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# HARMONIC $m$-PREINVEX FUNCTIONS AND INEQUALITIES

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Abstract. In this paper, we introduce a new class of harmonic functions, which is called harmonic $m$-preinvex functions for a fixed $m$. Some Hermite-Hadamard inequality for harmonic $m$-preinvex functions are derived. Several special cases are discussed as applications of the main results. The ideas and techniques of this paper may be starting point for further research.

## 1. Introduction

Convex functions and their variant forms are being used to study a wide class of problems which arises in various branches of pure and applied sciences. The concept of convexity have been generalize by many researchers using novel and innovative techniques and ideas. It is know that the convex function can be characterized by some integral inequalities, which are known as Hermite-Haramard inequalities. Hanson [11] introduced the concept of invex functions. Ben-Israel and Mond [5] introduced the concept of invex sets and peinvex functions. For the applications, properties and other aspects of the preinvex functions, see [2,3,17–24] and the references therein. Varosanec [36] introduced the class of $h$-convex functions. This class of functions unifies various classes of convex functions and is being used to discuss several concepts in a unified manners.

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Anderson et al. [1] and Iscan [13] have investigated various properties of harmonic convex function. Iscan [13] has obtained several Hermite-Hadamard inequalities for harmonic convex functions. Noor et al. [26] introduced and investigated another class of harmonic convex functions, which is called harmonic preinvex functions and can be viewed as significant generalization of both the harmonic convex functions and preinvex functions. For recent developments and other aspects of harmonic convex functions, see [14, 25–30].

Toader [34] define the concept of $m$-convexity, an intermediate between usual convexity and star shape functions. Noor et. al [30] have introduced the concept of harmonic $m$-convex function on a harmonic convex set. In particular, a function $f : I = [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic $m$-convex function with respect to an arbitrary nonnegative function $h$, where $m \in (0, 1]$, if

$$\begin{aligned} \frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{2ab}{a+b}\right) &\leq \frac{ab}{b-a}\int_a^b \frac{f(x)+mf(xm)}{x^2}\mathrm{d}x \\ &\leq \frac{1}{2}\bigg\{f(a)+f(b)+2m[f(am)+f(bm)] \\ &\quad +m^2[f(am^2)+f(bm^2)]\bigg\}\int_0^1 h(t)\mathrm{d}t, \end{aligned} \tag{1.1}$$

which is known as Hermite-Hadamard inequality for harmonic $m$-convex function with respect to an arbitrary nonnegative function $h$.

We would like to emphasize that $m$-convex functions and harmonic $h$-preinvex functions are two distinct classes of convex functions. It is natural to introduce a new class of convex functions, which unifies these concepts. Motivated and inspired by the on going research in the convexity theory, we introduce harmonic $m$-preinvex functions on a harmonic $m$-invex set. It is shown that several new classes of harmonic convex functions and harmonic preinvex functions can be obtained as special case. We have obtained several new Hermite-Hadamard inequality and related inequalities for harmonic $m$-preinvex functions. One can easily show that this new class includes harmonic $m$-convex functions, harmonic $m$-preinvex functions and harmonic beta $m$-preinvex functions as special cases. The techniques and the ideas of this paper may stimulate further research.

## 2. PRELIMINARIES

Let $I$ be a nonempty closed set in $\mathbb{R}$ . Let $f : I_\eta \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function and $\eta(\cdot, \cdot) : I_\eta \times I_\eta \to \mathbb{R}$ be a continuous bifunction.

**Definition 2.1.** [7]. *A set $I \subset \mathbb{R}$ is said to be $m$-convex set with respect to a fixed constant $m \in [0, 1]$, if*

$$(1-t)x + mty \in I, \qquad \forall x, y \in I, \ t \in [0, 1].$$

The $m$-convex set contains the line segment between points $x$ and $my$ for every pair of points $x$ and $y$ of $I$.

**Definition 2.2.** [34]. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be $m$-convex function, where $m \in [0,1]$, if*

$$f((1-t)x + mty) \leq (1-t)f(x) + mf(y), \qquad \forall x, y \in I,\ t \in [0,1].$$

If we take $m = 1$, then we recapture the concept of convex functions and if we take $t = 1$, then

$$f(my) \leq mf(y) \qquad \forall t \in [0,1], \quad y \in I.$$

This shows that the function $f$ is sub-homogeneous.

**Definition 2.3.** *A set $\mathcal{I}_\eta \subset \mathbb{R} \setminus \{0\}$ is said to be a harmonic $m$-invex set with respect to an arbitrary bifunction $\eta : I_\eta \times I_\eta \to \mathbb{R}$, if*

$$\frac{x(x + \eta(my, x))}{x + (1-t)\eta(my, x)} \in \mathcal{I}_\eta, \qquad \forall x, y \in \mathcal{I}_\eta,\ t \in [0,1].$$

The harmonic $m$-invex set contains the path between points $x$ and $x + \eta(my, x)$ for every pair of points $x$ and $y$ of $\mathcal{I}_\eta$. Every harmonic $m$-invex set is harmonic invex respecting the mapping $\eta(my, x) = my - x$.

We now introduce the concept of harmonic $m$-preinvex function as.

**Definition 2.4.** *Let $h : J = [0,1] \to \mathbb{R}$ be a nonnegative function. A function $f : \mathcal{I}_\eta \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic $m$-preinvex function, where $m \in (0,1]$, if*

$$f\left(\frac{x(x + \eta(my, x))}{x + (1-t)\eta(my, x)}\right) \leq h(1-t)f(x) + h(t)mf(y), \qquad \forall x, y \in \mathcal{I}_\eta,\ t \in (0,1).$$

If $t = \frac{1}{2}$, then we have

$$f\left(\frac{2x(x + \eta(my, x))}{2x + \eta(my, x)}\right) \leq h\left(\frac{1}{2}\right)[f(x) + mf(y)], \qquad \forall x, y \in \mathcal{I}_\eta.$$

The function $f$ is called Jensen type harmonic $m$-preinvex function.

Now we discuss some special cases of harmonic $m$-preinvex functions, which appears to be new ones.

**I.** If $\eta(my, x) = my - x$ in Definition 2.4, then it reduces to the Definition of harmonic $m$-convex function.

**Definition 2.5.** *A function $f : \mathcal{I} \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic $m$-convex, where $m \in (0,1]$, if*

$$f\left(\frac{mxy}{tx + (1-t)my}\right) \leq h(1-t)f(x) + h(t)mf(y), \qquad \forall x, y \in \mathcal{I},\ t \in [0,1].$$

**II.** If $h(t) = t^s$ in Definition 2.4, then it reduces to the Definition of harmonic $(s, m)$-preinvex function in the second sense.

**Definition 2.6.** *A function $f : \mathcal{I}_\eta \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonic $(s, m)$-preinvex, where $s, m \in (0, 1]$, if*

$$f\left(\frac{x(x+\eta(my,x))}{x+(1-t)\eta(my,x)}\right) \leq (1-t)^s f(x) + t^s m f(y), \qquad \forall x, y \in \mathcal{I}_\eta,\ t \in [0,1].$$

**III.** If $h(t) = t^p(1-t)^q$ in Definition 2.4, then it reduces to the Definition of harmonic beta-$m$-preinvex function.

**Definition 2.7.** *A function $f : \mathcal{I}_\eta \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonic beta-$m$-preinvex, where $m \in (0, 1]$ and $p, q \geq -1$, if*

$$f\left(\frac{x(x+\eta(my,x))}{x+(1-t)\eta(my,x)}\right) \leq m(1-t)^p t^q f(xm) + t^p(1-t)^q f(y), \qquad \forall x, y \in \mathcal{I}_\eta,\ t \in (0,1).$$

In brief, for suitable and appropriate choice of the functions, one can obtain several new and known classes of harmonic, preinvex and convex functions as special cases of harmonic $m$-preinvex functions. This shows that the class of harmonic $m$-preinvex functions is very general and unifying one.

**Definition 2.8.** [31]. *Two functions $f, g$ are said to be similarly ordered ($f$ is $g$-monotone), if and only if,*

$$\langle f(x) - f(y), g(x) - g(y)\rangle \geq 0, \qquad \forall x, y \in \mathbb{R}^n.$$

The Euler Beta function is a special function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \forall x, y > 0.$$

where $\Gamma(\cdot)$ is a Gamma function.

We now show that the product of two harmonic $m$-preinvex functions is again a harmonic $m$-preinvex function under certain condition, which is the main motivation of our next result.

**Lemma 2.1.** *Let $f$ and $g$ be two similarly ordered harmonic $m$-preinvex functions. If $h(1-t) + mh(t) \leq 1$, then the product $fg$ is again a harmonic $m$-preinvex function.*

*Proof.* Let $f$ and $g$ be harmonic $m$-preinvex functions. Then

$$\begin{aligned}
& f\left(\frac{x(x+\eta(my,x))}{x+(1-t)\eta(my,x)}\right)g\left(\frac{x(x+\eta(my,x))}{x+(1-t)\eta(my,x)}\right) \\
\leq\ & [h(1-t)f(x)+h(t)mf(y)][h(1-t)g(x)+h(t)mg(y))] \\
=\ & [h(1-t)]^2f(x)g(x)+mh(t)h(1-t)[f(x)g(y)+f(y)g(x)] \\
& +m^2[h(t)]^2f(y)g(y) \\
\leq\ & [h(1-t)]^2f(x)g(x)+mh(t)h(1-t)[f(x)g(x)+f(y)g(y)] \\
& +m^2[h(t)]^2f(y)g(y) \\
=\ & [h(1-t)f(x)g(x)+mh(t)f(y)g(y)][h(1-t)+mh(t)] \\
\leq\ & h(1-t)f(x)g(x)+h(t)mf(y)g(y),
\end{aligned}$$

where we have used the fact that $h(1-t)+mh(t)\leq 1$. This shows that product of two similarly ordered harmonic $m$-preinvex functions is again a harmonic $m$-preinvex function. □

## 3. MAIN RESULTS

In this section, we obtain Hermite-Hadamard inequalities for harmonic $m$-preinvex function. Throughout this section, we take $\mathcal{I}_\eta=[a,a+\eta(mb,a)]$ unless otherwise specified, where $a<a+\eta(mb,a)$.

**Theorem 3.1.** *Let $f:\mathcal{I}_\eta\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic $m$-preinvex function, where $m\in(0,1]$. If $f\in L[a,a+\eta(mb,a)]$, then*

$$\frac{a(a+\eta(mb,a))}{\eta(mb,a)}\int_a^{a+\eta(mb,a)}\frac{f(x)}{x^2}\mathrm{d}x\leq[f(a)+mf(b)]\int_0^1h(t)\mathrm{d}t.$$

*Proof.* Let $f$ be harmonic $m$-preinvex function. Then we have

$$f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)\leq h(1-t)f(a)+h(t)mf(b).$$

Integrating over $t\in[0,1]$, we obtain

$$\int_0^1 f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)\mathrm{d}t\leq[f(a)+mf(b)]\int_0^1h(t)\mathrm{d}t.$$

This implies

$$\frac{a(a+\eta(mb,a))}{\eta(mb,a)}\int_a^{a+\eta(mb,a)}\frac{f(x)}{x^2}\mathrm{d}x\leq[f(a)+mf(b)]\int_0^1h(t)\mathrm{d}t,$$

which is the required result. □

**Corollary 3.1.** *If $\eta(my,x)=my-x$, then Theorem 3.1 reduces to:*

$$\frac{mab}{mb-a}\int_a^{mb}\frac{f(x)}{x^2}\mathrm{d}x\leq[f(a)+mf(b)]\int_0^1h(t)\mathrm{d}t,$$

**Corollary 3.2.** *Under the assumptions of Theorem 3.1 and $h(t) = t^s$, we have*

$$\frac{a(a+\eta(mb,a))}{\eta(mb,a)}\int_a^{a+\eta(mb,a)}\frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+mf(b)}{s+1}.$$

**Corollary 3.3.** *Under the assumptions of Theorem 3.1 and $h(t) = t^p(1-t)^q$, we have*

$$\frac{a(a+\eta(mb,a))}{\eta(mb,a)}\int_a^{a+\eta(mb,a)}\frac{f(x)}{x^2}\mathrm{d}x \leq [f(a)+mf(b)]\beta(p+1,q+1).$$

**Theorem 3.2.** *Let $f, g : \mathcal{I}_\eta \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be harmonic $m$-preinvex functions, where $m \in (0,1]$. If $f \in L[a, a+\eta(mb,a)]$, then*

$$\frac{a(a+\eta(mb,a))}{\eta(mb,a)}\int_a^{a+\eta(mb,a)}\frac{f(x)g(x)}{x^2}\mathrm{d}x \leq M(a,b),$$

*where*

$$\begin{aligned} M(a,b) &= \big[f(a)g(a)+mf(b)mg(b)\big]\int_0^1 [h(t)]^2\mathrm{d}t \\ &\quad +\big[f(a)mg(b)+mf(b)g(a)\big]\int_0^1 h(t)h(1-t)\mathrm{d}t. \end{aligned} \tag{3.1}$$

*Proof.* Let $f, g$ be harmonic $m$-preinvex functions, we have

$$\begin{aligned} f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right) &\leq h(1-t)f(a)+h(t)mf(b) \\ g\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right) &\leq h(1-t)g(a)+h(t)mg(b). \end{aligned}$$

Now consider

$$\begin{aligned} & f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)g\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right) \\ \leq\ & \big[h(1-t)f(a)+h(t)mf(b)\big]\big[h(1-t)g(a)+h(t)mg(b)\big] \\ =\ & [h(1-t)]^2f(a)g(a)+h(t)h(1-t)[f(a)mg(b)+mf(b)g(a)] \\ & +[h(t)]^2mf(b)mg(b) \end{aligned}$$

Integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)g\left(\frac{a(a+\eta(mb,a)}{a+(1-t)t\eta(mb,a)}\right)\mathrm{d}t\\
\leq\ & f(a)g(a)\int_0^1[h(1-t)]^2\mathrm{d}t\\
&+[f(a)mg(b)+mf(b)g(a)]\int_0^1 h(t)h(1-t)\mathrm{d}t\\
&+mf(b)mg(b)\int_0^1[h(t)]^2\mathrm{d}t\\
=\ & \big[f(a)g(a)+mf(b)mg(b)\big]\int_0^1[h(t)]^2\mathrm{d}t\\
&+\big[f(a)mg(b)+mf(b)g(a)\big]\int_0^1 h(t)h(1-t)\mathrm{d}t.
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{a(a+\eta(mb,a))}{\eta(mb,a)}\int_a^{a+\eta(mb,a)}\frac{f(x)g(x)}{x^2}\mathrm{d}x\\
\leq\ & \big[f(a)g(a)+mf(b)mg(b)\big]\int_0^1[h(t)]^2\mathrm{d}t\\
&+\big[f(a)mg(b)+mf(b)g(a)\big]\int_0^1 h(t)h(1-t)\mathrm{d}t,
\end{aligned}$$

which is the required result. □

**Corollary 3.4.** *If $\eta(my,x)=my-x$, then Theorem 3.2 reduces to:*

$$\begin{aligned}
&\frac{mab}{mb-a}\int_a^{mb}\frac{f(x)g(x)}{x^2}\mathrm{d}x\\
\leq\ & \big[f(a)g(a)+mf(b)mg(b)\big]\int_0^1[h(t)]^2\mathrm{d}t\\
&+\big[f(a)mg(b)+mf(b)g(a)\big]\int_0^1 h(t)h(1-t)\mathrm{d}t,
\end{aligned}$$

**Corollary 3.5.** *Under the assumptions of Theorem 3.2 and $h(t)=t^s$, we have*

$$\begin{aligned}
\frac{a(a+\eta(mb,a))}{\eta(mb,a)}\int_a^{a+\eta(mb,a)}\frac{f(x)g(x)}{x^2}\mathrm{d}x \leq\ & \frac{f(a)g(a)+mf(b)mg(b)}{2s+1}\\
&+\beta(s+1,s+1)[f(a)mg(b)+mf(b)g(a)].
\end{aligned}$$

**Corollary 3.6.** *Under the assumptions of Theorem 3.2 and $h(t)=t^p(1-t)^q$, we have*

$$\frac{a(a+\eta(mb,a))}{\eta(mb,a)}\int_a^{a+\eta(mb,a)}\frac{f(x)g(x)}{x^2}\mathrm{d}x\leq M(a,b),$$

*where*

$$\begin{aligned}
M(a,b)\ =\ & [f(a)g(a)+mf(b)mg(b)]\beta(2p+1,2q+1)\\
&+[f(a)mg(b)+mf(b)g(a)]\beta(p+q+1,p+q+1).
\end{aligned}$$

**Theorem 3.3.** *Let $f, g : \mathcal{I}_\eta \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic $m$-preinvex functions, where $m \in (0,1]$. If $f \in L[a, a+\eta(mb,a)]$, then*

$$\begin{aligned} & \left(\frac{a(a+\eta(mb,a))}{\eta(mb,a)}\right)^2 \int_{\frac{1}{a+\eta(mb,a)}}^{\frac{1}{a}} h\left(x - \frac{1}{a+\eta(mb,a)}\right)\left[f(a)g\left(\frac{1}{x}\right) + g(a)f\left(\frac{1}{x}\right)\right]\mathrm{d}x \\ & \left(\frac{a(a+\eta(mb,a))}{\eta(mb,a)}\right)^2 \int_{\frac{1}{a+\eta(mb,a)}}^{\frac{1}{a}} h\left(\frac{1}{a} - x\right)\left[mf(b)g\left(\frac{1}{x}\right) + mg(b)f\left(\frac{1}{x}\right)\right]\mathrm{d}x \\ \leq\quad & M(a,b) + \frac{a(a+\eta(mb,a))}{\eta(mb,a)} \int_a^{a+\eta(mb,a)} \frac{f(x)g(x)}{x^2}\mathrm{d}x, \end{aligned}$$

*where $M(a,b)$ is given by (3.1).*

*Proof.* Let $f, g$ be harmonic $m$-preinvex functions, we have

$$\begin{aligned} f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right) &\leq h(1-t)f(a) + h(t)mf(b) \\ g\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right) &\leq h(1-t)g(a) + h(t)mg(b). \end{aligned}$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0, (x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$\begin{aligned} & f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)\left[h(1-t)g(a) + h(t)mg(b)\right] \\ & + g\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)\left[h(1-t)f(a) + h(t)mf(b)\right] \\ \leq\quad & \left[h(1-t)f(a) + h(t)mf(b)\right]\left[h(1-t)g(a) + h(t)mg(b)\right] \\ & + f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)g\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right) \end{aligned}$$

Thus

$$\begin{aligned} & g(a)h(1-t)f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right) + mg(b)h(t)f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right) \\ & + f(a)h(1-t)g\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right) + mf(b))h(t)g\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right) \\ \leq\quad & [h(1-t)]^2 f(a)g(a) + h(t)h(1-t)[f(a)mg(b) + mf(b)g(a)] \\ & + m^2[h(t)]^2 f(b)g(b) \\ & + f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)g\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right) \end{aligned}$$

Integrating the above inequality with respect to $t$ over $[0,1]$, we have

$$\begin{aligned}
& g(a)\int_0^1 h(1-t)f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)\mathrm{d}t \\
& +mg(b)\int_0^1 h(t)f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)\mathrm{d}t \\
& +f(a)\int_0^1 h(1-t)g\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)\mathrm{d}t \\
& +mf(b)\int_0^1 h(t)g\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)\mathrm{d}t \\
\leq\ & \big[f(a)g(a)+mmf(b)g(b)\big]\int_0^1 [h(t)]^2\mathrm{d}t \\
& +\big[f(a)mg(b)+mf(b)g(a)\big]\int_0^1 h(t)h(1-t)\mathrm{d}t \\
& +\int_0^1 f\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)g\left(\frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}\right)\mathrm{d}t
\end{aligned}$$

This implies

$$\begin{aligned}
& \left(\frac{a(a+\eta(mb,a))}{\eta(mb,a)}\right)^2\int_{\frac{1}{a+\eta(mb,a)}}^{\frac{1}{a}} h\left(x-\frac{1}{a+\eta(mb,a)}\right)\left[f(a)g\left(\frac{1}{x}\right)+g(a)f\left(\frac{1}{x}\right)\right]\mathrm{d}x \\
& \left(\frac{a(a+\eta(mb,a))}{\eta(mb,a)}\right)^2\int_{\frac{1}{a+\eta(mb,a)}}^{\frac{1}{a}} h\left(\frac{1}{a}-x\right)m\left[f(b)g\left(\frac{1}{x}\right)+g(b)f\left(\frac{1}{x}\right)\right]\mathrm{d}x \\
\leq\ & M(a,b)+\frac{a(a+\eta(mb,a))}{\eta(mb,a)}\int_a^{a+\eta(mb,a)}\frac{f(x)g(x)}{x^2}\mathrm{d}x,
\end{aligned}$$

which is the required result. □

**Corollary 3.7.** *If $\eta(my,x)=my-x$, then Theorem 3.3 reduces to:*

$$\begin{aligned}
& \left(\frac{mab}{mb-a}\right)^2\int_{\frac{1}{mb}}^{\frac{1}{a}} h\left(x-\frac{1}{mb}\right)\left[f(a)g\left(\frac{1}{x}\right)+g(a)f\left(\frac{1}{x}\right)\right]\mathrm{d}x \\
& \left(\frac{mab}{mb-a}\right)^2\int_{\frac{1}{mb}}^{\frac{1}{a}} h\left(\frac{1}{a}-x\right)m\left[f(b)g\left(\frac{1}{x}\right)+g(b)f\left(\frac{1}{x}\right)\right]\mathrm{d}x \\
\leq\ & M(a,b)+\frac{mab}{mb-a}\int_a^{mb}\frac{f(x)g(x)}{x^2}\mathrm{d}x,
\end{aligned}$$

**Lemma 3.1.** *Let $f:\mathcal{I}_\eta\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic $m$-preinvex function, where $m\in(0,1]$. Then*

$$f\left(\frac{a(a+\eta(mb,a)x}{(2a+\eta(mb,a))x-a(a+\eta(mb,a))}\right)\leq\big[f(a)+mf(b)\big][h(1-t)+h(t)]-f(x).$$

*Proof.* As we know that $x \in [a, a+\eta(mb,a)]$, can be represented as $x = \frac{a(a+\eta(mb,a))}{a+(1-t)\eta(mb,a)}$, $\forall t \in [0,1]$. Thus

$$\begin{aligned}
& f\left(\frac{a(a+\eta(mb,a)x}{(2a+\eta(mb,a))x - a(a+\eta(mb,a))}\right) \\
= \;& f\left(\frac{a(a+\eta(mb,a)}{a+t\eta(mb,a)}\right) \\
\leq \;& h(t)f(a) + mh(1-t)f(b) \\
= \;& h(1-t)\big[f(a)+mf(b)\big] + h(t)[f(a)+mf(b)] \\
& -[h(1-t)f(a) + h(t)mf(b)] \\
\leq \;& \big[f(a)+mf(b)\big][h(1-t)+h(t)] - f(x).
\end{aligned}$$

□

**Theorem 3.4.** *Let $f : \mathcal{I}_\eta \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic $m$-preinvex function, where $m \in (0,1]$. If $f \in L[a, a+\eta(mb,a)]$, then*

$$\begin{aligned}
& \frac{1}{2h(\frac{1}{2})} f\left(\frac{2a(a+\eta(mb,a))}{2a+\eta(mb,a)}\right) \int_a^{a+\eta(mb,a)} \frac{g(x)}{x^2} \mathrm{d}x \\
\leq \;& \frac{1}{2} \int_a^{a+\eta(mb,a)} \frac{(m+1)f(x)g(x)}{x^2} \mathrm{d}x \\
\leq \;& \frac{[f(a)+mf(b)]}{2} \int_a^b \left[h\left(\frac{(a+\eta(mb,a))(x-a)}{x(\eta(mb,a))}\right)\right. \\
& \left. + h\left(\frac{a((a+\eta(mb,a))-x)}{x(\eta(mb,a))}\right)\right] \frac{g(x)}{x^2} \mathrm{d}x + \frac{m-1}{2} \int_a^{a+\eta(mb,a)} \frac{f(x)g(x)}{x^2} \mathrm{d}x,
\end{aligned}$$

*where $g : [a, a+\eta(mb,a)] \subset \mathbb{R} \setminus \{0\}$ is nonnegative, integrable and satisfies*

$$g(x) = g\left(\frac{a(a+\eta(mb,a))x}{[2a+\eta(mb,a)]x - a(a+\eta(mb,a))}\right),$$

*for all $x \in [a, a+\eta(mb,a)]$.*

*Proof.* Using the given fact, we have

$$\begin{aligned}
& \frac{1}{2h(\frac{1}{2})} f\left(\frac{2a(a+\eta(mb,a))}{2a+\eta(mb,a)}\right) \int_a^{a+\eta(mb,a)} \frac{g(x)}{x^2} \mathrm{d}x \\
= \;& \frac{1}{2h(\frac{1}{2})} \int_a^b f\left(\frac{2a(a+\eta(mb,a))x}{(2a+\eta(mb,a))x - a(a+\eta(mb,a)) + a(a+\eta(mb,a))}\right) \frac{g(x)}{x^2} \mathrm{d}x \\
\leq \;& \frac{1}{2h(\frac{1}{2})} \int_a^b h\left(\frac{1}{2}\right)\left[f\left(\frac{a(a+\eta(mb,a))x}{(2a+\eta(mb,a))x - a(a+\eta(mb,a))}\right) + mf(x)\right] \frac{g(x)}{x^2} \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_a^{a+\eta(mb,a)} f\left(\frac{a(a+\eta(mb,a))x}{(2a+\eta(mb,a))x-a(a+\eta(mb,a))}\right)\frac{g(x)}{x^2}\mathrm{d}x \\
&\quad +\frac{m}{2}\int_a^{a+\eta(mb,a)} \frac{f(x)g(x)}{x^2}\mathrm{d}x \\
&= \frac{1}{2}\int_a^{a+\eta(mb,a)} \frac{(m+1)f(x)g(x)}{x^2}\mathrm{d}x
\end{aligned}
$$

To prove the other part of the inequality, we consider

$$
\begin{aligned}
&\frac{1}{2}\int_a^{a+\eta(mb,a)} \frac{(m+1)f(x)g(x)}{x^2}\mathrm{d}x \\
&= \frac{1}{2}\int_a^{b} f\left(\frac{a(a+\eta(mb,a))x}{(2a+\eta(mb,a))x-a(a+\eta(mb,a))}\right)\frac{g(x)}{x^2}\mathrm{d}x \\
&\quad +\frac{m}{2}\int_a^{a+\eta(mb,a)} \frac{f(x)g(x)}{x^2}\mathrm{d}x \\
&\leq \frac{1}{2}\int_a^{a+\eta(mb,a)} \left[\left[f(a)+mf(b)\right]\left[h(1-t)+h(t)\right]-f(x)\right]\frac{g(x)}{x^2}\mathrm{d}x \\
&\quad +\frac{m}{2}\int_a^{a+\eta(mb,a)} \frac{f(x)g(x)}{x^2}\mathrm{d}x \\
&\leq \frac{[f(a)+mf(b)]}{2}\int_a^{b}\left[h\left(\frac{(a+\eta(mb,a))(x-a)}{x(\eta(mb,a))}\right)\right. \\
&\quad \left.+h\left(\frac{a((a+\eta(mb,a))-x)}{x(\eta(mb,a))}\right)\right]\frac{g(x)}{x^2}\mathrm{d}x+\frac{m-1}{2}\int_a^{a+\eta(mb,a)} \frac{f(x)g(x)}{x^2}\mathrm{d}x.
\end{aligned}
$$

This completes the proof. □

**Corollary 3.8.** *If $\eta(my,x)=my-x$, then Theorem 3.4 reduces to:*

$$
\begin{aligned}
&\frac{1}{2h(\frac{1}{2})}f\left(\frac{2mab}{a+mb}\right)\int_a^{mb}\frac{g(x)}{x^2}\mathrm{d}x \\
&\leq \frac{1}{2}\int_a^{mb}\frac{(m+1)f(x)g(x)}{x^2}\mathrm{d}x \\
&\leq \frac{[f(a)+mf(b)]}{2}\int_a^{b}\left[h\left(\frac{mb(x-a)}{x(mb-a)}\right)\right. \\
&\quad \left.+h\left(\frac{a(mb-x)}{x(mb-a)}\right)\right]\frac{g(x)}{x^2}\mathrm{d}x+\frac{m-1}{2}\int_a^{mb}\frac{f(x)g(x)}{x^2}\mathrm{d}x.
\end{aligned}
$$

**Corollary 3.9.** *Under the assumptions of Theorem 3.4 with $g(x)=1$, we have*

$$
\begin{aligned}
\frac{1}{2h(\frac{1}{2})}f\left(\frac{2a(a+\eta(mb,a))}{2a+\eta(mb,a)}\right) &\leq \frac{a(a+\eta(mb,a))}{\eta(mb,a)}\int_a^{a+\eta(mb,a)}\frac{f(x)}{x^2}\mathrm{d}x \\
&\leq [f(a)+mf(b)]\int_0^1 h(t)\mathrm{d}t.
\end{aligned}
$$

## 4. CONCLUSION

In this paper we have introduced and studied a new class of harmonic preinvex functions with respect to an arbitrary non-negative function $h$ and the parameter $m$. It is shown that this class of harmonic $m$-preinvex functions is quite general, flexible and unifying one. New Hermite-Hadamard type inequalities are obtained. Special cases of the main results are discussed.

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