together imply that

$$\sup_{s<u}\frac{|x(u)-x(s)|}{(u-s)^{1-\alpha}}\leq\hat{c}_0(1+\Lambda_\alpha(g))\Big[1+\int_0^t(t-s)^{\alpha-1}J(v)dv\Big],$$

and

$$J(t)\leq\|\varphi\|_{1-\alpha}+\hat{c}_0(1+\Lambda_\alpha(g))\Big[1+\int_0^t(t-s)^{\alpha-1}J(s)ds\Big].$$

Using the Gronwall lemma (see [11]), we have since

$$J(t)\leq\|\varphi\|_{1-\alpha}+\hat{c}_0(1+\Lambda_\alpha(g))\Big[1+\int_0^t J(s)(t-s)^{\alpha-1}t^{1-\alpha}s^{-(1-\alpha)}ds\Big]$$

then

$$\|x\|_{1-\alpha}\leq\hat{c}_1(1+\|\varphi\|_{1-\alpha})\exp\left(\hat{c}_2\Lambda_\alpha^{1/\alpha}(g)\right).$$

□

The following result shows the dependance of the solution of Eq.(16) on the initial condition.

Lemma 5.3. *Let the assumptions* ($\mathbf{H}b$), ($\mathbf{H}\sigma_W$) *and* ($\mathbf{H}\sigma_H$) *be satisfied for the coefficients b, σ_h and σ_g respectively, $\varphi,\varphi^n\in\mathcal{C}^{1-\alpha}([-r,0])$. Let x be the solution of Eq.(16) and x^n be the solution of the same equation with φ^n in place of φ. Then for $1-\lambda<\alpha<\lambda$ and $\frac{1}{2}<\lambda<1$ we have*

$$\|x-x^n\|_{1-\alpha}\leq\hat{c}_1\|\varphi-\varphi^n\|_{1-\alpha}\exp\left(\hat{c}_2(\|x\|_{1-\alpha}^{1/\alpha}+\|x^n\|_{1-\alpha}^{1/\alpha})\right)\exp\left(\hat{c}_3\Lambda_\alpha^{1/\alpha}(g)\right),$$

where $\hat{c}_1$, $\hat{c}_2$ and $\hat{c}_3$ depend only on α, T and C.

Proof. For $t \leq 0$, let

$$J^n(t) = \sup_{-r\leq s\leq t} |x(s) - x^n(s)| + \sup_{s<u} \frac{|x(u) - x^n(u) - x(s) + x^n(s)|}{(u-s)^{1-\alpha}}.$$

Following the same lines as in Proof of Theorem 5.2. we obtain for $t \leq 0$

$$J^n(t) \leq \|\varphi - \varphi^n\|_{1-\alpha} + \hat{c}_0 \Big[1 + \Lambda_\alpha(g)(1 + \|x\|_{1-\alpha} + \|x^n\|_{1-\alpha})\Big] \int_0^t J^n(s)(t-s)^{\alpha-1} ds.$$

Therefore,

$$J^n(t) \leq \|\varphi - \varphi^n\|_{1-\alpha} + \hat{c}_0 \Big[1 + \Lambda_\alpha(g)(1 + \|x\|_{1-\alpha} + \|x^n\|_{1-\alpha})\Big] \int_0^t J(s)(t-s)^{\alpha-1} t^{1-\alpha} s^{-(1-\alpha)} ds.$$

By the Gronwall lemma ([11]) we get

$$\|x - x^n\|_{1-\alpha} \leq \hat{c}_1 \|\varphi - \varphi^n\|_{1-\alpha} \exp\Big(\hat{c}_3 \Lambda_\alpha^{1/\alpha}(g)\Big) \exp\Big(\hat{c}_2 (\|x\|_{1-\alpha}^{1/\alpha} + \|x^n\|_{1-\alpha}^{1/\alpha})\Big).$$

□

6. Functional equation driven by a Wiener process and fBm

In this section we apply the deterministic results in order to prove the main theorems of this article.

Proof.(Theorem 2.1)

The existence and uniqueness of the solution can be established following the same argument as in the deterministic Theorem 5.1.

Using Theorem 5.2., we get for $\alpha > 1 - H$

$$\|x\|_{1-\alpha} \leq \hat{c}_1 (1 + \|\phi\|_{1-\alpha}) \exp\Big(\hat{c}_2 \Lambda_\alpha^{1/\alpha}(B)\Big),$$

$\hat{c}_1$, $\hat{c}_1$ depend only on α, T and C.

Therefore, for all $p \geq 1$ we have

$$\mathbb{E}\|x\|_{1-\alpha}^p \leq \frac{1}{2} \hat{c}_1^{2p} \mathbb{E}(1 + \|\phi\|_{1-\alpha})^{2p} + \frac{1}{2} \mathbb{E} \exp\Big(2p\hat{c}_2 \Lambda_\alpha^{1/\alpha}(B)\Big). \tag{21}$$

Hence for any $0 < \gamma < 2$ we have by Fernique's theorem ([3])

$$\mathbb{E}\Big[\exp \Lambda_\alpha(B)^\gamma\Big] < \infty.$$

As consequence $\mathbb{E}||x||_{1-\alpha}^p < \infty$, $\forall p \geq 1$ such that $\dfrac{1}{\alpha} < 2$ with H should be greater than $\frac{3}{4}$ and $\alpha + H > \frac{3}{2}$.

□

Proof.(Theorem 2.2)

The almost-sure convergence can be obtained using Lemma 5.3. The $\mathbf{L}^p$-convergence can also be obtained by a dominated convergence argument since we have for any $n \geq 0$

$$\begin{aligned} \|x^n - x\|_{1-\alpha} &\leq \|x^n\|_{1-\alpha} + \|x\|_{1-\alpha} \\ &\leq \hat{c}_1 (2 + \|\phi\|_{1-\alpha} + \|\phi^n\|_{1-\alpha}) \exp\Big(\hat{c}_2 \Lambda_\alpha^{1/\alpha}(B)\Big) \end{aligned}$$

and $||\phi^n||_{1-\alpha}$ is bounded, the we can write

$$|\|x^n - x\| \leq \hat{c}_4 \exp\Big(\hat{c}_2 \Lambda_\alpha^{1/\alpha}(B)\Big) := Y$$

and $\mathbb{E}Y^p < \infty$, $\forall p \geq 1$.

$\square$

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