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ON DUAL CURVES OF $DAW(k)-$TYPE AND THEIR EVOLUTES

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Abstract. In this paper, we study to express the theory of curves including a wide section of Euclidean geometry in terms of dual vector calculus which has an important place in the three -dimensional dual space $\mathbb{D}^3$. In other words, we study $DAW(k)$-type curves $(1 \leq k \leq 3)$ by using Bishop frame defined as alternatively of these curves and give some of their properties in $\mathbb{D}^3$. Moreover, we define the notion of evolutes of dual spherical curves for ruled surfaces. Finally, we give some examples to illustrate our findings.

1. Introduction

The analytical tools in the study of 3-dimensional kinematics and differential geometry of ruled surfaces are based on dual vector calculus. Dual numbers were introduced in the 19th century by Clifford as a tool for his geometrical investigations. In addition, their applications to rigid body kinematics were generalized by Study in their principal of transference. The application of dual numbers to the lines of 3-space is carried out by the principle of transference which has been formulated by E. Study [1]. It allows a complete generalization of the mathematical expression for the spherical point geometry to the spatial line geometry by means of dual-number extension, i.e., replacing all ordinary quantities by the corresponding dual-number quantities.

In other words, in the Euclidean 3-space $\mathbb{E}^3$, lines combined with one of their two directions can be represented

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by unit dual vectors over the ring of dual numbers. The most important properties of real vector analysis are valid for the dual vectors and the oriented lines in $\mathbb{E}^3$ are in one-to-one correspondence with the points of a dual unit sphere.

A dual point in dual space $\mathbb{D}^3$ corresponds to an oriented line in $\mathbb{E}^3$ and two different points in $\mathbb{D}^3$ represent two skew-lines in $\mathbb{E}^3$ in general. A differentiable curve on dual unit sphere in $\mathbb{D}^3$ represents a ruled surface in $\mathbb{E}^3$ [2]. Ruled surfaces are those surfaces which are generated by moving a straight line continuously in the space [3]. In the light of this correspondence, dual spherical motion, expressed with the help of dual unit vectors, is closely analogous to real spherical motion, expressed with the help of real unit vectors. Therefore, the properties of elementary real spherical motion can also be carried over by analogy into the motion of lines in $\mathbb{E}^3$.

The notion of $AW(k)-$ type submanifolds was defined by K. Arslan and A. West [4]. After that, a lot of work related to curves of $AW(k)-$ type has been done (see for example [5–7]). In [8], the authors studied $DAW(k)-$ type curves on the dual unit sphere using Frenet frame.

In this paper, we investigate the $DAW(k)-$type curves and give the curvature conditions of these curves using a Bishop frame which has many properties that make it ideal for mathematical research. It also has applications in the area of biology and computer graphics, for example it may be possible to compute information about the shape of sequences of DNA using a curve defined by Bishop frame. Also, it may provide a new way to control virtual cameras in computer animations.

2. Fundamental concepts

In this section, we briefly give the mathematical formulations and basic concepts that are needed in our study.

2.1. **Bishop frame.** Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in the Euclidean 3-space $\mathbb{E}^3$ (i.e. its tangent vectors are normed over the whole parameter interval I). Therefore, the Frenet equations along α are defined as follows [9–11]:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{2.1}$$

where $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is the moving Frenet frame along α and the functions $\kappa(s)$ and $\tau(s)$ are respectively, the curvature and torsion of α.

The Bishop frame of 1-type or parallel transport frame is an alternative approach to define a moving frame that is well defined even when the curve has vanishing second derivative. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. To define the Frenet frame, curvature and torsion of a curve, this curve needs to be 3-times continuously differentiable non-degenerate. But the curvature function may vanish at some points on the curve, i.e., second derivative of the curve may be zero at some parameter values. In this situation, we need an alternative frame in $\mathbb{E}^3$.

We can parallel transport an orthonormal frame along $\alpha(s)$ simply by parallel transporting each component of the frame. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. This frame is denoted by $\{\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s)\}$, and its derivative is expressed using matrix coefficients k_1 and k_2.

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}_1' \\ \mathbf{N}_2' \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{pmatrix}. \tag{2.2}$$

Thereby the relation between Frenet and Bishop frames are given as follows

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{pmatrix}, \tag{2.3}$$

where

$$\begin{aligned} \theta(s) &= \arctan(\frac{k_2}{k_1}); k_1 \neq 0, \quad \tau(s) = \frac{d\theta(s)}{ds}, \quad \kappa(s) = \sqrt{k_1^2 + k_2^2}, \\ k_1 &= \kappa\cos\theta \ \textit{and} \ \ k_2 = \kappa\sin\theta, \end{aligned} \tag{2.4}$$

and k_1, k_2 are called the first and second Bishop curvatures and effectively correspond to a cartesian coordinate system for the polar coordinates κ, θ.

2.2. **Dual space.** Here, we give the notions of dual space and dual spherical curves of ruled surfaces. For more detailed descriptions, see [[12–16]].

Let a and a^* be two real numbers and $\varepsilon \neq 0$, $\varepsilon^2 = 0$. A dual number $\hat{a}$ is an ordered pair of the form $(a,\ a^*)$ for all $a,\ a^* \in \mathbb{R}$. Let $\mathbb{R} \times \mathbb{R}$ be a set denoted as $\mathbb{D}$, where

$$\mathbb{D} = \hat{a} = a + \varepsilon a^* : a,\ a^* \in \mathbb{R}, \tag{2.5}$$

and the dual numbers form a ring over the real number field.

Two inner operations and an equality on $\mathbb{D}$ are defined as follows:

(1) $\oplus : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ for $\hat{a} = (a,\ a^*), \hat{b} = (b,\ b^*)$ defined as $\hat{a} \oplus \hat{b} = (a +\ b) + \varepsilon(\ a^* +\ b^*)$, is called the addition in $\mathbb{D}$.

(2) $\odot : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ for $\hat{a} = (a,\ a^*), \hat{b} = (b,\ b^*)$ defined as $\hat{a} \odot \hat{b} = ab + \varepsilon(\ a^*b + ab^*)$, is called the multiplication in $\mathbb{D}$.

(3) For the equality of $\hat{a}$ and $\hat{b}$ we have $\hat{a} = \hat{b} \Leftrightarrow a = b$, and $\ a^* = b^*$.

The dual number $\hat{a} = a + \varepsilon a^*$ divided by the dual number $\hat{b} = b + \varepsilon b^*$ provided $b \neq 0$ can be defined as

$$\frac{\hat{a}}{\hat{b}} = \frac{a + \varepsilon\ a^*}{b + \varepsilon b^*} = \frac{a}{b} + \varepsilon \frac{a^* b - b^* a}{b^2}. \tag{2.6}$$

Obviously, the set $\mathbb{D}$ with the operations of addition, multiplication and equality on $\mathbb{D} = \mathbb{R} \times \mathbb{R}$ is a ring with non trivial zero devisors. In a dual number $\hat{a} = (a, a^*) \in \mathbb{D}$, the real number a is called the *real part* of $\hat{a}$ and the real number a^* is called the *dual part* of $\hat{a}$. The dual number $(1, 0) = 1$ is called a real unit in $\mathbb{D}$ and the dual number $(0, 1)$ is to be denoted with ε in short, and is called *dual unit* [9].

The set of

$$\begin{aligned} \mathbb{D}^3 &= \mathbb{D} \times \mathbb{D} \times \mathbb{D} = \{\hat{\mathbf{a}} : \hat{\mathbf{a}} = \mathbf{a} + \varepsilon \mathbf{a}^*, \mathbf{a}, \mathbf{a}^* \in \mathbb{R}^3\}; \\ \mathbf{a} &= (a_1, a_2, a_3), \mathbf{a}^* = (a_1^*, a_2^*, a_3^*) \end{aligned} \tag{2.7}$$

is a module over the ring $\mathbb{D}$.

For any $\hat{\mathbf{a}}, \hat{\mathbf{b}} \in \mathbb{D}^3$, the inner product and the vector product are defined as follows:

$$< \hat{\mathbf{a}}, \hat{\mathbf{b}} > = < \mathbf{a}, \mathbf{b} > + \varepsilon(< \mathbf{a}, \mathbf{b}^* > + < \mathbf{a}^*, \mathbf{b} >),$$

$$\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \mathbf{a} \times \mathbf{b} + \varepsilon(\mathbf{a} \times \mathbf{b}^* + \mathbf{a}^* \times \mathbf{b}), \tag{2.8}$$

respectively. If $\mathbf{a} \neq 0$, then the norm is defined by

$$\|\hat{\mathbf{a}}\| = \sqrt{< \hat{\mathbf{a}}, \hat{\mathbf{a}} >} = \|\mathbf{a}\| + \varepsilon \frac{< \mathbf{a}, \mathbf{a}^* >}{\|\mathbf{a}\|}. \tag{2.9}$$

A dual vector $\hat{\mathbf{a}}$ with norm 1 is called a *dual unit vector*. Let $\hat{\mathbf{a}} = \mathbf{a} + \varepsilon \mathbf{a}^* \in \mathbb{D}^3$, the set

$$\hat{S}^2 = \{\hat{\mathbf{a}} = \mathbf{a} + \varepsilon \mathbf{a}^* : \|\hat{\mathbf{a}}\| = (1, 0);\ \mathbf{a}, \mathbf{a}^* \in \mathbb{R}^3\}, \tag{2.10}$$

is called *the dual unit sphere* with center $\hat{O}$ in $\mathbb{D}^3$. Via this we have the following map (E. Study's map, c.f. Figure 1)*:* The set of all oriented lines in Euclidean space $\mathbb{E}^3$ is in one-to-one correspondence with set of points of dual unit sphere in $\mathbb{D}^3-$space [13, 14, 16].

Dual function of dual number presents a mapping of a dual number space on itself. Properties of dual functions were thoroughly investigated by Dimentberg [17]. He derived the general expression for dual analytic (differentiable) function as follows:

$$f(\hat{\mathbf{a}}) = f(\mathbf{a} + \varepsilon \mathbf{a}^*) = f(\mathbf{a}) + \varepsilon \mathbf{a}^*\ f'(\mathbf{a}),$$

where $f'(\mathbf{a})$ is the derivative of $f(\mathbf{a})$ and $\mathbf{a}, \mathbf{a}^* \in \mathbb{R}$. This definition allows us to write the dual forms of some well-known function as follows:

$$\begin{cases} \sin(\hat{\mathbf{a}}) = \sin(\mathbf{a} + \varepsilon \mathbf{a}^*) = \sin(\mathbf{a}) + \varepsilon \mathbf{a}^* \cos(\mathbf{a}), \\ \cos(\hat{\mathbf{a}}) = \cos(\mathbf{a} + \varepsilon \mathbf{a}^*) = \cos(\mathbf{a}) - \varepsilon \mathbf{a}^* \sin(\mathbf{a}), \\ \sqrt{\hat{\mathbf{a}}} = \sqrt{\mathbf{a} + \varepsilon \mathbf{a}^*} = \sqrt{\mathbf{a}} + \varepsilon \frac{\mathbf{a}^*}{2\sqrt{\mathbf{a}}}\ , (\mathbf{a} > 0). \end{cases}$$

The E. Study's map allows us to write a ruled surface by a dual vector function. So, ruled surfaces and dual curves are synonymous in this paper.

Consider the dual Serret–Frenet frame $\{\hat{\mathbf{T}}(\hat{s}),\ \hat{\mathbf{N}}(\hat{s}),\ \hat{\mathbf{B}}(\hat{s})\}$ associated with a dual curve $\hat{\alpha}(\hat{s})$, then the Serret–Frenet formulae read:

$$\begin{pmatrix} \hat{\mathbf{T}}' \\ \hat{\mathbf{N}}' \\ \hat{\mathbf{B}}' \end{pmatrix} = \begin{pmatrix} 0 & \hat{k} & 0 \\ -\hat{k} & 0 & \hat{\tau} \\ 0 & -\hat{\tau} & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{N}} \\ \hat{\mathbf{B}} \end{pmatrix}, \left(' = \frac{d}{d\hat{s}} \right), \tag{2.11}$$

where $\hat{k} = \hat{k}(\hat{s})$ and $\hat{\tau} = \hat{\tau}(\hat{s})$ are called the dual curvature function and the dual torsion function, respectively.

3. $DAW(k)-$type curves

In this section, we consider $AW(k)-$type curves in the dual space $\mathbb{D}^3$ and denote this type of curves by $DAW(k)-$type. For this purpose, let $\{\hat{\mathbf{T}}, \hat{\mathbf{N}}_1, \hat{\mathbf{N}}_2\}$ be a dual Bishop frame of $\hat{\alpha}(\hat{s})$. Then the Bishop formulas of $\hat{\alpha}$ are given by

$$\begin{pmatrix} \hat{\mathbf{T}}' \\ \hat{\mathbf{N}}_1' \\ \hat{\mathbf{N}}_2' \end{pmatrix} = \begin{pmatrix} 0 & \hat{k}_1 & \hat{k}_2 \\ -\hat{k}_1 & 0 & 0 \\ -\hat{k}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{N}}_1 \\ \hat{\mathbf{N}}_2 \end{pmatrix}, \tag{3.1}$$

where

$$\begin{aligned} \hat{\mathbf{T}} &= \mathbf{T} + \varepsilon \mathbf{T}^*,\ \hat{\mathbf{N}}_1 = \mathbf{N}_1 + \varepsilon \mathbf{N}_1^*\ ,\ \hat{\mathbf{N}}_2 = \mathbf{N}_2 + \varepsilon \mathbf{N}_2^*, \\ \hat{k}_1 &= k_1 + \varepsilon k_1^* \text{ and } \hat{k}_2 = k_2 + \varepsilon k_2^*. \end{aligned}$$

Proposition 3.1. *Let $\hat{\alpha}$ be a unit speed dual curve with arc length parameter $\hat{s}$ and $\{\hat{\mathbf{T}}, \hat{\mathbf{N}}_1, \hat{\mathbf{N}}_2\}$ be its Bishop frame. There within follows for its derivatives*

$$\begin{aligned} \hat{\alpha}^{`} &= \hat{\mathbf{T}}, \\ \hat{\alpha}^{``} &= \hat{k}_1 \hat{\mathbf{N}}_1 + \hat{k}_2 \hat{\mathbf{N}}_2, \\ \hat{\alpha}^{```} &= -\left(\hat{k}_1^2 + \hat{k}_2^2\right) \hat{\mathbf{T}} + \hat{k}_1^{`} \hat{\mathbf{N}}_1 + \hat{k}_2^{`} \hat{\mathbf{N}}_2, \\ \hat{\alpha}^{````} &= -\left(\left(\hat{k}_1^2 + \hat{k}_2^2\right)^{`} + \hat{k}_1 \hat{k}_1^{`} + \hat{k}_2 \hat{k}_2^{`}\right) \hat{\mathbf{T}} \\ &\quad + \left(\hat{k}_1^{``} - \hat{k}_1^3 - \hat{k}_1 \hat{k}_2^2\right) \hat{\mathbf{N}}_1 \\ &\quad + \left(\hat{k}_2^{``} - \hat{k}_2^3 - \hat{k}_1^2 \hat{k}_2\right) \hat{\mathbf{N}}_2. \end{aligned} \tag{3.2}$$

Notation 3.1. *From proposition* 3.1*, we can write*

$$\begin{aligned}\hat{\mathbf{V}}_1 &= \hat{k}_1\hat{\mathbf{N}}_1+\hat{k}_2\hat{\mathbf{N}}_2,\\ \hat{\mathbf{V}}_2 &= \hat{k}_1'\hat{\mathbf{N}}_1+\hat{k}_2'\hat{\mathbf{N}}_2,\\ \hat{\mathbf{V}}_3 &= \left(\hat{k}_1''-\hat{k}_1^3-\hat{k}_1\hat{k}_2^2\right)\hat{\mathbf{N}}_1\\ &\quad+\left(\hat{k}_2''-\hat{k}_2^3-\hat{k}_1^2\hat{k}_2\right)\hat{\mathbf{N}}_2.\end{aligned}\tag{3.3}$$

Definition 3.1. *[4] The unit speed dual curves of osculating order* 3 *are*

(i): *of Bishop $DAW(1)-$type if and only if*

$$\hat{\mathbf{V}}_3=0,$$

(ii): *of Bishop $DAW(2)-$type if and only if*

$$\left\|\hat{\mathbf{V}}_2\right\|^2\hat{\mathbf{V}}_3=\left\langle\hat{\mathbf{V}}_3,\hat{\mathbf{V}}_2\right\rangle\hat{\mathbf{V}}_2,$$

(iii): *of Bishop $DAW(3)-$type if and only if*

$$\left\|\hat{\mathbf{V}}_1\right\|^2\hat{\mathbf{V}}_3=\left\langle\hat{\mathbf{V}}_3,\hat{\mathbf{V}}_1\right\rangle\hat{\mathbf{V}}_1.\tag{3.4}$$

Proposition 3.2. *The unit speed dual curve is of Bishop $DAW(1)-$type if and only if Bishop curvature equations*

$$k_1''-k_1^3-k_1k_2^2=0,\ \ k_2''-k_2^3-k_1^2k_2=0,$$

$$k_1^{*''}-3k_1^2k_1^*-k_2^2k_1^*-2k_1k_2k_2^*=0,\ \ k_2^{*''}-3k_2^2k_2^*-k_1^2k_2^*-2k_1k_2k_1^*=0,\tag{3.5}$$

hold.

Proof. By the aid of definition 3.1 and notation 3.1, one can obtain

$$\left(\hat{k}_1''-\hat{k}_1^3-\hat{k}_1\hat{k}_2^2\right)\hat{\mathbf{N}}_1+\left(\hat{k}_2''-\hat{k}_2^3-\hat{k}_1^2\hat{k}_2\right)\hat{\mathbf{N}}_2=0.\tag{3.6}$$

Since $\hat{\mathbf{N}}_1$ and $\hat{\mathbf{N}}_2$ are linearly independent, then

$$\begin{aligned}\hat{k}_1''-\hat{k}_1^3-\hat{k}_1\hat{k}_2^2 &= 0,\\ \hat{k}_2''-\hat{k}_2^3-\hat{k}_1^2\hat{k}_2 &= 0.\end{aligned}\tag{3.7}$$

By separating Eqs. (3.7) into real and dual parts we obtain the desired equations. □

Proposition 3.3. *The unit speed dual curve is of Bishop $DAW(2)-$type if and only if the Bishop curvature equations*

$$\left(k_1^{\prime\prime} - k_1^3 - k_1 k_2^2\right) k_2^{\prime} = \left(k_2^{\prime\prime} - k_2^3 - k_1^2 k_2\right) k_1^{\prime},$$

$$\begin{aligned} & \left(k_1^{\prime\prime} - k_1^3 - k_1 k_2^2\right) k_2^{*\prime} + \left(k_1^{*\prime\prime} - 3k_1^2 k_1^* - k_1^* k_2^2 - 2k_1 k_2 k_1^*\right) k_2^{\prime} \\ = \; & \left(k_2^{\prime\prime} - k_2^3 - k_1^2 k_1\right) k_1^{*\prime} + \left(k_2^{*\prime\prime} - 3k_2^2 k_2^* - 2k_1 k_1^* k_2 - k_1^2 k_2^*\right) k_1^{\prime}. \end{aligned} \tag{3.8}$$

hold.

Proof. In the light of definition 3.1 and notation 3.1, it is easy to get

$$\begin{aligned} & \left(\hat{k}_2^{\prime 2} \hat{k}_1^{\prime\prime} - \hat{k}_1^3 \hat{k}_2^{\prime 2} - \hat{k}_1 \hat{k}_2^4\right) \hat{\mathbf{N}}_1 + \left(\hat{k}_1^{\prime 2} \hat{k}_2^{\prime\prime} - \hat{k}_2^3 \hat{k}_1^{\prime 2} - \hat{k}_1^2 \hat{k}_2 \hat{k}_1^{\prime 2}\right) \hat{\mathbf{N}}_2 \\ = \; & \left(\hat{k}_1^{\prime} \hat{k}_2^{\prime\prime} \hat{k}_2^{\prime} - \hat{k}_2^3 \hat{k}_1^{\prime} \hat{k}_2^{\prime} - \hat{k}_1^2 \hat{k}_2 \hat{k}_1^{\prime} \hat{k}_2^{\prime}\right) \hat{\mathbf{N}}_1 + \left(\hat{k}_1^{\prime} \hat{k}_2^{\prime\prime} - \hat{k}_2^3 \hat{k}_1^{\prime} - \hat{k}_1^2 \hat{k}_2 \hat{k}_1^{\prime}\right) \hat{\mathbf{N}}_2, \end{aligned}$$

then

$$\left(\hat{k}_1^{\prime\prime} - \hat{k}_1^3 - \hat{k}_1 \hat{k}_2^2\right) \hat{k}_2^{\prime} = \left(\hat{k}_2^{\prime\prime} - \hat{k}_2^3 - \hat{k}_1^2 \hat{k}_2\right) \hat{k}_1^{\prime}. \tag{3.9}$$

Similarly, by separating Eq. (3.9) into real and dual parts, we get Eq. (3.8). □

Proposition 3.4. *The unit speed dual curve is of Bishop $DAW(3)-$type if and only if the Bishop curvature equations*

$$\left(k_1^{\prime\prime} - k_1^3 - k_1 k_2^2\right) k_2 = \left(k_2^{\prime\prime} - k_2^3 - k_1^2 k_2\right) k_1,$$

$$\begin{aligned} & \left(k_1^{\prime\prime} - k_1^3 - k_1 k_2^2\right) k_2^* + \left(k_1^{*\prime\prime} - 3k_1^2 k_1^* - k_1^* k_2^2 - 2k_1 k_2 k_2^*\right) k_2 \\ = \; & \left(k_2^{\prime\prime} - k_2^3 - k_1^2 k_1\right) k_1^* + \left(k_2^{*\prime\prime} - 3k_2^2 k_2^* - 2k_1 k_1^* k_2 - k_1^2 k_2^*\right) k_1. \end{aligned} \tag{3.10}$$

hold.

Proof. Using definition 3.1 and notation 3.1, we have

$$\begin{aligned} & \left(\hat{k}_1^2 \hat{k}_1^{\prime\prime} - \hat{k}_1^3 \hat{k}_2^2 - \hat{k}_1^5 + \hat{k}_1 \hat{k}_2 \hat{k}_2^{\prime\prime} - \hat{k}_2^4 \hat{k}_1 - \hat{k}_1^3 \hat{k}_2^2\right) \hat{\mathbf{N}}_1 + \left(\left(\hat{k}_2^{\prime\prime} - \hat{k}_2^3 - \hat{k}_1^2 \hat{k}_2\right)\left(\hat{k}_1^2 + \hat{k}_1^2\right)\right) \hat{\mathbf{N}}_2 \\ = \; & \left(\hat{k}_1^2 \hat{k}_1^{\prime\prime} - \hat{k}_1^5 - \hat{k}_1^3 \hat{k}_2^2 + \hat{k}_1^{\prime\prime} \hat{k}_2^2 - \hat{k}_1^3 \hat{k}_2^2 - \hat{k}_1 \hat{k}_2^4\right) \hat{\mathbf{N}}_1 \\ & + \left(\hat{k}_2^2 \left(\hat{k}_2^{\prime\prime} - \hat{k}_2^3 - \hat{k}_1^2 \hat{k}_2\right) + \hat{k}_1 \hat{k}_2 \left(\hat{k}_1^{\prime\prime} - \hat{k}_1^3 - \hat{k}_1 \hat{k}_2^2\right)\right) \hat{\mathbf{N}}_2, \end{aligned}$$

it follows that

$$\left\{\hat{k}_1^{\prime\prime} - \hat{k}_1^3 - \hat{k}_1 \hat{k}_2^2\right\} \hat{k}_2 = \left\{\hat{k}_2^{\prime\prime} - \hat{k}_2^3 - \hat{k}_1^2 \hat{k}_2\right\} \hat{k}_1. \tag{3.11}$$

Separating Eq. (3.11) into real and dual parts, we obtain Eq. (3.10). □

4. Evolutes of dual spherical curves for ruled surfaces

In this section, we give the notions of dual spherical curves of ruled surfaces as well as evolutes of these curves. For more detailed descriptions (see [12, 15, 16, 18, 19]). A ruled surface is a surface swept out by a straight line L moving along a curve $\beta = \beta(t)$. The various positions of the generating lines L are called the rulings of the surface. Such a surface, thus, has a parametrization in the ruled form

$$\Psi(t, v) = \beta(t) + v\alpha(t); v \in \mathbb{R}. \tag{4.1}$$

Here, $\beta = \beta(t)$ is called the base curve, $\alpha = \alpha(t)$ is the unit vector giving the direction of generating line, and t is the motion parameter. The base curve is not unique, because any curve of the form

$$\gamma(t) = \beta(t) + \eta(t)\alpha(t), \tag{4.2}$$

may be used as its base curve, $\eta(t)$ is a smooth function. If there exists a common perpendicular to two neighboring rulings on Ψ , then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the striction curve. In Eq. (4.2), if

$$\eta(t) = -\frac{\langle \beta'(t), \alpha'(t)\rangle}{\|\alpha'(t)\|^2}, \tag{4.3}$$

then $\gamma(t)$ is called the striction curve on the ruled surface Ψ, and it is unique. The base curve $\beta(t)$ of the ruled surface is its striction curve if and only if $\langle \beta'(t), \alpha'(t)\rangle = 0$.
While a differentiable curve on the dual unit sphere $\hat{S}^2$ corresponds to a ruled surface in $\mathbb{E}^3$. A differentiable curve $\hat{\gamma}$ on a dual unit sphere, depending on real parameter t, represents a differentiable family of straight lines in $\mathbb{E}^3$, which we call a ruled surface. The ruled surface Ψ is written as the dual vector function $\hat{\gamma}$ given by (according to the E. Study's dual-line coordinates)

$$\hat{\gamma}(t) = \mathbf{U}(t) = \alpha(t) + \varepsilon\gamma(t) \wedge \alpha(t) = \alpha(t) + \varepsilon\alpha^*(t), \tag{4.4}$$

where α^* is the moment of α about the origin in $\mathbb{E}^3$, and ε is an indeterminate subject to the relation $\varepsilon^2 = 0$. Hence, ruled surfaces and dual curves are synonymous in this work. Because $\langle \hat{\gamma}(t), \hat{\gamma}(t)\rangle = 1$, thus, the ruled surface can be represented by the dual curve on the surface of a dual unit sphere $\hat{S}^2$ (see Fig. 1 and Fig. 2). Then $\hat{\gamma}(t)$ is called the dual spherical curve of ruled surface Ψ.

Now, as in real spherical geometry, we define an orthonormal moving frame along this dual curve as follows [18]:

$$\mathbf{U}_1 = \mathbf{U}(t), \qquad \mathbf{U}_2(t) = \frac{\mathbf{U}_1'}{\| \mathbf{U}_1'\|}, \qquad \mathbf{U}_3(t) = \mathbf{U}_1 \wedge \mathbf{U}_2. \tag{4.5}$$

From now on we consider the case without $\alpha(t) =$ constant vector and $\alpha^*(t) = 0$. In the case $\alpha(t) =$ constant vector, the ruled surface $\Psi(t, v)$ is a cylinder and in the case $\alpha^*(t) = 0$, the ruled surface $\Psi(t, v)$ is a cone.

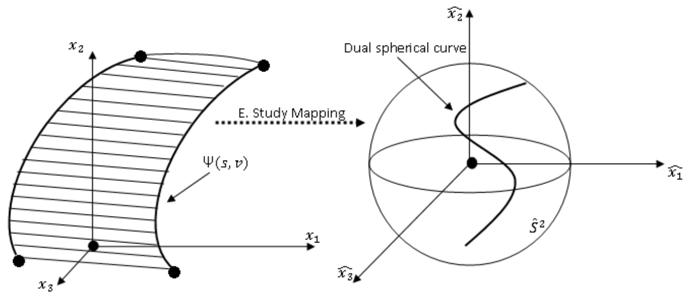


FIGURE 1. Ruled surface mapped to the dual spherical curve.

The dual unit vectors $\mathbf{U}_1, \mathbf{U}_2$ and $\mathbf{U}_3$ corresponds to three concurrent mutually orthogonal lines in $\mathbb{E}^3$. Their point of intersection is the central point on the ruling $\mathbf{U}_1$, $\mathbf{U}_3(t)$ is the limit position of the common perpendicular to $\mathbf{U}_1(t)$, and is called the central tangent of the ruled surface $\mathbf{U}_1 = \mathbf{U}(t)$ at the central point. The line $\mathbf{U}_2 = \mathbf{U}_2(t)$ is called the central normal of $\mathbf{U}_1 = \mathbf{U}(t)$ at the central point. Moreover, the dual planes which correspond to the subspaces $\mathrm{Sp}\{\mathbf{U}_1, \mathbf{U}_2\}$, $\mathrm{Sp}\{\mathbf{U}_3, \mathbf{U}_2\}$, and $\mathrm{Sp}\{\mathbf{U}_1, \mathbf{U}_3\}$, respectively, are called the tangent plane, asymptotic plane and normal plane. By construction, the Blaschke formula is

$$\begin{pmatrix} \mathbf{U}_1' \\ \mathbf{U}_2' \\ \mathbf{U}_3' \end{pmatrix} = \begin{pmatrix} 0 & P & 0 \\ -P & 0 & Q \\ 0 & -Q & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \end{pmatrix}, ' = \frac{d}{dt}, \tag{4.6}$$

where

$$\langle \mathbf{U}_2, \mathbf{U}_2 \rangle = 1 = \langle \mathbf{U}_3, \mathbf{U}_3 \rangle, \ \langle \mathbf{U}_1, \mathbf{U}_2 \rangle = 0,$$

and

$$P = p + \varepsilon p^* = \| \mathbf{U}_1' \|, \ Q = q + \varepsilon q^* = \frac{\det(\mathbf{U}_1, \mathbf{U}_1', \mathbf{U}_1'')}{\| \mathbf{U}_1' \|^3}, \tag{4.7}$$

are called the Blaschke's invariants of the dual curve $\mathbf{U}_1(t)$. One of the invariants of the dual curve $\mathbf{U}_1 = \mathbf{U}_1(t)$ is

$$\Sigma : \ = \frac{Q}{P}, \ P \neq 0, \tag{4.8}$$

which is well-known as the dual geodesic curvature in $\hat{S}^2$ [19,20]. Then, as in the case of real spherical curve, we may write for the dual curve $\mathbf{U}(t)$ the following formulas:

$$\mathcal{K} := \kappa + \varepsilon\kappa^* = \sqrt{1+\Sigma^2}, \mathcal{T} := \tau + \varepsilon\tau^* = \pm\frac{\Sigma'}{1+\Sigma^2}, \tag{4.9}$$

where $\mathcal{K} = \mathcal{K}(t)$ is the dual curvature, and $\mathcal{T} = \mathcal{T}(t)$ is the dual torsion of the dual curve $\mathbf{U} = \mathbf{U}(t)$. Due to [16], the evolute of the dual unit spherical curve $\hat{\gamma}$ is the locus of all its centers of geodesic curvature. So, it can be defined through the following form

$$\mathbf{E}_{\hat{\gamma}}(\hat{s}) = \frac{1}{\sqrt{1+\Sigma^2}}(\Sigma \mathbf{U}_1 + \ \mathbf{U}_3), \tag{4.10}$$

where $\hat{s}=\int\limits_{t_1}^{t}\|(\hat{\gamma}'(t))\|\,dt=s+\varepsilon s^*$ is the dual arc length of the curve $\hat{\gamma}(t)$ from t_1 to t.
Under the previous notations about dual spherical curves and their evolutes we can summarize the following results:

Corollary 4.1. *Let $\hat{\gamma}:I\subset\mathbb{D}\to\hat{S}^2$ be a dual regular unit spherical curve of a ruled surface, then $\hat{\gamma}$ and its osculating dual circle have a four-point contact at $\hat{\gamma}(\hat{s}_0)$ if and only if $\Sigma'(\hat{s})=0$ and $\Sigma''(\hat{s})\neq 0$.*

Corollary 4.2. *[16] The evolute of the dual unit spherical curve $\hat{\gamma}$ at $\hat{s}_0$ is diffeomorphic to the ordinary cusp if $\Sigma'(\hat{s}_0)=0$ and $\Sigma''(\hat{s}_0)\neq 0$. The ordinary cusp is $\hat{C}=\{(\hat{a}_1,\hat{a}_2)\,|\hat{a}_1^2=\hat{a}_2^3)\}$.*

Lemma 4.1. *The dual spherical curve $\hat{\gamma}(\hat{s})$ is a great circle if the dual geodesic curvature function $\Sigma(\hat{s})$ of $\hat{\gamma}$ is identically zero, and then the ruled surface is a right helicoid and the striction curve is a geodesic curve.*

5. Examples

Example 5.1. *Let $\hat{\alpha}$ be a dual curve in $\mathbb{D}^3$ defined by*

$$\begin{aligned}\hat{\alpha}\left(\hat{s}\right) &= \left(\sin\hat{s},\sin\hat{s}\ \cos\hat{s},\cos^2\hat{s}\right)\\ &= \left(\sin s,\sin s\ \cos s,\cos^2 s\right)+\varepsilon s^*\left(\cos s,\cos 2s,-\sin 2s\right);\hat{s}=s+\varepsilon s^*.\end{aligned}$$

The corresponding ruled surface is given by

$$\begin{aligned}r_\alpha(s,v) &= \alpha\wedge\alpha^*+v\alpha\\ &= \left(-\cos^2 s+v\ \sin s,\cos^3 s+\sin s\sin 2s+v\cos s\sin s,-\sin^3 s+v\ \cos^2 s\right).\end{aligned}$$

After some calculations, we obtain

$$\hat{\mathbf{T}}=\left(\frac{\sqrt{2}\cos\hat{s}}{\sqrt{3+\cos 2\hat{s}}},\frac{\sqrt{2}\cos 2\hat{s}}{\sqrt{3+\cos 2\hat{s}}},-\frac{\sqrt{2}\sin 2\hat{s}}{\sqrt{3+\cos 2\hat{s}}}\right),$$

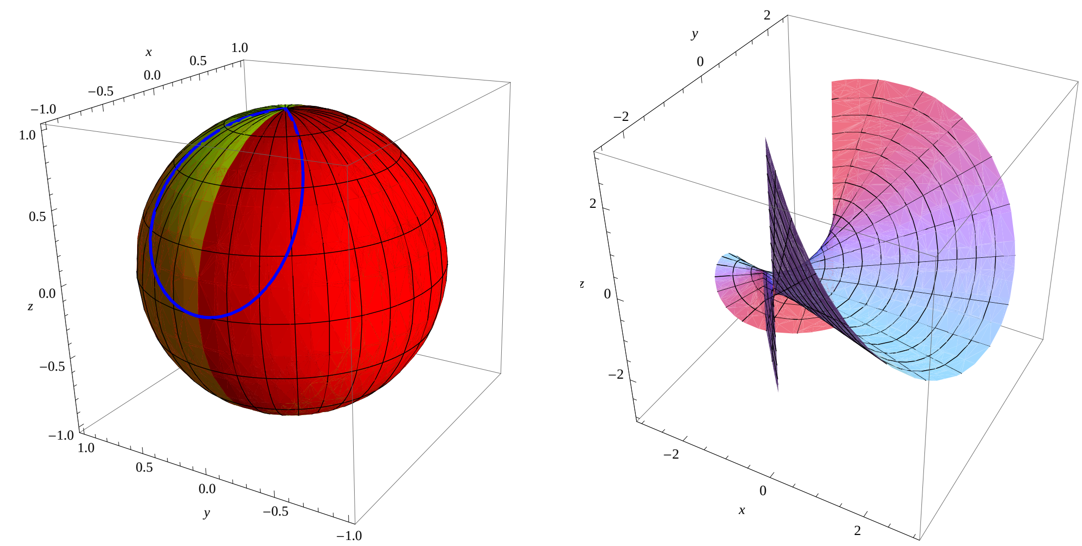
$$\hat{\mathbf{N}}=\left(-\frac{2\sin\hat{s}}{\sqrt{3+\cos 2\hat{s}}\sqrt{13+3\cos 2\hat{s}}},-\frac{12\sin 2\hat{s}+\sin 4\hat{s}}{2\sqrt{3+\cos 2\hat{s}}\sqrt{13+3\cos 2\hat{s}}},-\frac{4\left(\cos^4\hat{s}+\cos 2\hat{s}\right)}{\sqrt{3+\cos 2\hat{s}}\sqrt{13+3\cos 2\hat{s}}}\right),$$

$$\hat{\mathbf{B}}=\left(-\frac{2\sqrt{2}}{\sqrt{13+3\cos 2\hat{s}}},\frac{2\sqrt{2}\cos^3\hat{s}}{\sqrt{13+3\cos 2\hat{s}}},\frac{-3\sin\hat{s}-\sin 3\hat{s}}{\sqrt{26+6\cos 2\hat{s}}}\right).$$

And

$$\begin{aligned}\hat{\kappa} &= \frac{2\sqrt{13+3\cos 2\hat{s}}}{(3+\cos 2\hat{s})^{3/2}},\quad \hat{\tau}=-\frac{12\cos\hat{s}}{13+3\cos 2\hat{s}},\\ \hat{\theta} &= \int\hat{\tau}(\hat{s})d\hat{s}=\frac{1}{2}\sqrt{\frac{3}{2}}\ln\left(\frac{8\sqrt{6}-12\sin\hat{s}}{2\sqrt{6}+3\sin\hat{s}}\right),\end{aligned}$$

$$\hat{k}_1=\hat{\kappa}\cos\hat{\theta}=\frac{2\sqrt{13+3\cos 2\hat{s}}}{(3+\cos 2\hat{s})^{3/2}}\left(\cos\left[\frac{1}{2}\sqrt{\frac{3}{2}}\ln\left(\frac{8\sqrt{6}-12\sin\hat{s}}{2\sqrt{6}+3\sin\hat{s}}\right)\right]\right).$$



(A) (B)

FIGURE 2. The ruled surface corresponding to the dual unit spherical curve $\hat{\alpha}$.

$$\hat{k}_2 = \hat{\kappa}\sin\hat{\theta} = \frac{2\sqrt{13+3\cos 2\hat{s}}}{(3+\cos 2\hat{s})^{3/2}}\left(\sin\left[\frac{1}{2}\sqrt{\frac{3}{2}}\ln\left(\frac{8\sqrt{6}-12\sin\hat{s}}{2\sqrt{6}+3\sin\hat{s}}\right)\right]\right).$$

In the case that $\hat{s} = 3\pi/2$*, we get*

$$\hat{k}_1 = -0.34,\ \hat{k}_2 = 2.21,\ \hat{k}'_1(s) = 0,\ \hat{k}'_2(s) = 0,\ \hat{k}''_1(s) = 3.46,\ \hat{k}''_2(s) = -4.90.$$

According to Proposition 3.3*, the curve* $\hat{\alpha}$ *is of Bishop DAW*(2)*-type because*

$$\left(\hat{k}''_1(s) - \hat{k}^3_1(s) - \hat{k}_1(s)\,\hat{k}^2_2(s)\right)\hat{k}'_2(s) = \left(\hat{k}''_2(s) - \hat{k}^3_2(s) - \hat{k}^2_1(s)\,\hat{k}_2(s)\right)\hat{k}'_1(s) = 0.$$

Also, according to Propositions 3.2 *and* 3.4*,* $\hat{\alpha}$ *is neither of Bishop DAW*(1)*-type nor Bishop DAW*(3)*-type because*

$$\hat{k}''_1(s) - \hat{k}^3_1(s) - \hat{k}_1(s)\,\hat{k}^2_2(s) \neq 0, \quad \hat{k}''_2(s) - \hat{k}^3_2(s) - \hat{k}^2_1(s)\,\hat{k}_2(s) \neq 0,$$

and

$$\left\{\hat{k}''_1(s) - \hat{k}^3_1(s) - \hat{k}_1(s)\,\hat{k}^2_2(s)\right\}\hat{k}_2(s) \neq \left\{\hat{k}''_2(s) - \hat{k}^3_2(s) - \hat{k}^2_1(s)\,\hat{k}_2(s)\right\}\hat{k}_1(s).$$

Example 5.2. *Let* $\Psi(s,v)$ *be a ruled surface of* $\mathbb{E}^3$ *defined by [16]*

$$\Psi(s,v) = \gamma(s) + v\delta(s);\, v \in \mathbb{R},$$

where

$$\gamma(s) = \left(\frac{-2}{1+\cos^2 s}, \frac{2\cos^3 s}{1+\cos^2 s}, \frac{-\sin s - 2\sin s\ \cos^2 s}{1+\cos^2 s}\right)$$

and

$$\delta(s) = \left(\sin s, \sin s\ \cos s, \cos^2 s\right).$$

The ruled surface Ψ *is written as the dual vector function* $\hat{\gamma}$ *given by*

$$\hat{\gamma}(s) = \hat{\mathbf{U}}(s) = \delta(s) + \varepsilon\ \gamma(s) \wedge \delta\left(s\right),$$

which can be expressed as follows

$$\hat{\gamma}\left(\hat{s}\right) = \hat{\mathbf{U}}(\hat{s}) = \left(\sin\hat{s}, \sin\hat{s}\ \ \cos\hat{s}, \cos^2\hat{s}\right).$$

Now, we can write the orthonormal moving frame $\left\{\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2, \hat{\mathbf{U}}_3\right\}$ *along this dual curve as follows*

$$\hat{\mathbf{U}}_1 = \hat{\mathbf{U}}(\hat{s}), \qquad \hat{\mathbf{U}}_2(\hat{s}) = \frac{\hat{\mathbf{U}}_1'}{\left\|\hat{\mathbf{U}}_1'\right\|}, \qquad \hat{\mathbf{U}}_3(\hat{s}) = \hat{\mathbf{U}}_1 \wedge \hat{\mathbf{U}}_2.$$

$$\begin{aligned}
\hat{\mathbf{U}}_1 &= \left\{\sin\hat{s}, \cos\hat{s}\sin\hat{s}, \cos^2\hat{s}\right\}, \\
\hat{\mathbf{U}}_2 &= \left\{\frac{\sqrt{2}\cos\hat{s}}{\sqrt{3+\cos 2\hat{s}}}, \frac{\sqrt{2}\cos 2\hat{s}}{\sqrt{3+\cos 2\hat{s}}}, -\frac{\sqrt{2}\sin 2\hat{s}}{\sqrt{3+\cos 2\hat{s}}}\right\}, \\
\hat{\mathbf{U}}_3 &= \left\{-\frac{\sqrt{2}\cos^2\hat{s}}{\sqrt{3+\cos 2\hat{s}}}, -\frac{-5\cos\hat{s}+\cos 3\hat{s}}{2\sqrt{2}\sqrt{3+\cos 2\hat{s}}}, -\frac{\sqrt{2}\sin^3\hat{s}}{\sqrt{3+\cos 2\hat{s}}}\right\},
\end{aligned}$$

and

$$\begin{aligned}
P &= \left\|\hat{\mathbf{U}}_1'\right\| = \frac{\sqrt{3+\cos 2\hat{s}}}{\sqrt{2}}, \\
Q &= \frac{\det(\ \mathbf{U}_1,\ \mathbf{U}_1', \mathbf{U}_1'')}{\|\ \mathbf{U}_1'\|^3} = -\frac{\sqrt{2}(5+\cos 2\hat{s})\sin\hat{s}}{(3+\cos 2\hat{s})^{3/2}},
\end{aligned}$$

hence, the dual geodesic curvature

$$\hat{\Sigma} :\ = \frac{Q}{P} = -\frac{2(5+\cos 2\hat{s})\sin\hat{s}}{(3+\cos 2\hat{s})^2},$$

the dual curvature $\mathcal{K} = \mathcal{K}(\hat{s})$ *and the dual torsion* $\mathcal{T} = \mathcal{T}(\hat{s})$ *of* $\hat{\gamma}$ *are calculated as follows*

$$\begin{aligned}
\mathcal{K} &= \sqrt{1+\hat{\Sigma}^2} = \sqrt{1+\frac{4(5+\cos 2\hat{s})^2\sin^2\hat{s}}{(3+\cos 2\hat{s})^4}}, \\
\mathcal{T} &= \pm\frac{\hat{\Sigma}'}{1+\hat{\Sigma}^2} = \frac{-106\cos\hat{s}+9\cos 3\hat{s}+\cos 5\hat{s}}{2(3+\cos 2\hat{s})^3\left(1+\frac{4(5+\cos 2\hat{s})^2\sin^2\hat{s}}{(3+\cos 2\hat{s})^4}\right)}.
\end{aligned}$$

We obtain the evolute of the dual unit spherical curve of ruled surface as follows (see Fig. 3)

$$\mathbf{E}_{\hat{\gamma}}\left(\hat{s}\right) = \left(\hat{A}_1\left(\hat{s}\right), \hat{A}_2\left(\hat{s}\right), \hat{A}_3\left(\hat{s}\right)\right),$$

where

$$\hat{A}_1\left(\hat{s}\right) = \frac{1-\frac{8}{(3+\cos 2\hat{s})^2}-\frac{2}{3+\cos 2\hat{s}}+\frac{\sqrt{2}}{\sqrt{3+\cos 2\hat{s}}}-\frac{\sqrt{3+\cos 2\hat{s}}}{\sqrt{2}}}{\sqrt{1+\frac{4(5+\cos 2\hat{s})^2\sin^2\hat{s}}{(3+\cos 2\hat{s})^4}}},$$

$$\hat{A}_2\left(\hat{s}\right)=\frac{\frac{\sqrt{2}\cos^3\hat{s}}{\sqrt{3+\cos 2\hat{s}}}-\frac{2\cos\hat{s}\ \left(5+\cos 2\hat{s}\right)\sin^2\hat{s}}{\left(3+\cos 2\hat{s}\right)^2}+\frac{\sqrt{2}\sin\hat{s}\ \ \sin 2\hat{s}}{\sqrt{3+\cos 2\hat{s}}}}{\sqrt{1+\frac{4\left(5+\cos 2\hat{s}\right)^2\sin^2\hat{s}}{\left(3+\cos 2\hat{s}\right)^4}}},$$

$$\hat{A}_3\left(\hat{s}\right)=\frac{\left(-1+\frac{4}{\left(3+\cos 2\hat{s}\right)^2}-\frac{2\sqrt{2}}{\sqrt{3+\cos 2\hat{s}}}+\frac{\sqrt{3+\cos 2\hat{s}}}{\sqrt{2}}\right)\sin\hat{s}}{\sqrt{1+\frac{4\left(5+\cos 2\hat{s}\right)^2\sin^2\hat{s}}{\left(3+\cos 2\hat{s}\right)^4}}}.$$

Also, in the case that $\hat{s}=\frac{3\pi}{2}$, *we get*

$$\hat{\Sigma}'_{\mathbf{E}}(\frac{3\pi}{2})=0,\ \ \hat{\Sigma}''_{\mathbf{E}}(\frac{3\pi}{2})=-8.$$

Then the evolute of the dual unit spherical curve $\hat{\gamma}$ *at* $\hat{s}=\frac{3\pi}{2}$ *is diffeomorphic to the ordinary cusp and hence, the corollary 4.2 is satisfied.*

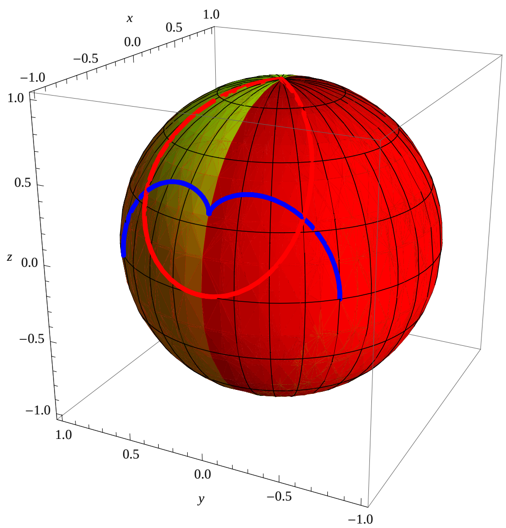


FIGURE 3. The dual spherical curve $\hat{\gamma}$ (the red color) of the ruled surface Ψ and its evolute (the blue color).

6. Conclusion

In this work, we have studied dual curves in dual space $\mathbb{D}^3$ due to the notion of $AW(k)$-type curves which was defined by K. Arslan and A. West [4] and denote it by $DAW(k)$ curves. Besides, some conditions on curvatures of these curves to be of $DAW(k)$-type using Bishop frame were introduced. In addition, according to the E. Study of the correspondence between the oriented lines in Euclidean three space and the dual points of the dual unit sphere in dual three space, we have obtained evolutes of dual spherical curves for ruled surfaces. Finally, the obtained results were confirmed by giving two examples.

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