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ON Λ-TYPE DUALITY OF FRAMES IN BANACH SPACES

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Abstract. Frames are redundant system which are useful in the reconstruction of certain classes of spaces. The dual of a frame (Hilbert) always exists and can be obtained in a natural way. Due to the presence of three Banach spaces in the definition of retro Banach frames (or Banach frames) duality of frames in Banach spaces is not similar to frames for Hilbert spaces. In this paper we introduce the notion of Λ-type duality of retro Banach frames. This can be generalized to Banach frames in Banach spaces. Necessary and sufficient conditions for the existence of the dual of retro Banach frames are obtained. A special class of retro Banach frames which always admit a dual frame is discussed.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle .,. \rangle$. A countable system $\mathcal{E} \equiv \{f_k\} \subset \mathcal{H}$ in a separable is called a *frame* (Hilbert) for $\mathcal{H}$ if there exists positive constants A and B such that

$$\text{A}\|f\|^2 \leq \|\{\langle f, f_k \rangle\}\|_{\ell^2}^2 \leq \text{B}\|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

The positive constants A and B are called the *lower* and *upper frame bounds* of the frame $\mathcal{E}$, respectively (the largest A and smallest B for which (1.1) holds are the *optimal* frame bounds). They are not unique. A frame $\mathcal{E} \equiv \{f_n\}$ for $\mathcal{H}$ is called *tight* if it is possible to choose $A = B$ and *normalized tight* if $A = B = 1$. If removal of one f_n renders the collection $\mathcal{E} \equiv \{f_k\}$ no longer a frame for $\mathcal{H}$, then $\mathcal{E}$ is called an *exact frame* for $\mathcal{H}$. Let $\mathcal{E} \equiv \{f_k\}$ be a frame(Hilbert) for $\mathcal{H}$. The operator $T : \ell^2 \to \mathcal{H}$ given by

$$T(\{c_k\}) = \sum_{k=1}^{\infty} c_k f_k, \ \{c_k\} \in \ell^2,$$

is called the *synthesis operator* or *pre-frame operator* of $\mathcal{E}$.

Adjoint of T is the operator $T^* : \mathcal{H} \to \ell^2$ given by

$$T^*(f) = \{\langle f, f_k \rangle\}$$

and is called the *analysis operator* of the frame $\mathcal{E}$. Composing T and T^* we obtain the *frame operator* $S = TT^* : \mathcal{H} \to \mathcal{H}$ given by

$$S(f) = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, f \in \mathcal{H}.$$

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It can be easily verified that the frame operator S is a positive continuous invertible linear operator from $\mathcal{H}$ onto $\mathcal{H}$. Furthermore, every vector $f \in \mathcal{H}$ can be written as:

$$f = SS^{-1}f = \sum_{k=1}^{\infty} \langle S^{-1}f, f_k \rangle f_k. \tag{1.2}$$

The series given in (1.2) converges unconditionally for all $f \in \mathcal{H}$ and is called *frame decomposition* or *reconstruction formula* for the frame. Thus, frames are redundant building blocks which can recover the underlying space and which have basis like property. One may observe that the frame decomposition shows that all the information about a given signal (vector) $f \in \mathcal{H}$ is contained in the system $\{\langle S^{-1}f, f_k \rangle\}$. The scalars $\langle S^{-1}f, f_k \rangle$ are called the *frame coefficients* associated with the frame $\mathcal{E} \equiv \{f_k\}$.

Frames for Hilbert space were first introduced by Duffin and Schaeffer in [7], while working in deep problem in non-harmonic Fourier series. For some reason the work of Duffin and Schaeffer was not continued until 1986, when the fundamental work of Daubechies, Grossmann and Mayer [6] brought this all back to life, right at the dawn of the "wavelet era". More precisely, Daubechies et al. in [6] observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$ which are very similar to the expansions using orthonormal bases. This was probably the time, when many mathematicians started to see the potential of the topic. Since then, the theory of frames has been more widely studied. Gröchenig in [10], generalized Hilbert frames to Banach spaces. Before the concept of Banach frames was formalized, it appeared in the foundational work of Feichtinger and Gröchenig [9] related to *atomic decompositions*. Atomic decompositions appeared in the field of applied mathematics providing many applications. An atomic decomposition allow a representation of every vector of the space via a series expansion in terms of a fixed sequence of vectors which we call *atoms*. On the other hand Banach frame for a Banach space ensure reconstruction via a bounded linear operator or *synthesis operator*. In last one decade, various types of frames and reconstruction system in Banach spaces have been introduced and studied. Retro Banach frames were introduced and studied in [15] and further studied in [18, 19]. An excellent approach towards the utility of frames in different direction is given in the book by Casazza and Kutynoik [1] and in the paper by Heil and Walnut [12]. The basic theory of frames can be found in [2, 3, 5, 11, 13, 21].

A warm up on duality of frames in Hilbert spaces: Let us give now a brief discussion on duality of frames in Hilbert spaces. Suppose that $\mathcal{E} \equiv \{f_k\}$ is a frame for a Hilbert space $\mathcal{H}$. A system $\mathcal{E}' \equiv \{g_k\} \subset \mathcal{H}$ is called a *dual frame* of $\mathcal{E}$ if $\mathcal{E}'$ is a frame for $\mathcal{H}$ and

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \text{ for all } f \in \mathcal{H}. \tag{1.3}$$

In short we say that $(\mathcal{E}, \mathcal{E}')$ is a dual pair. Let $\mathcal{E} \equiv \{f_k\}$ be a frame for $\mathcal{H}$ with frame bounds $0 < A,\ B < \infty$ and with frame operator S. Then $\mathcal{E}_{(natural)} \equiv \{S^{-1}f_k\}$ is a frame for $\mathcal{H}$ with frame bounds A^{-1}, B^{-1}. Furthermore, the condition in (1.3) follows immediately follows from (1.2). Therefore, the system $\mathcal{E}_{(natural)} \equiv \{S^{-1}f_k\}$ is a dual of $\mathcal{E} \equiv \{f_k\}$. The dual frame $\mathcal{E}_{(natural)} \equiv \{S^{-1}f_k\}$ is known as *canonical dual* (or *natural dual*) of $\mathcal{E} \equiv \{f_k\}$. Thus, every frame for a Hilbert space admit a

dual (at least one). An interesting property of dual frames is that dual of a frame need not be unique. In fact, there may be infinitely many duals for a non-exact frame. A frame has a unique dual if and only if it is exact. For more technical details about duality of frames (Hilbert), one may refer to [5, 13].
The following example shows that the dual of a frame (Hilbert) need not be unique. Infact, it may have infinitely many duals.

Example 1.1. Let $\{\chi_n\}$ be an orthogonal basis for a Hilbert space $\mathcal{H}$. Then, $\{\chi_n\}$ is a Parseval frame for $\mathcal{H}$. Consider a system $\mathcal{E} \equiv \{f_k\} = \{\chi_1, \chi_1, \chi_2, \chi_3, \chi_4, ...\}$. One can easily verify that

$$\|f\|^2 \leq \|\{\langle f, f_k\rangle\}\|^2_{\ell^2} \leq 2\|f\|^2, \text{ for all } f \in \mathcal{H}.$$

Therefore, $\mathcal{E}$ is a frame for $\mathcal{H}$ with one of the choice of bounds $A = 1, B = 2$. Let S be the frame operator of $\mathcal{E}$. Then, the canonical dual of $\mathcal{E}$ is given by

$$\mathcal{E}' \equiv \{S^{-1}(f_k)\} = \{\frac{1}{2}\chi_1, \frac{1}{2}\chi_1, \chi_2, \chi_2, ...\}.$$

Other dual of $\mathcal{E}$ are $\mathcal{E}'' \equiv \{0, \chi_1, 0, \chi_2, ...\}$ and $\mathcal{E}''' \equiv \{\frac{1}{3}\chi_1, \frac{2}{3}\chi_1, \chi_2, \chi_3, \chi_4, ...\}$. Therefore, infinitely many dual of a frame (Hilbert) can be constructed.

Motivation: Let us have a look on Example 1.1, where is observed that a vector (signal) can be represented in infinitely many ways associated with a given frame. Thus, there is an opportunity that even if the canonical dual frame is difficult to find, there exists other duals that are easy to find. Now dual frames (like frames) represents a vector in a concern space as series. On the other hand in case of retro Banach frames a vector can be reconstructed by pre-frame operator(a bounded linear operator). More precisely, retro Banach frame recover the underlying space via an operator and not by mean of an infinite series (see Definition 1.2). Due to the presence of three Banach spaces in the definition of a retro Banach frame, it is difficult to define its dual which preserve the property of reconstruction via infinite series (as case of duals of frames for Hilbert spaces). Not only this, the dual of a Hilbert frame has a *relation* with original frame that can be seen in (1.3). Some author have work in the direction of dual of frame [4, 8, 11].

In this paper we introduce the notion of dual (or simply Λ-dual) of retro Banach frames in Banach spaces. This can be generalized to Banach frames in Banach spaces. It is observed that even an exact retro Banach frame does not admit a dual retro Banach frame for the underlying space. The said situation is entirely different from the dual of frames (Hilbert) for Hilbert spaces, where an exact frame admit a unique dual. Necessary and sufficient conditions for the existence of retro dual frames are obtained. Counterexamples are also given to defend results and remarks. A special class of retro Banach frames which always admit retro dual frames is discussed. In fact, we discussed an application of Young's result given in [20].

Now we give basic definitions and notations which will be required in this paper. Throughout this paper $\mathcal{X}$ will denote an infinite dimensional Banach space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $\mathcal{X}^*$ the conjugate space of $\mathcal{X}$ and $\mathcal{Z}_d$ is a Banach space of scalar-valued sequences indexed by $\mathbb{N}$ which is associated with $\mathcal{X}^*$. For a sequence $\{\Phi_n\}$ in $\mathcal{X}$, $[\Phi_n]$ denotes the closure of the linear hull of $\{\Phi_n\}$ in the norm topology of $\mathcal{X}$. As usual $\delta_{k,m}$ denote the Kronecker delta which is defined as $\delta_{k,m} = 0$, if

$k \neq m$ and $\delta_{k,m} = 1$, if $k = m$. The class of all bounded linear operator from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$ is denoted by $\mathfrak{B}(\mathcal{X}, \mathcal{Y})$. Unless otherwise stated all systems of the form $\{\Phi_k\}$ are indexed by $\mathbb{N}$.

Definition 1.2. [18, at page 83] A system $\mathcal{F} \equiv (\{\Phi_k\}, \Theta)$ $(\{\Phi_k\} \subset \mathcal{X}, \Theta : \mathcal{Z}_d \to \mathcal{X}^*)$ is called a *retro Banach frame* for $\mathcal{X}^*$ with respect to an associated sequence space $\mathcal{Z}_d$ if,

(i) $\{\Phi^*(\Phi_k)\} \in \mathcal{Z}_d$, for each $\Phi^* \in \mathcal{X}^*$.

(ii) There exist positive constants $(0 < \mathrm{A}_0 \leq \mathrm{B}_0 < \infty)$ such that

$$\mathrm{A}_0 \|\Phi^*\| \leq \|\{\Phi^*(\Phi_k)\}\|_{\mathcal{Z}_d} \leq \mathrm{B}_0 \|\Phi^*\|, \text{ for each } \Phi^* \in \mathcal{X}^*.$$

(iii) Θ is a bounded linear operator such that $\Theta(\{\Phi^*(\Phi_k)\} = \Phi^*, \Phi^* \in \mathcal{X}^*$

The positive constant $\mathrm{A}_0, \mathrm{B}_0$ are called the *lower* and *upper retro frame bounds* of $(\{\Phi_k\}_{k=1}^{\infty}, \Theta)$, respectively. As in case of frames (Hilbert) for Hilbert spaces, they are not unique. The operator $\Theta : \mathcal{Z}_d \to X^*$ is called *retro pre-frame operator* (or simply reconstruction operator) associated with $\{\Phi_k\}_{k=1}^{\infty}$. A retro Banach $\mathcal{F} \equiv (\{\Phi_k\}, \Theta)$ is said to be *exact* if for each $m \in \mathbb{N}$ there exists no reconstruction operator Θ_m such that $(\{\Phi_k\}, \Theta_m)_{k \neq m}$ is retro Banach frame for $\mathcal{X}^*$. Retro Banach frames were further studied in [18, 19].

Lemma 1.3. [18, Lemma 2 at page 83] *Let $\mathcal{F} \equiv (\{\Phi_k\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$. Then, $\mathcal{F}$ is exact if and only if $\Phi_n \notin [\Phi_k]_{k \neq n}$, for all $n \in \mathbb{N}$.*

Lemma 1.4. *Let $\mathcal{X}$ be a Banach space and $\{\Phi_n^*\} \subset \mathcal{X}^*$ be a sequence such that $\{\Phi \in \mathcal{X} : \Phi_n^*(\Phi) = 0,$ for all $n \in \mathbb{N}\} = \{0\}$. Then, $\mathcal{X}$ is linearly isometric to the Banach space $\mathcal{Z} = \{\{\Phi_n^*(\Phi)\} : \Phi \in \mathcal{X}\}$, where the norm is given by*

$$\|\{\Phi_n^*(\Phi)\}\|_{\mathcal{Z}} = \|\Phi\|_{\mathcal{X}}, \ \Phi \in \mathcal{X}.$$

2. Dual of Retro Banach Frames

Definition 2.1. Let $\mathcal{F} \equiv (\{\Phi_k\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$ and let Λ be a fixed subset of $\mathbb{N}$. A system $\{\Phi_k^*\} \subset \mathcal{X}^*$ is called *dual retro Banach frame* of $\mathcal{F}$ (or simply Λ-dual of $\mathcal{F}$) if

(i) $\Phi_j^*(\Phi_l) = \delta_{j,l}$, for all $l \in \mathbb{N}$ and for all $j \in \mathbb{N} \setminus \Lambda$,

(ii) there exists a reconstruction operator $\Theta^{\times}$ such that $\mathcal{G} \equiv (\{\Phi_k^*\}, \Theta^{\times})$ is a retro Banach frame for $\mathcal{X}^{**}$.

If $\mathcal{G} \equiv (\{\Phi_k^*\}, \Theta^{\times})$ is a dual of $\mathcal{F} \equiv (\{\Phi_k\}, \Theta)$, then we say that $\mathcal{F}$ admits a dual with respect to the system $\{\Phi_k^*\}$. In short we say that $(\mathcal{F}, \mathcal{G})$ is a *retro dual pair.*

Remark 2.2. In Definition 2.1, if $\Lambda = \emptyset$, then $\mathcal{G}$ is called the strong dual of $\mathcal{F}$.

Remark 2.3. Recall that the dual of a frame (Hilbert) $\mathcal{E}$ for a Hilbert space $\mathcal{H}$ is associated with $\mathcal{E}$ via series expansions (see equation (1.3)). The condition (i), in Definition 2.1, gives a relation between a given retro Banach frame $\mathcal{F}$ and its dual $\mathcal{G}$. There may be other relation but at present we discuss Λ-type duality.

Example 2.4. Let $\{\Phi_k\} = \{\chi_k\}$ be an orthogonal basis for a Hilbert space $\mathcal{H}$. Then

$$\|\{\Phi^*(\Phi_k)\}\|_{\ell^2} = \|\Phi^*\|_{\mathcal{X}^*}, \ \Phi^* \in \mathcal{X}^*, \text{ for all } \Phi^* \in \mathcal{X}^*.$$

Define $\Theta : \ell^2 \to \mathcal{X}^*$ by $\Theta(\{\Phi^*(\Phi_k)\}) = \Phi^*$. Then, Θ is reconstruction operator such that $\mathcal{F} \equiv (\{\Phi_k\}, \Theta)$ is a normalized tight exact retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}_d = \ell^2$.
Consider the system $\{\Phi_k^*\} \equiv \{\chi_1^*, \chi_1^*, \chi_2^*, \chi_3^*, \chi_4^*, ...\} \subset \mathcal{H}^*$. Choose $\Lambda = \{1\} \subset \mathbb{N}$. Then

$$\Phi_j^*(\Phi_l) = \delta_{j,l}, \text{ for all } l \in \mathbb{N} \text{ and for all } j \in \mathbb{N} \setminus \Lambda.$$

Furthermore, there exists a reconstruction operator $\Theta^\times$ such that $\mathcal{G} \equiv (\{\Phi_k^*\}, \Theta^\times)$ is a retro Banach frame for $\mathcal{X}^{**}$ with respect to $\mathcal{Z}_d = \ell^2$ and with one of the choice of bounds $A = 1, B = \sqrt{2}$. Hence $(\mathcal{F}, \mathcal{G})$ is a dual pair. It may be noted that we can construct infinitely many dual retro Banach frames for $\mathcal{F}$. More precisely, there are infinitely many dual pair of the form $(\mathcal{F}, \mathcal{G})$.

Let $\mathcal{F}$ be an exact retro Banach frame for $\mathcal{X}^*$. Then, in general, $\mathcal{F}$ has no retro dual frame. The following example provides the existence of an exact retro Banach frame $\mathcal{F}$ which does not admits a dual retro Banach frame.

Example 2.5. Consider the measure space $\mathcal{X} = L^2(\Omega)$ with counting measure, where $\Omega = \mathbb{N}$. Let $\{\chi_n\}$ be an orthonormal basis of $\mathcal{X}$.
Define $\{\Phi_k\} \subset \mathcal{X}$ by

$$\Phi_k = \chi_{k+1} + \chi_1, \ k \in \mathbb{N}.$$

Let $\mathcal{Z} = \{\{\Phi^*(\Phi_k)\} : \Phi^* \in \mathcal{X}^*\}$. Then, by using Lemma 1.4, $\mathcal{Z}$ is a Banach space with norm given by

$$\|\{\Phi^*(\Phi_k)\}\|_{\mathcal{Z}} = \|\Phi^*\|_{\mathcal{X}^*}, \ \Phi^* \in \mathcal{X}^*.$$

Define $\Theta : \mathcal{Z} \to \mathcal{X}^*$ by

$$\Theta(\{\Phi^*(\Phi_k)) = \Phi^*, \ \Phi^* \in \mathcal{X}^*.$$

Then, Θ is a bounded linear operator such that $\mathcal{F} \equiv (\{\Phi_k\}, \Theta)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}$.

To show $\mathcal{F}$ is exact. Choose $\Phi_k^* = \chi_{k+1}^*, k \in \mathbb{N}$. Then, $\{\Phi_k^*\} \subset \mathcal{X}^*$ and we observe that

$$\Phi_n^*(\Phi_m) = \delta_{n,m}, \text{ for all } n, m \in \mathbb{N}.$$

Therefore, $\Phi_n \notin [\Phi_k]_{k \neq n}$, for all $n \in \mathbb{N}$. Thus, by using Lemma 1.3, we conclude that $\mathcal{F}$ is an exact retro Banach frame for $\mathcal{X}^*$. Note the if $\mathcal{F}$ is exact, then the condition (i) given in definition 2.1 is satisfied for $\Lambda = \emptyset$.

To show that the condition (ii) given in Definition 2.1 is not satisfied. Let $\Theta^\times$ be the reconstruction operator such that $(\{\Phi_k^*\}, \Theta^\times)$ is a retro Banach frame for $\mathcal{X}^{**}$. Let $A^0, \ B^0$ be a choice of retro bounds for $(\{\Phi_k^*\}, \Theta^\times)$.
Then

$$A^0\|\Phi^{**}\| \leq \|\{\Phi^{**}(\Phi_k^*)\}\|_{\mathcal{Z}_d} \leq B^0\|\Phi^*\|, \text{ for all } \Phi^{**} \in \mathcal{X}^{**}. \tag{2.1}$$

In particular for $\Phi^{**} = \chi_1$, we have $\Phi^{**}(\Phi_k^*) = 0$, for all $k \in \mathbb{N}$. Therefore, by using retro frame inequality (2.1), we obtain $\Phi^{**} = 0$, a contradiction. Hence $\mathcal{F}$ has no dual retro Banach frame. Furthermore, there is no other system $\{\Psi_k^*\} \subset \mathcal{X}^*$ such that $\mathcal{F}$ admits a dual with respect to $\{\Psi_k^*\}$.

Remark 2.6. The second dual of a Banach space $\mathcal{X}$ may have a retro Banach frame from its pre-dual space with respect to which a retro Banach frame for $\mathcal{X}^*$ does not admit strong duality, but it may have a dual pair.

The following example defend Remark 2.6

Example 2.7. Let $\mathcal{F} \equiv (\{\Phi_k\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$ given in Example 2.5. Choose $\Phi_k^* = \chi_k^*, k \in \mathbb{N}$ and let $\mathcal{Z}^0 = \{\{\Phi^{**}(\Phi_k^*)\} : \Phi^{**} \in \mathcal{X}^{**}\}$. Then, $\mathcal{Z}^0$ is a Banach space of sequences of scalars with norm given by

$$\|\{\Phi^{**}(\Phi_k^*)\}\|_{\mathcal{Z}^0} = \|\Phi^{**}\|_{\mathcal{X}^{**}},\ \Phi^{**} \in \mathcal{X}^{**}.$$

Therefore, $\Theta^\times : \{\Phi^{**}(\Phi_k^*)\} \to \Phi^{**}$ is a bounded linear operator from $\mathcal{Z}^0$ onto $\mathcal{X}^{**}$ such that $\mathcal{G}_0 \equiv (\{\Phi_k^*\}, \Theta^\times)$ is a retro Banach frame for $\mathcal{X}^{**}$ with respect to $\mathcal{Z}^0$ with bounds $A = B = 1$. Choose $\Lambda = \{1\}$. Then, $\Phi_j^*(\Phi_l) = \delta_{j,l}$, for all $l \in \mathbb{N}$ and for all $j \in \mathbb{N} \setminus \Lambda$. Therefore, $(\mathcal{F}, \mathcal{G}_0)$ is a retro dual pair, which is not strong.

Remark 2.8. By Example 2.7 we observe that a retro Banach frame $\mathcal{F}$ for $\mathcal{X}^*$ may have dual pair but does not admit a strong dual (even) with respect to an orthonormal basis for $\mathcal{X}^*$.

The following proposition provides sufficient conditions for the existence of the dual retro Banach frames. It is sufficient to prove the result for the existence of strong dual retro Banach frame. We can extend the same construction for the existence of arbitrary dual pair.

Proposition 2.9. *Let $\mathcal{F} \equiv (\{\Phi_k\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$. Then, $\mathcal{F}$ admits a strong dual retro Banach frame if there exists a system $(\{\Phi_k^*\} \subset \mathcal{X}^*$ such that $\Phi_n^*(\Phi_m) = \delta_{n,m}$, for all $n, m \in \mathbb{N}$ and there exists an injective closed linear operator $\Phi^{**} \to \{\Phi^{**}(\Phi_k^*)\}$ with closed range from $\mathcal{X}^{**}$ to $\widehat{\mathcal{Z}}_0$, where $\widehat{\mathcal{Z}}_0$ is some Banach space of scalar valued sequences. These conditions are not necessary.*

Proof. Let $U : \mathcal{X}^{**} \to \widehat{\mathcal{Z}}_0$ be given by $U(\Phi^{**}) = \{\Phi^{**}(\Phi_k^*)\}, \Phi^{**} \in \mathcal{X}^{**}$. Then, by hypothesis, U is injective and closed linear operator with closed range $R(U)$. Therefore, $U^{-1} : R(U) \subset \widehat{\mathcal{Z}}_0 \to \mathcal{X}^{**}$ is closed [14]. Thus, by using Closed Graph Theorem [14], there exists a constant $c > 0$ such that

$$\|U(\Phi^{**})\| \geq c\|\Phi^{**}\|, \text{ for all } \Phi^{**} \in \mathcal{X}^{**}.$$

Let, if possible, there exists no reconstruction operator $\Theta^\times$ such that $(\{\Phi_k^*\}, \Theta^\times)$ is a retro Banach frame for $\mathcal{X}^{**}$. Then, by using Hahn-Banach Theorem there exists a non-zero functional $\Phi_0^{**} \in \mathcal{X}^{**}$ such that $\Phi_0^{**}(\Phi_k^*) = 0$, for all $k \in \mathbb{N}$.
Therefore

$$0 = \|U(\Phi_0^{**})\| \geq c\|\Phi_0^{**}\|.$$

This gives $\Phi_0^{**} = 0$, a contradiction. Therefore, there exists a reconstruction operator $\Theta^\times$ such that $(\{\Phi_k^*\}, \Theta^\times)$ is a retro Banach frame for $\mathcal{X}^{**}$ with respect to some associated Banach space of scalar valued sequences. Hence $\mathcal{F}$ admits a dual retro Banach frame for the underlying space.

To show that the conditions are not necessary. Let $\mathcal{X}$ be the measure space given in Example 2.5. Let $\Phi_k = k^2\chi_k, k \in \mathbb{N}$, where $\{\chi_k\}$ is an orthonormal basis for $\mathcal{X}$. Then, there exists a bounded linear operator Θ such that $\mathcal{F} = (\{\Phi_k\}, \Theta)$ is a retro Banach frame for $\mathcal{X}^*$.
Choose $\Phi_k^* = \frac{1}{k^2}\chi_k, k \in \mathbb{N}$. Then, $\Phi_i^*(\Phi_j) = \delta_{i,j}$ for all $i, j \in \mathbb{N}$. By the nature of the system $\{\Phi_k^*\}$, we conclude that there exists a reconstruction operator $\Theta^\times$ such that $\mathcal{G} \equiv (\{\Phi_k^*\}, \Theta^\times)$ is a retro Banach frame for $\mathcal{X}^{**}$ with respect to $\widehat{\mathcal{Z}}_o = \ell^2$. Thus, $(\mathcal{F}, \mathcal{G})$ is a dual pair.

Define $\widetilde{\Theta}: \mathcal{X}^{**} \to \widehat{\mathcal{Z}}_o (= \ell^2)$ by

$$\widetilde{\Theta}(\Phi^{**}) = \{\Phi^{**}(\Phi_j^*)\} = \{\frac{\xi_j}{j^2}\},\ \Phi^* = \{\xi_j\} \in \mathcal{X}^{**}.$$

It can be verified that the range of the operator $\widetilde{\Theta}$ is not closed. $\square$

The following theorem gives necessary and sufficient condition for a given retro Banach frame to admit its dual.

Theorem 2.10. *Let $\mathcal{F} \equiv (\{\Phi_n\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$. Then, $\mathcal{F}$ has a dual retro Banach frame if and only if there exists a system $\{\Phi_n^*\} \subset \mathcal{X}^*$ such that $\Phi_j^*(\Phi_l) = \delta_{j,l}$, for all $l \in \mathbb{N}$ and for all $j \in \mathbb{N} \setminus \Lambda$ and dist$(\Phi^*, L_n) \to 0$ as $n \to \infty$, for all $\Phi^* \in \mathcal{X}^*$, where $L_n = [\Phi_1^*, \Phi_2^*, ..., \Phi_n^*]$, for all $n \in \mathbb{N}$.*

Proof. Suppose first that $\mathcal{F} \equiv (\{\Phi_n\}, \Theta)$ has a dual retro Banach frame. Then, by definition we can find a system $\{\Phi_n^*\} \subset \mathcal{X}^*$ such that $\Phi_j^*(\Phi_l) = \delta_{j,l}$, for all $l \in \mathbb{N}$ and for all $j \in \mathbb{N} \setminus \Lambda$ and a reconstruction operator $\Theta^\times$ such that $\mathcal{G} \equiv (\{\Phi_k^*\}, \Theta^\times)$ is a retro Banach frame for $\mathcal{X}^{**}$. Let A_0 and B_0 be a choice of bounds for $\mathcal{G}$. Then

$$A_0\|\Phi^{**}\|_{\mathcal{X}^{**}} \leq \|\{\Phi^{**}(\Phi_n^*)\}\|_{(\mathcal{X}^{**})_d} \leq B_0\|\Phi^{**}\|_{\mathcal{X}^{**}}, \quad \text{for all} \quad \Phi^{**} \in \mathcal{X}^{**}. \tag{2.2}$$

Suppose that the condition dist$(\Phi, L_n) \to 0$ as $n \to \infty$, for all $\Phi^* \in \mathcal{X}^*$, is not satisfied. Then, there exists a non zero functional $\Phi_0^* \in \mathcal{X}^*$ such that

$$\lim_{n\to\infty} \text{dist}(\Phi_0^*, L_n) \neq 0.$$

Note that dist$(\Phi_0^*, L_n) \geq$ dist(Φ_0^*, L_{n+1}) for all $n \in \mathbb{N}$. This is because $\{L_n\}$ is a nested system of subspaces and by definition of distance of a point from a set. Now $\{\text{dist}(\Phi_0^*, L_n)\}$ is bounded below monotone decreasing sequence. So, $\{\text{dist}(\Phi_0^*, L_n)\}$ is convergent and its limit is a positive real number, since otherwise dist$(\Phi_0^*, L_n) \to 0$ as $n \to \infty$ (which is not possible). Let $\lim_{n\to\infty} \text{dist}(\Phi_0^*, L_n) = \xi > 0$.

Choose $D = \bigcup_n L_n$. Then, by using the fact that dist$(\Phi_0^*, D) = \inf\{\text{dist}(\Phi_0^*, L_n)\}$, we obtain

$$\text{dist}(\Phi_0^*, D) \geq \xi > 0. \tag{2.3}$$

Now we show that $\Phi_0^* \notin \overline{D}$. Let if possible, $\Phi_0^* \in \overline{D}$. Then, we can find a sequence $\{\zeta_n\} \subset D$ such that dist$(\zeta_n, \Phi_0^*) \to 0$ as $n \to \infty$.

By using (2.3), we have dist$(\Phi_0^*, D) \geq \xi$. Therefore, dist$(\zeta_n, \Phi_0^*) \geq \xi > 0$. This is a contradiction to the fact that dist$(\zeta_n, \Phi_0^*) \to 0$ as $n \to \infty$. Hence $\Phi_0^* \notin \overline{D}$. Thus, by using Hahn-Banach theorem, there exists a non zero functional $\Phi_0^{**} \in \mathcal{X}^{**}$ such that

$$\Phi_0^{**}(\Phi_n^*) = 0,\ \text{for all}\ n \in \mathbb{N}.$$

Therefore, by using retro frame inequality (2.2), we have $\Phi_0^{**} = 0$, a contradiction. Hence dist$(\Phi^*, L_n) \to 0$ as $n \to \infty$, for all $\Phi^* \in \mathcal{X}^*$.

To prove the converse part, assume that there exists a system $\{\Phi_n^*\} \subset \mathcal{X}^*$ is such that

$$\Phi_j^*(\Phi_l) = \delta_{j,l},\ \text{for all}\ l \in \mathbb{N}\ \text{and for all}\ j \in \mathbb{N} \setminus \Lambda,$$

and

$$\text{dist}(\Phi^*, L_n) \to 0\ \text{as}\ n \to \infty,\ \text{for all}\ \Phi^* \in \mathcal{X}^*.$$

Then, in particular, for each $\epsilon > 0$ and for each $\Phi^* \in \mathcal{X}^*$, we can find a Φ_j^* from some L_k such that

$$\|\Phi^* - \Phi_j^*\| < \epsilon.$$

Therefore, by using Lemma 1.4, $\mathcal{Z} = \{\{\Phi^{**}(\Phi_n^*)\} : \Phi^{**} \in \mathcal{X}^{**}\}$ is a Banach space of sequences of scalars with norm given by

$$\|\{\Phi^{**}(\Phi_n^*)\}\|_{\mathcal{Z}} = \|\Phi^{**}\|_{\mathcal{X}^{**}},\ \Phi^{**} \in \mathcal{X}^{**}.$$

Define $\Theta_0 : \mathcal{Z} \to \mathcal{X}^{**}$ by

$$\Theta_0(\{\Phi^{**}(\Phi_k^*)\}) = \Phi^{**},\ \Phi^{**} \in \mathcal{X}^{**}.$$

Then, Θ_0 is a bounded linear operator such that such that $(\{\Phi_n^*\}, \Theta_0)$ is a retro Banach frame for $\mathcal{X}^{**}$ with respect to $\mathcal{Z}$. Hence $\mathcal{F}$ has a dual retro Banach frame. □

The existence of duality with respect to certain sequence space is also interested. The following theorem provides the necessary and sufficient condition for an exact retro Banach frame to admit a dual frame with respect to a given sequence space.

Theorem 2.11. *Let $\mathcal{F} \equiv (\{\Phi_n\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}_{\mathfrak{d}}$ and let $\mathcal{A}_d = \{\{\Phi^{**}(\Phi_k^*)\} : \Phi^{**} \in \mathcal{X}^{**}\}$. Then, $\mathcal{F}$ has a dual retro Banach frame with respect to the sequence space $\mathcal{A}_d$ if and only if there exists a system $\{\Phi_n^*\} \subset \mathcal{X}^*$ such that $\Phi_j^*(\Phi_l) = \delta_{j,l}$, for all $l \in \mathbb{N}$ and for all $j \in \mathbb{N} \setminus \Lambda$ and the analysis operator $\mathcal{U} : \Phi^{**} \to \{\Phi^{**}(\Phi_k^*)\}$ is a bounded below continuous linear operator from $\mathcal{X}^{**}$ onto $\mathcal{A}_d$.*

Proof. Suppose first that $\mathcal{F}$ has a dual retro Banach frame $\mathcal{G}$ with respect to $\mathcal{A}_d$. Then, there exists a reconstruction operator $\Theta^\times$ such that $\mathcal{G} \equiv (\{\Phi_k^*\}, \Theta^\times)$ is a retro Banach frame for $\mathcal{X}^{**}$ with respect to $\mathcal{A}_d$. Therefore, there are positive constants a_0 and b_0 such that

$$\mathrm{a}_0\|\Phi^{**}\| \leq \|\{\Phi^{**}(\Phi_k^*)\}\|_{\mathcal{A}_d} \leq \mathrm{b}_0\|\Phi^{**}\|,\ \text{for each } \Phi^{**} \in \mathcal{X}^{**}. \tag{2.4}$$

Now consider the analysis operator $\mathcal{U} : \mathcal{X}^{**} \to \mathcal{A}_d$ which is given by

$$\mathcal{U}(\Phi^{**}) = \{\Phi^{**}(\Phi_k^*)\},\ \Phi^{**} \in \mathcal{X}^{**}.$$

Then, linearity and ontoness of $\mathcal{U}$ is obvious. By using upper retro frame inequality in (2.4), we have

$$\|\mathcal{U}(\Phi^{**})\|_{\mathcal{A}_d} \leq \mathrm{b}_0\|\Phi^{**}\|,\ \text{for each } \Phi^{**} \in \mathcal{X}^{**}.$$

Therefore, $\|\mathcal{U}\| \leq \mathrm{b}_0$. Hence $\mathcal{U}$ is continuous. Similarly, by using lower retro frame inequality in (2.4), we have

$$\|\mathcal{U}(\bullet)\|_{\mathcal{A}_d} \geq \mathrm{a}_0\| \bullet \|_{\mathcal{X}^{**}}.$$

Hence $\mathcal{U}$ is bounded below.
For the reverse part, assume that $\mathcal{U}$ is bounded below. Then, using Lemma 1.4, $\mathcal{A}_d$ is a Banach space with the norm given by

$$\|\{\Phi^{**}(\Phi_k^*)\}\|_{\mathcal{A}_d} = \|\Phi^{**}\|_{\mathcal{X}^{**}},\ \Phi^{**} \in \mathcal{X}^{**}.$$

Define $\Theta^\times : \mathcal{A}_d \to X^{**}$ by $\Theta^\times(\{\Phi^{**}(\Phi_k^*)\}) = \Phi^{**}$. Then, $\Theta^\times$ is a bounded linear operator such that $(\{\Phi_k^*\}, \Theta^\times)$ is a retro Banach for $\mathcal{X}^{**}$ with respect to $\mathcal{A}_d$. The theorem is proved. □

3. Applications

In this section we give some applications of the duality of retro Banach frames. First two two examples provides an application of the Theorem 2.10.

Example 3.1. In this example, we give an application of Theorem 2.10 in duality of retro Banach frames in finite dimensional Banach spaces. We start with a well known frame in a finite dimensional Banach space, namely Mercedes-Benz frame. Mercedes-Benz frame is one of the popular frame in finite dimensional frame theory. It consists of three vectors which are equiangular situated on the unit circle. More precisely, a Mercedes-Benz frame is a system consisting of three vectors in $\mathbb{R}^2$, namely,

$$\mathcal{A} = \left\{ \Phi_1 = (0,1), \quad \Phi_2 = (\frac{-\sqrt{3}}{2}, \frac{-1}{2}), \Phi_3 = (\frac{\sqrt{3}}{2}, \frac{-1}{2}) \right\}.$$

Let $\mathcal{Z}_{d_0} = \{[\Phi^*(\Phi_1), \Phi^*(\Phi_2), \Phi^*(\Phi_3)]^t : \Phi \in \mathcal{X}^*\}$. Then, $\mathcal{Z}_{d_0}$ is a Banach space with the norm given by

$$\|[\Phi^*(\Phi_1), \Phi^*(\Phi_2), \Phi^*(\Phi_3)]^t\|_{\mathcal{Z}_{d_0}} = \|\Phi^*\|_{\mathcal{X}^*}, \ \Phi^* \in \mathcal{X}^*.$$

Define $\Theta_0 : \mathcal{Z}_{d_0} \to \mathcal{X}^*$ by

$$\Theta_0([\Phi^*(\Phi_1), \Phi^*(\Phi_2), \Phi^*(\Phi_3)]^t) = \Phi^*, \ \Phi^* \in \mathcal{X}^*.$$

Then, Θ_0 is a bounded linear operator such that $\mathcal{F}_0 \equiv (\{\Phi_i\}_{i=1}^3, \Theta_0)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}_{d_0}$ and with bounds $A = B = 1$.

By using Theorem 2.10 and of above argument we conclude that every proper subfamily of the Mercedes-Benz frame containing two vectors from $\mathcal{A}$ is a retro Banach frame for $(\mathbb{R}^2)^*$ and admits a dual (strong) retro Banach frame for the underlying space. This can be extended to arbitrary finite dimensional Banach space. More precisely, every retro Banach frame for a finite dimensional space contains a strong dual retro Banach frame for the underlying space.

For more applications of Mercedes-Benz frame an interested reader may refer to [16, 17].

Example 3.2. Now we discuss Banach frames of wavelet system. Let $\mathcal{X} = L^2(\mathbb{R})$. Consider the Haar function Φ which is defined by

$$\Phi(x) = 1, \text{if } x \in [0, \frac{1}{2}), \ \Phi(x) = -1, \text{if } x \in [\frac{1}{2}, 1) \text{ and } \Phi(x) = 0, \text{otherwise}.$$

Let $\psi_{j,k}(x) = 2^{\frac{1}{2}}\Phi(2^j x - k), j, k \in \mathbb{Z}$ and $x \in \mathbb{R}$. The system $\{\psi_{j,k}(x)\}$ is called a wavelet system associated with the window function Φ. Without loss of generality, we can write $\{\Phi_n(\bullet)\} \equiv \{\psi_{j,k}(\bullet)\}$. Let $\mathcal{Z}_d = \{\{\psi(\Phi_n)\} : \psi \in \mathcal{X}^*\}$. Then, $\mathcal{Z}_d$ is Banach space of scalar valued sequences with the norm given by

$$\|\{\psi(\Phi_n)\}\|_{\mathcal{Z}_d} = \|\psi\|_{\mathcal{X}^*}, \psi \in \mathcal{X}^*. \tag{3.1}$$

Define $\Theta : \mathcal{Z}_d \to \mathcal{X}^*$ by

$$\Theta(\{\psi(\Phi_n)\}) = \psi, \ \psi \in \mathcal{X}^*.$$

Then, $\Theta \in \mathfrak{B}(\mathcal{Z}_d, \mathcal{X}^*)$. Hence Θ is a bounded linear operator such that $\mathcal{F} \equiv (\{\Phi_n\}, \Theta)$ is a retro Banach frame for $\mathcal{X}^*$ with bounds $A = B = 1$.

By the nature of construction of the system $\{\Phi_n\}$, it can be easily verified that

$$\Phi_n \notin [\Phi_m]_{m \neq n}, \ \text{for each } n \in \mathbb{N}.$$

Thus, by Hahn-Banach theorem, there exists a $\{\Phi_n^*\} \subset \mathcal{X}^*$ such that

$$\Phi_n^*(\Phi_m) = \delta_{n,m}, \text{ for all } n, m \in \mathbb{N}.$$

Hence $\mathcal{F}$ satisfying one of the sufficient conditions in Theorem 2.10, with $\Lambda = \emptyset$. Let $L_n = [\Phi_1^*, \Phi_2^*, ..., \Phi_n^*]$, for all $n \in \mathbb{N}$. Note that each $\Phi_j^* \in L^2(\mathbb{R})$ is given by

$$\Phi_j^*(\Phi) = \int \Phi \Psi_\Phi d\mu,$$

where Ψ_Φ is the representative of Φ.
Now by using retro frame inequality (3.1), we have

$$\begin{aligned}
\operatorname{dist}(\Psi^*, L_n) &= \inf_{F^* \in L_n} \|\Psi^* - F^*\| \\
&= \inf_{F^* \in L_n} \left(\int |\Psi^* - F^*|^2 d\mu \right)^{\frac{1}{2}} \\
&\to 0 \text{ as } n \to \infty, \text{ for all } \Psi^* \in \ \mathcal{X}^*.
\end{aligned}$$

Hence by Theorem 2.10, $\mathcal{F}$ admits a dual (strong) retro Banach frame . We can construct a dual of $\mathcal{F}$ which is not strong.

It is interested to know the class of retro Banach frames which always admits dual retro Banach frames. It is difficult to characterize the class of retro Banach frames which have dual retro Banach frames. A special class of retro Banach frames consisting of complex exponentials in $\mathcal{X} = L^2(-\pi, \pi)$ always admits a dual retro Banach frame. This can be proved by using a result by Robert Young in [20 at page 538] . Recall that two sequences $\{\Phi_n\}$ and $\{\Psi_n\}$ of elements from a Hilbert space $\mathcal{H}$ are said to be *biorthogonal* if $\langle \Phi_n, \Psi_m \rangle = \delta_{n,m}$ for all $n, m \in \mathbb{N}$. A sequence that admits a biorthogonal sequence will be called *minimal*. A sequence $\{\Phi_n\} \subset \mathcal{H}$ is said to be *complete* in $\mathcal{H}$ if zero vector is alone is perpendicular to every Φ_n. A sequence that is both minimal and complete will be called *exact*. Robert Young considered a Paley-Wiener space consisting of all entire functions of exponential type at most π that are square-integrable on the real axis and using Paley-Wiener theorem proved the following result.

Theorem 3.3. [20, at page 538] *If the sequence of complex exponential* $\{e^{i\lambda_k t}\}$ *is exact in* $L^2(-\pi, \pi)$*, then its biorthogonal sequence is also exact.*

To conclude the paper we show that retro Banach frames consisting of complex exponentials in $\mathcal{X} = L^2(-\pi, \pi)$ always admits a dual retro Banach frame.

Proposition 3.4. *Let* $\mathcal{X} = L^2(-\pi, \pi)$ *and* $\Phi_k = e^{i\lambda_k t}, k \in \mathbb{N}$. *If* $\mathcal{F} \equiv (\{\Phi_k\}, \Theta)$ *be an exact retro Banach frame for* $\mathcal{X}^*$ *with admissible system* $\{\Phi_k^*\} \subset \mathcal{X}^*$*, then there exists a reconstruction operator* $\Theta^\times$ *such that* $(\{\Phi_k^*\}, \Theta^\times)$ *is a retro Banach frame for* $\mathcal{X}^{**}$. *More precisely,* $\mathcal{F}$ *has a dual (strong) retro Banach frame.*

Proof. Suppose that there is no $\Theta^\times \in \mathfrak{B}(Z_\lrcorner, \mathcal{X}^{**})$ such that $(\{\Phi_k^*\}, \Theta^\times)$ is a retro Banach frame for $\mathcal{X}^{**}$, where $Z_\lrcorner$ is associated Banach space of scalar-valued sequences. Then, there exists a non-zero $\Phi^{**} \in \mathcal{X}^{**}(\equiv \mathcal{X})$ such that

$$\Phi^{**}(\Phi_k^*) = \int_{-\pi}^{\pi} \Phi^{**}\Phi_k^* dt = 0, \text{ for all } k \in \mathbb{N}.$$

By using construction in the proof of Theorem 3.3 (for proof see [20] at page 538), we can show that $\Phi^{**} = 0$. This is a contradiction. Hence there exists a reconstruction operator $\Theta^{\times} \in \mathfrak{B}(Z_{\lrcorner}, \mathcal{X}^{**})$, where $Z_{\lrcorner} = \{\{\Phi^{**}(\Phi_k^*)\} : \Phi^{**} \in \mathcal{X}^{**}\}$, such that $\mathcal{G} \equiv (\{\Phi_k^*\}, \Theta^{\times})$ is a retro Banach frame for $\mathcal{X}^{**}$. $\square$

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