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WIJSMAN ROUGH LACUNARY STATISTICAL CONVERGENCE ON I CESÀRO TRIPLE SEQUENCES

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Abstract. In this paper, we defined concept of Wijsman I-Cesàro summability for sequences of sets and investigate the relationships between the concepts of Wijsman strongly I-Cesàro summability and Wijsman statistical $I-$ Cesàro summability by using the concept of a triple sequence spaces.

1. Introduction

The idea of statistical convergence was introduced by Steinhaus and also independently by Fast for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence.

Let K be a subset of the set of positive integers $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, and let us denote the set $\{(m,n,k) \in K : m \leq u, n \leq v, k \leq w\}$ by K_{uvw}. Then the natural density of K is given by $\delta(K) = lim_{uvw \to \infty} \frac{|K_{uvw}|}{uvw}$, where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, a finite subset has natural density zero, and we have $\delta(K^c) = 1 - \delta(K)$ where $K^c = \mathbb{N} \backslash K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}, m,n,k \in \mathbb{N}$.

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A triple sequence $x = (x_{mnk})$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st - lim\, x = 0$, provided that the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}, 0| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence x.

If a triple sequence is statistically convergent, then for every $\epsilon > 0$, infinitely many terms of the sequence may remain outside the $\epsilon-$ neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence $x = (x_{mnk})$ satisfies some property P for all m, n, k except a set of natural density zero, then we say that the triple sequence x satisfies P for almost all (m,n,k) and we abbreviate this by a.a. (m,n,k).

Let $\left(x_{m_i n_j k_\ell}\right)$ be a sub sequence of $x = (x_{mnk})$. If the natural density of the set $K = \left\{(m_i, n_j, k_\ell) \in \mathbb{N}^3 : (i,j,\ell) \in \mathbb{N}^3\right\}$ is different from zero, then $\left(x_{m_i n_j k_\ell}\right)$ is called a non thin sub sequence of a triple sequence x.

$c \in \mathbb{R}$ is called a statistical cluster point of a triple sequence $x = (x_{mnk})$ provided that the natural density of the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - c| < \epsilon\right\}$$

is different from zero for every $\epsilon > 0$. We denote the set of all statistical cluster points of the sequence x by Γ_x.

A triple sequence $x = (x_{mnk})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geq M\right\}\right) = 0$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by [8], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence [7] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

Let (X, ρ) be a metric space. For any non empty closed subsets $A, A_{mnk} \subset X\, (m,n,k \in \mathbb{N})$, we say that the triple sequence (A_{mnk}) is Wijsman statistical convergent to A is the triple sequence $(d(x, A_{mnk}))$ is statistically convergent to $d(x, A)$, i.e., for $\epsilon > 0$ and for each $x \in X$

$$lim_{rst} \frac{1}{rst} |\{m \leq r, n \leq s, k \leq t : |d(x, A_{mnk}) - d(x, A)| \geq \epsilon\}| = 0.$$

In this case, we write $St-lim_{mnk} A_{mnk} = A$ or $A_{mnk} \longrightarrow A(WS)$. The triple sequence (A_{mnk}) is bounded if $sup_{mnk} d(x, A_{mnk}) < \infty$ for each $x \in X$.

In this paper, we introduce the notion of Wijsman rough statistical convergence of triple sequences. Defining the set of Wijsman rough statistical limit points of a triple sequence, we obtain to Wijsman statistical convergence criteria associated with this set. Later, we prove that this set of Wijsman statistical cluster points and the set of Wijsman rough statistical limit points of a triple sequence.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by ([9], [10]), ([2], [3], [4]), [5], [11], [6], [12] and many others.

Throughout the paper let r be a nonnegative real number.

2. Definitions and Preliminaries

Definition 2.1. *A triple sequence $x = (x_{mnk})$ of real numbers is said to be statistically convergent to $l \in \mathbb{R}^3$ if for any $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where*

$$A(\epsilon) = \left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - l| \geq \epsilon\right\}.$$

Definition 2.2. *A triple sequence $x = (x_{mnk})$ is said to be statistically convergent to $l \in \mathbb{R}^3$, written as $st - limx = l$, provided that the set*

$$\left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - l| \geq \epsilon\right\},$$

has natural density zero for every $\epsilon > 0$.

In this case, l is called the statistical limit of the sequence x.

Definition 2.3. *Let $x = (x_{mnk})_{m,n,k \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}}$ be a triple sequence in a metric space $(X, |.,.|)$ and r be a non-negative real number. A triple sequence $x = (x_{mnk})$ is said to be $r-$convergent to $l \in X$, denoted by $x \to^r l$, if for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that for all $m, n, k \geq N_\epsilon$ we have*

$$|x_{mnk} - l| < r + \epsilon$$

In this case l is called an $r-$ limit of x.

Remark 2.1. *We consider $r-$ limit set x which is denoted by LIM_x^r and is defined by*

$$LIM_x^r = \{l \in X : x \to^r l\}.$$

Definition 2.4. *A triple sequence $x = (x_{mnk})$ is said to be $r-$ convergent if $LIM_x^r \neq \phi$ and r is called a rough convergence degree of x. If $r = 0$ then it is ordinary convergence of triple sequence.*

Definition 2.5. *Let $x = (x_{mnk})$ be a triple sequence in a metric space $(X, |.,.|)$ and r be a non-negative real number is said to be $r-$ statistically convergent to l, denoted by $x \rightarrow^{r-st_3} l$, if for any $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where*

$$A(\epsilon) = \{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{mnk} - l| \geq r + \epsilon\}.$$

In this case l is called $r-$ statistical limit of x. If $r = 0$ then it is ordinary statistical convergent of triple sequence.

Definition 2.6. *A class I of subsets of a nonempty set X is said to be an ideal in X provided*
(i) $\phi \in I$
(ii) $A, B \in I$ implies $A \bigcup B \in I$.
(iii) $A \in I, B \subset A$ implies $B \in I$.
I is called a nontrivial ideal if $X \notin I$.

Definition 2.7. *A nonempty class F of subsets of a nonempty set X is said to be a filter in X. Provided*
(i) $\phi \in F$.
(ii) $A, B \in F$ implies $A \bigcap B \in F$.
(iii) $A \in F, A \subset B$ implies $B \in F$.

Definition 2.8. *I is a non trivial ideal in X, $X \neq \phi$, then the class*

$$F(I) = \{M \subset X : M = X \backslash A \; for \; some \; A \in I\}$$

is a filter on X, called the filter associated with I.

Definition 2.9. *A non trivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$.*

Case 2.1. *If I is an admissible ideal, then usual convergence in X implies I convergence in X.*

Remark 2.2. *If I is an admissible ideal, then usual rough convergence implies rough $I-$ convergence.*

Definition 2.10. *Let $x = (x_{mnk})$ be a triple sequence in a metric space $(X, |.,.|)$ and r be a non-negative real number is said to be rough ideal convergent or $rI-$ convergent to l, denoted by $x \rightarrow^{rI} l$, if for any $\epsilon > 0$ we have*

$$\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{mnk} - l| \geq r + \epsilon\} \in I.$$

In this case l is called $rI-$ limit of x and a triple sequence $x = (x_{mnk})$ is called rough $I-$ convergent to l with r as roughness of degree. If $r = 0$ then it is ordinary $I-$ convergent.

Case 2.2. *Generally, a triple sequence $y = (y_{mnk})$ is not $I-$ convergent in usual sense and $|x_{mnk} - y_{mnk}| \leq r$ for all $(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ or*

$$\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{mnk} - y_{mnk}| \geq r\} \in I.$$

for some $r > 0$. Then the triple sequence $x = (x_{mnk})$ is $rI-$ convergent.

Case 2.3. *It is clear that $rI-$ limit of x is not necessarily unique.*

Definition 2.11. *Consider $rI-$ limit set of x, which is denoted by*

$$I - LIM_x^r = \left\{L \in X : x \to^{rI} l\right\},$$

then the triple sequence $x = (x_{mnk})$ is said to be $rI-$ convergent if $I - LIM_x^r \neq \phi$ and r is called a rough $I-$ convergence degree of x.

Definition 2.12. *A triple sequence $x = (x_{mnk}) \in X$ is said to be $I-$ analytic if there exists a positive real number M such that*

$$\left\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{mnk}|^{1/m+n+k} \geq M\right\} \in I.$$

Definition 2.13. *A point $L \in X$ is said to be an $I-$ accumulation point of a triple sequence $x = (x_{mnk})$ in a metric space (X, d) if and only if for each $\epsilon > 0$ the set*

$$\left\{(m, n, k) \in \mathbb{N}^3 : d\left(x_{mnk}, l\right) = |x_{mnk} - l| < \epsilon\right\} \notin I.$$

We denote the set of all $I-$ accumulation points of x by $I\left(\Gamma_x\right)$.

Definition 2.14. *A triple sequence $x = (x_{mnk})$ is said to be Wijsman $r-$ convergent to A denoted by $A_{mnk} \to^r A$, provided that*
$\forall \epsilon > 0 \, \exists \left(m_\epsilon, n_\epsilon, k_\epsilon\right) \in \mathbb{N}^3 : m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon \Longrightarrow$
$lim_{rst} \frac{1}{rst} \left|\{m \leq r, n \leq s, k \leq t : \left|d\left(x, A_{mnk}\right) - d\left(x, A\right)\right| < r + \epsilon\}\right| = 0$
The set

$$LIM^r A = \left\{L \in \mathbb{R}^3 : A_{mnk} \to^r A\right\}$$

is called the Wijsman $r-$ limit set of the triple sequences.

Definition 2.15. *A triple sequence $x = (x_{mnk})$ is said to be Wijsman $r-$ convergent if $LIM^r A \neq \phi$. In this case, r is called the Wijsman convergence degree of the triple sequence $x = (x_{mnk})$. For $r = 0$, we get the ordinary convergence.*

Definition 2.16. *A triple sequence (x_{mnk}) is said to be Wijsman $r-$ statistically convergent to A, denoted by $A_{mnk} \to^{rst} A$, provided that the set*

$$lim_{rst} \frac{1}{rst} \left|\left\{(m, n, k) \in \mathbb{N}^3 : \left|d\left(x, A_{mnk}\right) - d\left(x, A\right)\right| \geq r + \epsilon\right\}\right| = 0$$

has natural density zero for every $\epsilon > 0$, or equivalently, if the condition

$$st - lim\ sup\ \left|d\left(x, A_{mnk}\right) - d\left(x, A\right)\right| \leq r$$

is satisfied.

In addition, we can write $A_{mnk} \rightarrow^{rst} A$ *if and only if the inequality*

$$lim_{rst} \frac{1}{rst} \left|\left\{m \leq r, n \leq s, k \leq t : \left|d\left(x, A_{mnk}\right) - d\left(x, A\right)\right| < r + \epsilon\right\}\right| = 0$$

holds for every $\epsilon > 0$ *and almost all* (m, n, k). *Here* r *is called the wijsman roughness of degree. If we take* $r = 0$, *then we obtain the ordinary Wijsman statistical convergence of triple sequence.*

Definition 2.17. *A triple sequence* (x_{mnk}) *is said to be Wijsman Cesáro convergent to* A, *denoted by* $A_{mnk} \rightarrow^{ces} A$, *provided that the set*

$$lim_{rst} \frac{1}{rst} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} d\left(x, A_{mnk}\right) = d\left(x, A\right).$$

Definition 2.18. *A triple sequence* (x_{mnk}) *is said to be Wijsman strongly Cesáro convergent to* A, *denoted by* $A_{mnk} \rightarrow^{stces} A$, *provided that the set*

$$lim_{rst} \frac{1}{rst} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} \left|d\left(x, A_{mnk}\right) - d\left(x, A\right)\right| = 0.$$

Definition 2.19. *A triple sequence* (x_{mnk}) *is said to be Wijsman* $p-$ *Cesáro convergent to* A, *denoted by* $A_{mnk} \rightarrow^{stces} A$, *if for each* p *positive real number and for each* $x \in X$,

$$lim_{rst} \frac{1}{rst} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} \left|d\left(x, A_{mnk}\right) - d\left(x, A\right)\right|^{p} = 0.$$

Definition 2.20. *The triple sequence* $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ *is called triple lacunary if there exist three increasing sequences of integers such that*

$$m_0 = 0, h_i = m_i - m_{i-1} \rightarrow \infty \text{ as } i \rightarrow \infty \text{ and}$$
$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \rightarrow \infty \text{ as } \ell \rightarrow \infty.$$
$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \rightarrow \infty \text{ as } j \rightarrow \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, *and* $\theta_{i,\ell,j}$ *is determine by* $I_{i,\ell,j} = \{(m, n, k) : m_{i-1} < m < m_i \, and \, n_{\ell-1} < n \leq n_\ell \, and \, k_{j-1} < k \leq k_j\}, q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}$. *Let* $\theta_{i,\ell,j}$ *be a lacunary sequence. A triple sequence* (x_{mnk}) *is said to be Wijsman strongly lacunary convergent to* A, *denoted by* $A_{mnk} \rightarrow^{stlac} A$, *if*

$$lim_{rst} \frac{1}{h_{rst}} \sum_{(m,n,k) \in I_{rst}} \left|d\left(x, A_{mnk}\right) - d\left(x, A\right)\right| = 0.$$

In a similar fashion to the idea of classic Wijsman rough convergence, the idea of Wijsman rough statistical convergence of a triple sequence spaces can be interpreted as follows:

Assume that a triple sequence $y = (y_{mnk})$ *is Wijsman statistically convergent and cannot be measured or calculated exactly; one has to do with an approximated (or Wijsman statistically approximated) triple sequence* $x = (x_{mnk})$ *satisfying* $\left|d\left(x - y, A_{mnk}\right) - d\left(x - y, A\right)\right| \leq r$ *for all* m, n, k *(or for almost all* (m, n, k), *i.e.,*

$$\delta\left(lim_{rst} \frac{1}{rst} \left|\left\{m \leq r, n \leq s, k \leq t : \left|d\left(x - y, A_{mnk}\right) - d\left(x - y, A\right)\right| > r\right\}\right|\right) = 0.$$

Then the triple sequence x is not statistically convergent any more, but as the inclusion

$$lim_{rst}\frac{1}{rst}\left\{\left|d\left(y,A_{mnk}\right)-d\left(y,A\right)\right|\geq\epsilon\right\}\supseteq lim_{rst}\frac{1}{rst}\left\{\left|d\left(x,A_{mnk}\right)-d\left(x,A\right)\right|\geq r+\epsilon\right\} \tag{2.1}$$

holds and we have

$$\delta\left(lim_{rst}\tfrac{1}{rst}\left|\left\{(m,n,k)\in\mathbb{N}^3:\left|y_{mnk}-l\right|\geq\epsilon\right\}\right|\right)=0,$$

i.e., we get

$$\delta\left(lim_{rst}\tfrac{1}{rst}\left|\left\{m\leq r,n\leq s,k\leq t:\left|d\left(x,A_{mnk}\right)-d\left(x,A\right)\right|\geq r+\epsilon\right\}\right|\right)=0,$$

i.e., the triple sequence spaces x is Wijsman $r-$ statistically convergent in the sense of definition (2.21)

In general, the Wijsman rough statistical limit of a triple sequence may not unique for the Wijsman roughness degree $r>0$. So we have to consider the so called Wijsman $r-$ statistical limit set of a triple sequence $x=(x_{mnk})$, which is defined by

$$st-LIM^rA_{mnk}=\left\{L\in\mathbb{R}:A_{mnk}\to^{rst}A\right\}.$$

The triple sequence x is said to be Wijsman $r-$ statistically convergent provided that $st-LIM^rA_{mnk}\neq\phi$. It is clear that if $st-LIM^rA_{mnk}\neq\phi$ for a triple sequence $x=(x_{mnk})$ of real numbers, then we have

$$st-LIM^rA_{mnk}=\left[st-lim\ sup\ A_{mnk}-r,st-lim\ inf\ A_{mnk}+r\right] \tag{2.2}$$

We know that $LIM^r=\phi$ for an unbounded triple sequence $x=(x_{mnk})$. But such a triple sequence might be Wijsman rough statistically convergent. For instance, define

$$d\left(x,A_{mnk}\right)=\left\{\begin{array}{cc}(-1)^{mnk}, & if(m,n,k)\neq(i,j,\ell)^2\,(i,j,\ell\in\mathbb{N}),\\ (mnk), & otherwise\\ & \end{array}\right\}.$$

in $\mathbb{R}$. Because the set $\{1,64,739,\cdots\}$ has natural density zero, we have

$$st-LIM^rA_{mnk}=\left\{\begin{array}{cc}\phi, & if\,r<1,\\ \left[1-r,r-1\right], & otherwise\\ & \end{array}\right\}$$

and $LIM^rA_{mnk}=\phi$ for all $r\geq0$.

As can be seen by the example above, the fact that $st-LIM^rA_{mnk}\neq\phi$ does not imply $LIM^rA_{mnk}\neq\phi$. Because a finite set of natural numbers has natural density zero, $LIM^rA_{mnk}\neq\phi$ implies $st-LIM^rA_{mnk}\neq\phi$. Therefore, we get $LIM^rA_{mnk}\subseteq st-LIM^rA_{mnk}$. This obvious fact means $\{r\geq0:LIM^rA_{mnk}\neq\phi\}\subseteq\{r\geq0:st-LIM^rA_{mnk}\neq\phi\}$ in this language of sets and yields immediately

$$inf\left\{r\geq0:LIM^rA_{mnk}\neq\phi\right\}\geq inf\left\{r\geq0:st-LIM^rA_{mnk}\neq\phi\right\}.$$

Moreover, it also yields directly $diam\left(LIM^rA_{mnk}\right)\leq diam\left(st-LIM^rA_{mnk}\right)$.

Throughout the paper, we let $(X;\rho)$ be a separable metric space, $I\subseteq 2^{N^3}$ be an admissible ideal and $A;A_{mnk}$ be any non-empty closed subsets of X.

Definition 2.21. *A triple sequence (x_{mnk}) is said to be Wijsman $r-I$ convergent to A, if for every $\epsilon>0$ and for each $x\in X$,*

$$A(x,\epsilon)=\left\{(m,n,k)\in\mathbb{N}^3:|d(x,A_{mnk})-d(x,A)|\geq r+\epsilon\right\}\in I$$

Definition 2.22. *A triple sequence (x_{mnk}) is said to be Wijsman $r-I$ statistical convergent to A, if for every $\epsilon>0,\delta>0$ and for each $x\in X$,*

$$\left\{(r,s,t)\in\mathbb{N}^3:\tfrac{1}{rst}\left|\{(r,s,t)\leq(m,n,k):|d(x,A_{mnk})-d(x,A)|\geq r+\epsilon\}\right|\geq\delta\right\}\in I.$$

In this case, we write $A_{mnk}\rightarrow^{s(I_W)}A$.

Definition 2.23. *Let θ be a lacunary sequence. A triple sequence (x_{mnk}) is said to be Wijsman strongly $r-I$ convergent to A, if for every $\epsilon>0$ and for each $x\in X$,*

$$\left\{(r,s,t)\in\mathbb{N}^3:\tfrac{1}{h_{rst}}\sum_{(m,n,k)\in I_{rst}}|d(x,A_{mnk})-d(x,A)|\geq r+\epsilon\right\}\in I.$$

In this case, we write $A_{mnk}\rightarrow^{N_\theta(I_W)}A$.

Definition 2.24. *A triple sequence (x_{mnk}) is said to be Wijsman $r-I$ Cesáro convergent to A, if for every $\epsilon>0$ and for each $x\in X$,*

$$\left\{(r,s,t)\in\mathbb{N}^3:\left|\tfrac{1}{rst}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t}d(x,A_{mnk})-d(x,A)\right|\geq r+\epsilon\right\}\in I.$$

In this case, we write $A_{mnk}\rightarrow^{C(I_W)}A$.

Definition 2.25. *A triple sequence (x_{mnk}) is said to be Wijsman strongly $r-I$ Cesáro convergent to A, if for every $\epsilon>0$ and for each $x\in X$,*

$$\left\{(r,s,t)\in\mathbb{N}^3:\tfrac{1}{rst}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t}|d(x,A_{mnk})-d(x,A)|\geq r+\epsilon\right\}\in I.$$

In this case, we write $A_{mnk}\rightarrow^{C(I_W)}A$.

Definition 2.26. *A triple sequence (x_{mnk}) is said to be Wijsman p strongly $r-I$ Cesáro convergent to A, if for each p positive real number, if for every $\epsilon>0$ and for each $x\in X$,*

$$\left\{(r,s,t)\in\mathbb{N}^3:\tfrac{1}{rst}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t}|d(x,A_{mnk})-d(x,A)|^p\geq r+\epsilon\right\}\in I.$$

In this case, we write $A_{mnk}\rightarrow^{C_p(I_W)}A$.

3. Main Results

Theorem 3.1. *Let the triple sequence $(A_{mnk})\in\Lambda^3$. If (A_{mnk}) is Wijsman $r-I$ statistical convergent to A, then (A_{mnk}) is Wijsman p strongly $r-I$ Cesáro convergent to A.*

Proof: *Suppose that (A_{mnk}) is triple analytic and $A_{mnk}\rightarrow^{S(I_W)}A$. Then, there is an $M>0$ such that*

$$|d(x,A_{mnk})-d(x,A)|^{1/m+n+k}\leq M,$$

for each $x \in X$ and for all m, n, k. Given $\epsilon > 0$, we have
$\frac{1}{rst} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right|^{p/m+n+k} =$
$\frac{1}{rst} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1, |d(x,A_{mnk})-d(x,A)| \geq r+\epsilon}^{t} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right|^{p/m+n+k} +$
$\frac{1}{rst} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1, |d(x,A_{mnk})-d(x,A)| < r+\epsilon}^{t} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right|^{p/m+n+k}$
$\leq \frac{1}{rst} M^{p/m+n+k} \left| \left\{ (m,n,k) \leq (r,s,t) : \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right|^{1/m+n+k} \geq r+\epsilon \right\} \right| +$
$\frac{1}{rst} \epsilon^{p/m+n+k} \left| \left\{ (m,n,k) \leq (r,s,t) : \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right|^{1/m+n+k} < r+\epsilon \right\} \right|$
$\leq \frac{M^{p/m+n+k}}{rst} \left| \left\{ (m,n,k) \leq (r,s,t) : \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right|^{1/m+n+k} \geq r+\epsilon \right\} \right| + \epsilon^{p/m+n+k}$. *Then for any* $\delta > 0$
$\left\{ (r,s,t) \in \mathbb{N}^3 : \frac{1}{rst} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right|^{p/m+n+k} \geq \delta \right\} \subseteq$
$\left\{ (r,s,t) \in \mathbb{N}^3 : \frac{1}{rst} \left| \left\{ (m,n,k) \leq (r,s,t) \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| \geq r+\epsilon \right\} \right| \geq \frac{\delta^{p/m+n+k}}{M^{p/m+n+k}} \right\} \in I$, *for each* $x \in X$.
Hence $A_{mnk} \rightarrow^{C_p(I_W)} A$.

Theorem 3.2. *Let the triple sequence (A_{mnk}) is Wijsman p strongly $r - I$ Cesáro convergent to A then (A_{mnk}) is Wijsman $r - I$ statistical convergent to A.*

Proof: Let $A_{mnk} \rightarrow^{C_p(I_W)} A$ *and given* $\epsilon > 0$. *Then, we have*
$\sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right|^{p} \geq$
$\sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1, |d(x,A_{mnk})-d(x,A)| \geq r+\epsilon}^{t} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right|^{p}$
$\epsilon^{p} \left| \left\{ (m,n,k) \leq (r,s,t) : \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| \geq r+\epsilon \right\} \right|$
for each $x \in X$ and so
$\frac{1}{\epsilon^{p}(rst)} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right|^{p} \geq$
$\frac{1}{rst} \left| \left\{ (m,n,k) \leq (r,s,t) : \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| \geq r+\epsilon \right\} \right|.$
Hence for given $\delta > 0$
$\left\{ (r,s,t) \in \mathbb{N}^3 : \frac{1}{rst} \left| \left\{ (m,n,k) \leq (r,s,t) \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| \geq r+\epsilon \right\} \right| \geq \delta \right\} \subseteq$
$\left\{ (r,s,t) \in \mathbb{N}^3 : \frac{1}{rst} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right|^{p} \geq (r+\epsilon)^{p} \delta \right\} \in I$, *for each* $x \in X$. *Hence* $A_{mnk} \rightarrow^{S(I_W)} A$.

Theorem 3.3. *Let θ be a triple lacunary sequence (A_{mnk}). If $liminf_{uvw} q_{uvw} > 1$ then, $A_{mnk} \rightarrow^{C_1(I_W)} A \Rightarrow A_{mnk} \rightarrow^{N_\theta(I_W)} A$.*

Proof: If $liminf_{uvw} q_{uvw} > 1$, then there exists $\delta > 0$ such that $q_{uvw} \geq 1 + \delta$ for all $u, v, w \geq 1$. Since $h_{uvw} = (m_u n_v k_w) - (m_{u-1} n_{v-1} k_{w-1})$, we have

$$\frac{m_u n_v k_w}{h_{u-1,v-1,w-1}} \leq \frac{1+\delta}{\delta} \text{ and } \frac{m_{u-1} n_{v-1} k_{w-1}}{h_{u-1,v-1,w-1}} \leq \frac{1}{\delta}.$$

Let $\epsilon > 0$ and we define the set
$S = \left\{ (m_u n_v k_w) \in \mathbb{N}^3 : \frac{1}{m_u n_v k_w} \sum_{m=1}^{m_u} \sum_{n=1}^{n_v} \sum_{k=1}^{k_w} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| < r+\epsilon \right\}$, *for each $x \in X$ and also $S \in F(I)$, which is a filter of the ideal I, we have*
$\frac{1}{h_{uvw}} \sum_{mnk \in I_{uvw}} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| = \frac{1}{h_{uvw}} \sum_{m=1}^{m_u} \sum_{n=1}^{n_v} \sum_{k=1}^{k_w} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| -$

$$= \frac{m_r n_v k_w}{h_{uvw}} \frac{1}{m_r n_v k_w} \sum_{m=1}^{m_u} \sum_{n=1}^{n_v} \sum_{k=1}^{k_w} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| -$$
$$\leq \left(\frac{1+\delta}{\delta}\right)\left(r+\epsilon^{'}\right) - \frac{1}{\delta}\left(r+\epsilon^{'}\right)$$

for each $x \in X$ *and* $(m_u n_v k_w) \in S$. *Choose* $\eta = \left(\frac{1+\delta}{\delta}\right)\left(r+\epsilon^{'}\right) + \frac{1}{\delta}\left(r+\epsilon^{'}\right)$. *Therefore, for each* $x \in X$

$$\left\{(u,v,w) \in \mathbb{N}^3 : \frac{1}{h_{uvw}} \sum_{mnk \in I_{uvw}} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| < \eta \right\} \in F\left(I\right).$$

Theorem 3.4. *Let* θ *be a triple lacunary sequence* $\left(A_{mnk}\right)$. *If* $limsup_{uvw} q_{uvw} < \infty$ *then,* $A_{mnk} \to^{N_\theta(I_W)} A \Rightarrow A_{mnk} \to^{C_1(I_W)} A$.

Proof: *If* $limsup_{uvw} q_{uvw} < \infty$ *then there exists* $M > 0$ *such that* $q_{uvw} < M$, *for all* $u,v,w \geq 1$. *Let* $A_{mnk} \to^{N_\theta(I_W)} A$ *and we define the sets* T *and* R *such that*

$$T = \left\{(u,v,w) \in \mathbb{N}^3 : \frac{1}{h_{uvw}} \sum_{mnk \in I_{uvw}} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| < r + \epsilon_1 \right\}$$

and

$$R = \left\{(a,b,c) \in \mathbb{N}^3 : \frac{1}{rst} \sum_{m=1}^{a} \sum_{n=1}^{b} \sum_{k=1}^{c} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| < r + \epsilon_2 \right\},$$

for every $\epsilon_1, \epsilon_2 > 0$ *for each* $x \in X$. *Let*

$$a_j = \frac{1}{h_j} \sum_{mnk \in I_j} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| < r + \epsilon_1$$

for each $x \in X$ *and for all* $j \in T$. *It is obvious that* $T \in F\left(I\right)$. *Choose* (a,b,c) *in any integer with* $\left(m_{u-1} n_{v-1} k_{w-1}\right) < (a,b,c) < \left(m_u n_v k_w\right)$, *where* $(u,v,w) \in T$. *Then, we have*

$$\frac{1}{h_{uvw}} \sum_{mnk \in I_{uvw}} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| \leq \frac{1}{m_{u-1} n_{v-1} k_{w-1}} \sum_{m=1}^{m_u} \sum_{n=1}^{n_v} \sum_{k=1}^{k_w} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right|$$
$$= \frac{1}{m_{u-1} n_{v-1} k_{w-1}}$$
$$\begin{pmatrix} \sum_{(m,n,k) \in I_{111}} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| + \sum_{(m,n,k) \in I_{222}} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| + \\ \cdots + \qquad \sum_{(m,n,k) \in I_{uvw}} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| \end{pmatrix}$$
$$= \frac{m_1 n_1 k_1}{m_{u-1} n_{v-1} k_{w-1}} \left(\frac{1}{h_{111}} \sum_{(m,n,k) \in I_{111}} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| \right) +$$
$$\frac{(m_2 n_2 k_2) - (m_1 n_1 k_1)}{m_{u-1} n_{v-1} k_{w-1}} \left(\frac{1}{h_{222}} \sum_{(m,n,k) \in I_{222}} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| \right) + \cdots$$
$$\frac{(m_u n_v k_w) - (m_{u-1} n_{v-1} k_{w-1})}{m_{u-1} n_{v-1} k_{w-1}} \left(\frac{1}{h_{uvw}} \sum_{(m,n,k) \in I_{uvw}} \left| d\left(x, A_{mnk}\right) - d\left(x, A\right) \right| \right)$$
$$= \frac{m_1 n_1 k_1}{m_{u-1} n_{v-1} k_{w-1}} a_{111} + \frac{(m_2 n_2 k_2) - (m_1 n_1 k_1)}{m_{u-1} n_{v-1} k_{w-1}} a_{222} + \cdots + \frac{(m_u n_v k_w) - (m_{u-1} n_{v-1} k_{w-1})}{m_{u-1} n_{v-1} k_{w-1}} a_{uvw}$$

$\leq \left(sup_{j \in T} a_j\right) \cdot \frac{m_1 n_1 k_1}{m_{u-1} n_{v-1} k_{w-1}} < r + \epsilon_1 \cdot M$ *for each* $x \in X$. *Choose* $r + \epsilon_2 = \frac{r + \epsilon_1}{M}$ *and the fact that*

$$\bigcup \left\{(a,b,c) : m_{u-1} < a < m_u, n_{v-1} < b < n_v, k_{w-1} < c < k_w, (u,v,w) \in T\right\} \subset R,$$

where $T \in F\left(I\right)$. *It follows from our assumption on* θ *that the set* $R \in F\left(I\right)$.

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