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DONOHO-STARK UNCERTAINTY PRINCIPLE ASSOCIATED WITH A SINGULAR SECOND-ORDER DIFFERENTIAL OPERATOR

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Abstract. For a class of singular second-order differential operators Δ, we prove a continuous-time principles for L^1 theory and L^2 theory, respectively. Another version of continuous-time principle using $L^1 \cap L^2$ theory is given.

1. Introduction

The classical uncertainty principle says that if a function $f(t)$ is essentially zero outside an interval of length δt and its Fourier transform $\widehat{f}(w)$ is essentially zero outside an interval of length δw, then

$$\delta t.\delta w \geq 1;$$

a function and its Fourier transform cannot both be highly concentrated. The uncertainty principle is widely known for its "philosophical" applications: in quantum mechanics, of course, it shows that a particle's position and momentum cannot be determined simultaneously [10]; in signal processing it establishes limits on the extent to which the "instantaneous frequency" of a signal can be measured [9]. However, it also has technical applications, for example in the theory of partial differential equations [8].

Here we consider the second-order differential operator defined on $]0,\infty[$ by

$$\Delta u = u'' + \frac{A'}{A}u' + \rho^2 u,$$

where A is a nonnegative function satisfying certain conditions and ρ is a nonnegative real number. This operator plays an important role in analysis. For example, many special functions (orthogonal polynomials) are eigenfunctions of an operator of Δ type. The radial part of the Beltrami-Laplacian in a symmetric space is also of Δ type. Many aspects of such operators have been studied; we mention, in chronological order, in 1979 Chébli [2]; in 1981 Trimèche [15]; in 1989 Zeuner [18]; in 1994 Xu [17]; in 1997 Trimèche [16]; in 1998 Nessibi et al. [13]. In particular, the first two of these references investigate standard constructions of harmonic analysis, such as translation operators, convolution product, and Fourier transform, in connection with Δ.

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Many uncertainty principles have already been proved for the Sturm-Liouville operarator Δ, namely by Rösler and Voit [14] who established an uncertainty principle for Hankel transforms. Bouattour and Trimèche [1] proved a Beurling's theorem for the Sturm-Liouville transform. Daher et al. [3, 4, 5, 6] give some related versions of the uncertainty principle for the Sturm-Liouville transform (Titchmarsh's theorem, Hardy's theorem and Miyachi's theorem). Ma [11, 12] proved a Heisenberg uncertainty principle for the Sturm-Liouville transform.

Building on the ideas of Donoho and Stark [7] we show a continuous-time principle for the L^1 theory. The analogous of this uncertainty principle in the L^2 theory is also given. We prove another versions of continuous-time principle for the L^2 theory and for the $L^1 \cap L^2$ theory.

This paper is organized as follows. In Section 2 we recall some basic properties of the Fourier transform $\mathcal{F}$ associated to Δ (Plancherel theorem, inversion formula,...). In Section 3 we prove a continuous-time principle for L^1 theory. The last section of this paper is devoted to show another versions of continuous-time principles using L^2 theory and $L^1 \cap L^2$ theory.

2. The operator Δ

We consider the second-order differential operator Δ defined on $]0, \infty[$ by

$$\Delta u = u'' + \frac{A'}{A} u' + \rho^2 u,$$

where ρ is a nonnegative real number and

$$A(x) = x^{2\alpha+1} B(x), \quad \alpha > -1/2,$$

for B a positive, even, infinitely differentiable function on $\mathbb{R}$ such that $B(0) = 1$. Moreover we assume that A and B satisfy the following conditions:

(i) A is increasing and $\lim_{x\to\infty} A(x) = \infty$.

(ii) $\frac{A'}{A}$ is decreasing and $\lim_{x\to\infty} \frac{A'(x)}{A(x)} = 2\rho$.

(iii) There exists a constant $\delta > 0$ such that

$$\frac{A'(x)}{A(x)} = 2\rho + D(x)\exp(-\delta x) \quad \text{if } \rho > 0,$$

$$\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + D(x)\exp(-\delta x) \quad \text{if } \rho = 0,$$

where D is an infinitely differentiable function on $]0, \infty[$, bounded and with bounded derivatives on all intervals $[x_0, \infty[$, for $x_0 > 0$. This operator was studied in [2, 13, 15], and the following results have been established:

(I) For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} \Delta u = -\lambda^2 u \\ u(0) = 1, \ \ u'(0) = 0 \end{cases}$$

admits a unique solution, denoted by φ_λ, with the following properties:

φ_λ satisfies the product formula

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z) w(x, y, z) A(z) \mathrm{d}z \quad \text{for } \ x, y \geq 0;$$

where $w(x, y, .)$ is a measurable positive function on $[0, \infty[$, with support in $[|x - y|, x + y]$, satisfying

$$\int_0^\infty w(x, y, z)A(z)\mathrm{d}z = 1,$$

$$w(x, y, z) = w(y, x, z) \quad \text{for} \;\; z \geq 0,$$

$$w(x, y, z) = w(x, z, y) \quad \text{for} \;\; z > 0;$$

for $x \geq 0$, the function $\lambda \to \varphi_\lambda(x)$ is analytic on $\mathbb{C}$;
for $\lambda \in \mathbb{C}$, the function $x \to \varphi_\lambda(x)$ is even and infinitely differentiable on $\mathbb{R}$;
for all $\lambda, x \in \mathbb{R}$,

$$|\varphi_\lambda(x)| \leq 1; \tag{2.1}$$

for all $\lambda, x > 0$,

$$\varphi_\lambda(x) = \frac{1}{\sqrt{B(x)}} j_\alpha(\lambda x) + \frac{1}{\sqrt{A(x)}} \theta_\lambda(x),$$

where j_α is defined by $j_\alpha(0) = 1$ and $j_\alpha(s) = 2^\alpha \Gamma(\alpha + 1)s^{-\alpha} J_\alpha(s)$ if $s \neq 0$ (with J_α the Bessel function of first kind), and the function θ_λ satisfies

$$|\theta_\lambda(x)| \leq \frac{c_1}{\lambda^{\alpha+\frac{3}{2}}} \Big(\int_0^x |Q(s)|\mathrm{d}s \Big) \exp\Big(\frac{c_2}{\lambda} \int_0^x |Q(s)|\mathrm{d}s \Big)$$

with c_1, c_2 positive constants and Q the function defined on $]0, \infty[$ by

$$Q(x) = \frac{\frac{1}{4} - \alpha^2}{x^2} + \frac{1}{4}\Big(\frac{A'(x)}{A(x)}\Big)^2 + \frac{1}{2}\Big(\frac{A'(x)}{A(x)}\Big)' - \rho^2.$$

(II) For nonzero $\lambda \in \mathbb{C}$, the equation $\Delta u = -\lambda^2 u$ has a solution Φ_λ satisfying

$$\Phi_\lambda(x) = \frac{1}{\sqrt{A(x)}} \exp(i\lambda x) V(x, \lambda),$$

with $\lim_{x\to\infty} V(x, \lambda) = 1$. Consequently there exists a function (spectral function)

$$\lambda \mapsto c(\lambda),$$

such that

$$\varphi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda} \quad \text{for nonzero} \;\; \lambda \in \mathbb{C}.$$

Moreover there exist positive constants k_1, k_2, k_3 such that

$$k_1|\lambda|^{\alpha+1/2} \leq |c(\lambda)|^{-1} \leq k_2|\lambda|^{\alpha+1/2}$$

for all λ such that $\mathrm{Im}\lambda \leq 0$ and $|\lambda| \geq k_3$.

Notation 2.1. We denote by

μ the measure defined on $[0, \infty[$ by $\mathrm{d}\mu(x) := A(x)\mathrm{d}x$; and by $L^p(\mu)$, $1 \leq p \leq \infty$, the space of measurable functions f on $[0, \infty[$, such that

$$\|f\|_{L^p(\mu)} := \Big(\int_0^\infty |f(x)|^p \mathrm{d}\mu(x) \Big)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\mu)} := \operatorname*{ess\,sup}_{x\in[0,\infty[} |f(x)| < \infty;$$

ν the measure defined on $[0, \infty[$ by $\mathrm{d}\nu(\lambda) := \dfrac{\mathrm{d}\lambda}{2\pi|c(\lambda)|^2}$; and by $L^p(\nu)$, $1 \leq p \leq \infty$, the space of measurable functions f on $[0, \infty[$, such that $\|f\|_{L^p(\nu)} < \infty$.

The Fourier transform associated with the operator Δ is defined on $L^1(\mu)$ by

$$\mathcal{F}(f)(\lambda) := \int_0^\infty \varphi_\lambda(x) f(x) \mathrm{d}\mu(x) \quad \text{for } \lambda \in \mathbb{R}.$$

Some of the properties of the Fourier transform $\mathcal{F}$ are collected bellow (see [2, 13, 15, 16, 17]).

(a) $L^1 - L^\infty$-*boundedness.* For all $f \in L^1(\mu)$, $\mathcal{F}(f) \in L^\infty(\nu)$ and

$$\|\mathcal{F}(f)\|_{L^\infty(\nu)} \le \|f\|_{L^1(\mu)}. \tag{2.2}$$

(b) *Inversion theorem.* Let $f \in L^1(\mu)$, such that $\mathcal{F}(f) \in L^1(\nu)$. Then

$$f(x) = \int_0^\infty \varphi_\lambda(x) \mathcal{F}(f)(\lambda) \mathrm{d}\nu(\lambda), \quad \text{a.e. } x \in [0,\infty[. \tag{2.3}$$

(c) *Plancherel theorem.* The Dunkl transform $\mathcal{F}$ extends uniquely to an isometric isomorphism of $L^2(\mu)$ onto $L^2(\nu)$. In particular,

$$\|f\|_{L^2(\mu)} = \|\mathcal{F}(f)\|_{L^2(\nu)}. \tag{2.4}$$

Let T be measurable set of $[0,\infty[$. We introduce the time-limiting operator P_T by

$$P_T f(t) := \begin{cases} f(t), & t \in T \\ 0, & t \in [0,\infty[\backslash T. \end{cases} \tag{2.5}$$

This operator is bounded from $L^p(\mu)$, $1 \le p \le \infty$ into itself and

$$\|P_T f\|_{L^p(\mu)} \le \|f\|_{L^p(\mu)}, \quad f \in L^p(\mu). \tag{2.6}$$

Let W be measurable set of $[0,\infty[$. We introduce the partial sum operator S_W by

$$\mathcal{F}(S_W f) = \mathcal{F}(f) 1_W. \tag{2.7}$$

This operator is bounded from $L^2(\mu)$ into itself and

$$\|S_W f\|_{L^2(\mu)} \le \|f\|_{L^2(\mu)}, \quad f \in L^2(\mu). \tag{2.8}$$

Theorem 2.2. *If $\nu(W) < \infty$ and $f \in L^1(\mu)$ or $f \in L^2(\mu)$,*

$$S_W f(x) = \int_W \varphi_\lambda(x) \mathcal{F}(f)(\lambda) \mathrm{d}\nu(\lambda). \tag{2.9}$$

Proof. If $f \in L^1(\mu)$, then by (2.2),

$$\|\mathcal{F}(f) 1_W\|_{L^1(\nu)} = \int_W |\mathcal{F}(f)(w)| \mathrm{d}\nu(w) \le \nu(W) \|f\|_{L^1(\mu)},$$

and

$$\|\mathcal{F}(f) 1_W\|_{L^2(\nu)} = \Big(\int_W |\mathcal{F}(f)(w)|^2 \mathrm{d}\nu(w) \Big)^{1/2} \le \sqrt{\nu(W)} \|f\|_{L^1(\mu)}.$$

Thus $\mathcal{F}_k(f) 1_W \in L^1(\nu) \cap L^2(\nu)$ and by (2.7),

$$S_W f = \mathcal{F}^{-1}\Big(\mathcal{F}(f) 1_W \Big).$$

This combined with (2.3) gives the result.

If $f \in L^2(\mu)$, then by (2.4),

$$\|\mathcal{F}(f) 1_W\|_{L^1(\nu)} \le \sqrt{\nu(W)} \|f\|_{L^2(\mu)},$$

and

$$\|\mathcal{F}(f) 1_W\|_{L^2(\nu)} \le \|f\|_{L^2(\mu)}.$$

Thus $\mathcal{F}(f)1_W \in L^1(\nu) \cap L^2(\nu)$. This yields the desired result. □

3. An L^1 uncertainty principle

Let T and W be measurable sets of $[0,\infty[$. We say that a function $f \in L^1(\mu)$ is ε-concentrated to T if there is a measurable function $g(t)$ vanishing outside T such that $\|f-g\|_{L^1(\mu)} \leq \varepsilon\|f\|_{L^1(\mu)}$.

If f is ε_T-concentrated on T in $L^1(\mu)$-norm (g being the vanishing function) then

$$\|f - P_T f\|_{L^1(\mu)} = \int_{[0,\infty[\setminus T} |f(t)|\mathrm{d}\mu(t) \leq \|f-g\|_{L^1(\mu)} \leq \varepsilon_T\|f\|_{L^1(\mu)}$$

and therefore f is ε_T-concentrated to T in $L^1(\mu)$-norm if and only if $\|f-P_Tf\|_{L^1(\mu)} \leq \varepsilon_T\|f\|_{L^1(\mu)}$.

Let $B_1(W)$ denote the set of functions $g \in L^1(\mu)$ that are bandlimited to W (i.e. $g \in B_1(W)$ implies $S_W g = g$).

We say that f is ε-bandlimited to W in $L^1(\mu)$-norm if there is a $g \in B_1(W)$ with $\|f-g\|_{L^1(\mu)} \leq \varepsilon\|f\|_{L^1(\mu)}$.

The space $B_1(W)$ satisfies the following property.

Lemma 3.1. *Let T and W be measurable sets of $[0,\infty[$. For $g \in B_1(W)$,*

$$\frac{\|P_T g\|_{L^1(\mu)}}{\|g\|_{L^1(\mu)}} \leq \mu(T)\nu(W).$$

Proof. If $\mu(T) = \infty$ or $\nu(W) = \infty$, the inequality is clear. Assume that $\mu(T) < \infty$ and $\nu(W) < \infty$. For $g \in B_1(W)$, from Theorem 2.2,

$$g(t) = \int_W \varphi_w(t)\mathcal{F}(g)(w)\mathrm{d}\nu(w)$$

and by (2.1) and (2.2),

$$|g(t)| \leq \nu(W)\|g\|_{L^1(\mu)}.$$

Hence

$$\|P_T g\|_{L^1(\mu)} = \int_T |g(t)|\mathrm{d}\mu(t) \leq \mu(T)\nu(W)\|g\|_{L^1(\mu)}.$$

Therefore, for $g \in B_1(W)$,

$$\frac{\|P_T g\|_{L^1(\mu)}}{\|g\|_{L^1(\mu)}} \leq \mu(T)\nu(W),$$

which yields the result. □

It is useful to have uncertainty principle for the $L^1(\mu)$-norm.

Theorem 3.2. *Let T and W be measurable sets of $[0,\infty[$ and $f \in L^1(\mu)$. If f is ε_T-concentrated to T and ε_W-bandlimited to W in $L^1(\mu)$-norm, then*

$$\mu(T)\nu(W) \geq \frac{1-\varepsilon_T-\varepsilon_W}{1+\varepsilon_W}.$$

Proof. Let $f \in L^1(\mu)$. The triangle inequality gives

$$\|P_T f\|_{L^1(\mu)} \geq \|f\|_{L^1(\mu)} - \|f - P_T f\|_{L^1(\mu)}.$$

Since f is ε_T-concentrated to T in $L^1(\mu)$-norm,

$$\|P_T f\|_{L^1(\mu)} \geq (1-\varepsilon_T)\|f\|_{L^1(\mu)}. \tag{3.1}$$

On the other hand, f is ε_W-bandlimited to W in $L^1(\mu)$-norm, by definition there is a g in $B_1(W)$ with $\|f-g\|_{L^1(\mu)} \leq \varepsilon_W \|f\|_{L^1(\mu)}$. For this g and by (2.6), we have

$$\begin{aligned}\|P_T g\|_{L^1(\mu)} &\geq \|P_T f\|_{L^1(\mu)} - \|P_T(f-g)\|_{L^1(\mu)}\\ &\geq \|P_T f\|_{L^1(\mu)} - \varepsilon_W \|f\|_{L^1(\mu)}\end{aligned}$$

and also

$$\|g\|_{L^1(\mu)} \leq (1+\varepsilon_W)\|f\|_{L^1(\mu)}.$$

So that

$$\frac{\|P_T g\|_{L^1(\mu)}}{\|g\|_{L^1(\mu)}} \geq \frac{\|P_T f\|_{L^1(\mu)} - \varepsilon_W \|f\|_{L^1(\mu)}}{(1+\varepsilon_W)\|f\|_{L^1(\mu)}}.$$

Thus, by (3.1) we deduce

$$\frac{\|P_T g\|_{L^1(\mu)}}{\|g\|_{L^1(\mu)}} \geq \frac{1-\varepsilon_T-\varepsilon_W}{1+\varepsilon_W}.$$

This combined with Lemma 3.1 proves Theorem 3.2. □

4. An L^2 uncertainty principles

Let T and W be measurable sets of $[0,\infty[$. We say that a function $f \in L^2(\mu)$ is ε-concentrated to T if there is a measurable function $g(t)$ vanishing outside T such that $\|f-g\|_{L^2(\mu)} \leq \varepsilon \|f\|_{L^2(\mu)}$. Similarly, we say that $\mathcal{F}(f)$ is ε-concentrated to W if there is a function $h(w)$ vanishing outside W with $\|\mathcal{F}(f)-h\|_{L^2(\nu)} \leq \varepsilon \|f\|_{L^2(\mu)}$.

If f is ε_T-concentrated to T in $L^2(\mu)$-norm (g being the vanishing function) then

$$\|f-P_T f\|_{L^2(\mu)} = \Big(\int_{[0,\infty[\setminus T} |f(t)|^2 \mathrm{d}\mu(t)\Big)^{1/2} \leq \|f-g\|_{L^2(\mu)} \leq \varepsilon_T \|f\|_{L^2(\mu)}$$

and therefore f is ε_T-concentrated to T in $L^2(\mu)$-norm if and only if $\|f-P_T f\|_{L^2(\mu)} \leq \varepsilon_T \|f\|_{L^2(\mu)}$.

From (2.7) it follows as for P_T that $\mathcal{F}(f)$ is ε_W-concentrated to W in $L^2(\nu)$-norm if and only if

$$\|\mathcal{F}(f)-\mathcal{F}(S_W f)\|_{L^2(\nu)} = \|f-S_W f\|_{L^2(\mu)} \leq \varepsilon_W \|f\|_{L^2(\mu)}.$$

Let $B_2(W)$ denote the set of functions $g \in L^2(\mu)$ that are bandlimited to W (i.e. $g \in B_2(W)$ implies $S_W g = g$).

We say that f is ε-bandlimited to W in $L^2(\mu)$-norm if there is a $g \in B_2(W)$ with $\|f-g\|_{L^2(\mu)} \leq \varepsilon \|f\|_{L^2(\mu)}$.

The space $B_2(W)$ satisfies the following property.

Lemma 4.1. *Let T and W be measurable sets of $[0,\infty[$. For $g \in B_2(W)$,*

$$\frac{\|P_T g\|_{L^2(\mu)}}{\|g\|_{L^2(\mu)}} \leq \sqrt{\mu(T)\nu(W)}.$$

Proof. Assume that $\mu(T) < \infty$ and $\nu(W) < \infty$. For $g \in B_2(W)$, from (2.9),

$$g(t) = \int_W \varphi_w(t)\mathcal{F}(g)(w)\mathrm{d}\nu(w)$$

and by (2.1) and Hölder's inequality,

$$|g(t)| \leq \sqrt{\nu(W)}\|g\|_{L^2(\mu)}.$$

Hence

$$\|P_T g\|_{L^2(\mu)} = \Big(\int_T |g(t)|^2 \mathrm{d}\mu(t)\Big)^{1/2} \leq \sqrt{\mu(T)\nu(W)}\|g\|_{L^2(\mu)}.$$

Therefore, for $g \in B_2(W)$,

$$\frac{\|P_T g\|_{L^2(\mu)}}{\|g\|_{L^2(\mu)}} \leq \sqrt{\mu(T)\nu(W)}\,,$$

which yields the result. □

It is useful to have uncertainty principle for the $L^2(\mu)$-norm.

Theorem 4.2. *Let T and W be measurable sets of $[0,\infty[$ and $f \in L^2(\mu)$. If f is ε_T-concentrated to T and ε_W-bandlimited to W in $L^2(\mu)$-norm, then*

$$\sqrt{\mu(T)\nu(W)} \geq \frac{1-\varepsilon_T-\varepsilon_W}{1+\varepsilon_W}.$$

Proof. Let $f \in L^2(\mu)$. The triangle inequality gives

$$\|P_T f\|_{L^2(\mu)} \geq \|f\|_{L^2(\mu)} - \|f - P_T f\|_{L^2(\mu)}.$$

Since f is ε_T-concentrated to T in $L^2(\mu)$-norm,

$$\|P_T f\|_{L^2(\mu)} \geq (1-\varepsilon_T)\|f\|_{L^2(\mu)}. \tag{4.1}$$

On the other hand, f is ε_W-bandlimited to W in $L^2(\mu)$-norm, by definition there is a g in $B_2(W)$ with $\|f-g\|_{L^2(\mu)} \leq \varepsilon_W\|f\|_{L^2(\mu)}$. For this g and by (2.6), we have

$$\begin{aligned}\|P_T g\|_{L^2(\mu)} &\geq \|P_T f\|_{L^2(\mu)} - \|P_T(f-g)\|_{L^2(\mu)}\\ &\geq \|P_T f\|_{L^2(\mu)} - \varepsilon_W\|f\|_{L^2(\mu)}\end{aligned}$$

and also

$$\|g\|_{L^2(\mu)} \leq (1+\varepsilon_W)\|f\|_{L^2(\mu)}.$$

So that

$$\frac{\|P_T g\|_{L^2(\mu)}}{\|g\|_{L^2(\mu)}} \geq \frac{\|P_T f\|_{L^2(\mu)} - \varepsilon_W\|f\|_{L^2(\mu)}}{(1+\varepsilon_W)\|f\|_{L^2(\mu)}}.$$

Thus, by (4.1) we deduce

$$\frac{\|P_T g\|_{L^2(\mu)}}{\|g\|_{L^2(\mu)}} \geq \frac{1-\varepsilon_T-\varepsilon_W}{1+\varepsilon_W}.$$

This combined with Lemma 4.1 proves Theorem 4.2. □

Lemma 4.3. *Let T and W be measurable sets of $[0,\infty[$. For $f \in L^2(\mu)$,*

$$\frac{\|S_W P_T f\|_{L^2(\mu)}}{\|f\|_{L^2(\mu)}} \leq \sqrt{\mu(T)\nu(W)}.$$

Proof. Assume that $\mu(T) < \infty$ and $\nu(W) < \infty$.
Let $f \in L^2(\mu)$. From (2.5) and (2.9),

$$\begin{aligned}S_W P_T f(s) &= \int_W \varphi_w(s)\mathcal{F}(P_T f)(w)\mathrm{d}\nu(w)\\ &= \int_W \varphi_w(s)\int_T \varphi_w(t)f(t)\mathrm{d}\mu(t)\mathrm{d}\nu(w).\end{aligned}$$

Since by (2.1),

$$\int_W\int_T \Big|\varphi_w(s)\varphi_w(t)f(t)\Big|\mathrm{d}\mu(t)\mathrm{d}\nu(w) \leq \nu(W)\sqrt{\mu(T)}\|f\|_{L^2(\mu)} < \infty$$

by Fubini's theorem,

$$S_W P_T f(s) = \int_T f(t) \int_W \varphi_w(s)\varphi_w(t)\mathrm{d}\nu(w)\mathrm{d}\mu(t),$$

so that

$$S_W P_T f(s) = \int_T q(s,t) f(t)\mathrm{d}\mu(t), \tag{4.2}$$

where

$$q(s,t) = \int_W \varphi_w(s)\varphi_w(t)\mathrm{d}\nu(w), \quad t \in T, s \in [0,\infty[.$$

For $t \in T$, let

$$g_t(s) = q(s,t) = \int_W \varphi_w(s)\varphi_w(t)\mathrm{d}\nu(w).$$

Then the inversion formula (2.3) shows that

$$\mathcal{F}(g_t)(w) = 1_W \varphi_w(t).$$

By Plancherel's formula (2.4) it then follows

$$\int_0^\infty |q(s,t)|^2 \mathrm{d}\mu(s) = \int_0^\infty |g_t(s)|^2 \mathrm{d}\mu(s) = \int_0^\infty |\mathcal{F}(g_t)(w)|^2 \mathrm{d}\nu(w) \le \nu(W).$$

By applying Hölder's inequality to (4.2),

$$|S_W P_T f(s)|^2 \le \|f\|^2_{L^2(\mu)} \int_T |q(s,t)|^2 \mathrm{d}\mu(t).$$

Hence

$$\|S_W P_T f\|_{L^2(\mu)} \le \|f\|_{L^2(\mu)} \Big(\int_0^\infty \int_T |q(s,t)|^2 \mathrm{d}\mu(t)\mathrm{d}\mu(s) \Big)^{1/2}.$$

By Fubini-Tonnelli's theorem,

$$\|S_W P_T f\|_{L^2(\mu)} \le \|f\|_{L^2(\mu)} \Big(\int_T \int_0^\infty |q(s,t)|^2 \mathrm{d}\mu(s)\mathrm{d}\mu(t) \Big)^{1/2} \le \|f\|_{L^2(\mu)} \sqrt{\mu(T)\nu(W)}.$$

Thus, the proof is complete. □

Another uncertainty principle for $L^2(\mu)$-norm is obtained.

Theorem 4.4. *Let T and W be measurable sets of $[0,\infty[$ and $f \in L^2(\mu)$. If f is ε_T-concentrated to T in $L^2(\mu)$-norm and $\mathcal{F}(f)$ is ε_W-concentrated to W in $L^2(\nu)$-norm, then*

$$\sqrt{\mu(T)\nu(W)} \ge 1 - \varepsilon_T - \varepsilon_W.$$

Proof. Let $f \in L^2(\mu)$. From (2.8) it follows

$$\begin{aligned} \|f - S_W P_T f\|_{L^2(\mu)} &\le \|f - S_W f\|_{L^2(\mu)} + \|S_W f - S_W P_T f\|_{L^2(\mu)} \\ &\le \varepsilon_W \|f\|_{L^2(\mu)} + \|f - P_T f\|_{L^2(\mu)} \\ &\le (\varepsilon_T + \varepsilon_W)\|f\|_{L^2(\mu)}. \end{aligned}$$

The triangle inequality gives

$$\|S_W P_T f\|_{L^2(\mu)} \ge \|f\|_{L^2(\mu)} - \|f - S_W P_T f\|_{L^2(\mu)} \ge (1 - \varepsilon_W - \varepsilon_T)\|f\|_{L^2(\mu)}.$$

It then follows that $\|S_W P_T f\|_{L^2(\mu)} \ge (1 - \varepsilon_W - \varepsilon_T)\|f\|_{L^2(\mu)}$. The Lemma 4.3 show that

$$\sqrt{\mu(T)\nu(W)}\|f\|_{L^2(\mu)} \ge (1 - \varepsilon_T - \varepsilon_W)\|f\|_{L^2(\mu)},$$

which gives the desired result. □

An uncertainty principle for $L^1(\mu)\cap L^2(\mu)$ theory is obtained.

Theorem 4.5. *Let T and W be measurable sets of $[0,\infty[$ and $f\in L^1(\mu)\cap L^2(\mu)$. If f is ε_T-concentrated to T in $L^1(\mu)$-norm and $\mathcal{F}(f)$ is ε_W-concentrated to W in $L^2(\nu)$-norm, then*

$$\sqrt{\mu(T)\nu(W)}\geq(1-\varepsilon_T)(1-\varepsilon_W).$$

Proof. Assume that $\mu(T)<\infty$ and $\nu(W)<\infty$.
Let $f\in L^1(\mu)\cap L^2(\mu)$. Since $\mathcal{F}(f)$ is ε_W-concentrated to W in $L^2(\nu)$-norm, then

$$\begin{aligned}\|f\|_{L^2(\mu)} &\leq \varepsilon_W\|f\|_{L^2(\mu)}+\Big(\int_W|\mathcal{F}(f)(w)|^2\mathrm{d}\nu(w)\Big)^{1/2}\\ &\leq \varepsilon_W\|f\|_{L^2(\mu)}+\sqrt{\nu(W)}\|\mathcal{F}(f)\|_{L^\infty(\nu)}.\end{aligned}$$

Thus by (2.2),

$$(1-\varepsilon_W)\|f\|_{L^2(\mu)}\leq\sqrt{\nu(W)}\|f\|_{L^1(\mu)}. \tag{4.3}$$

On the other hand, since f is ε_T-concentrated to T in $L^1(\mu)$-norm,

$$\begin{aligned}\|f\|_{L^1(\mu)} &\leq \varepsilon_T\|f\|_{L^1(\mu)}+\int_T|f(t)|\mathrm{d}\mu(t)\\ &\leq \varepsilon_T\|f\|_{L^1(\mu)}+\sqrt{\mu(T)}\|f\|_{L^2(\mu)}.\end{aligned}$$

Thus

$$(1-\varepsilon_T)\|f\|_{L^1(\mu)}\leq\sqrt{\mu(T)}\|f\|_{L^2(\mu)}. \tag{4.4}$$

Combining (4.3) and (4.4) we obtain the result of this theorem. □

References

[1] L. Bouattour and K. Trimèche, Beurling-Hörmander's theorem for the Chébli-Trimèche transform, Glob. J. Pure Appl. Math. **1**(3) (2005) 342–357

[2] H. Chébli, Théorème de Paley-Wiener associ??un op érateur différentiel singulier sur $(0,\infty)$, J. Math. Pures Appl. **58**(1) (1979) 1–19.

[3] R. Daher, An analog of Titchmarsh's theorem of Jacobi transform, Int. J. Math. Anal. **6**(20) (2012) 975 –981.

[4] R. Daher and T. Kawazoe, Generalized of Hardy's theorem for Jacobi transform, Hiroshima J. Math. **36**(3) (2006) 331–337.

[5] R. Daher and T. Kawazoe, An uncertainty principle on Sturm-Liouville hypergroups, Proc. Japan Acad. **83** Ser. A (2007) 167–169.

[6] R. Daher, T. Kawazoe and H. Mejjaoli, A generalization of Miyachi's theorem, J. Math. Soc. Japan **61**(2) (2009) 551–558.

[7] D.L. Donoho and P.B. Stark, Uncertainty principles and signal recovery, SIAM J. Appl. Math. **49**(3) (1989) 906–931.

[8] C.L. Fefferman, The uncertainty principle, Bull. Amer. Math. Soc. **9** (1983) 129–206.

[9] D. Gabor, Theory of communication, J. Inst. Elec. Engrg. **93** (1946) 429–457.

[10] W. Heisenberg, The Physical Principles of the Quantum Theory, Dover, NewYork, 1949 (The University of Chicago Press, 1930).

[11] R. Ma, Heisenberg inequalities for Jacobi transforms, J. Math. Anal. Appl. **332** (2007) 155–163.

[12] R. Ma, Heisenberg uncertainty principle on Chébli-Trimèche hypergroups, Pacific J. Math. **235**(2) (2008) 289–296.

[13] M.M. Nessibi, L.T. Rachdi and K. Trimèche, The local central limit theorem on the product of the Chébli-Trimèche hypergroup and the Euclidean hypergroup $\mathbb{R}^n$, J. Math. Sci. (Calcutta) **9**(2) (1998) 109–123.

[14] M. Rösler and M. Voit, An uncertainty principle for Hankel transforms, Proc. Amer. Math. Soc. **127**(1) (1999) 183–194.

[15] K. Trimèche, Transformation intégrale de Weyl et théorème de Paley-Wiener associés ?un opérateur différentiel singulier sur $(0,\infty)$, J. Math. Pures Appl. **60**(1) (1981) 51–98.

[16] K. Trimèche, Inversion of the Lions transmutation operators using generalized wavelets, Appl. Comput. Harmon. Anal. **4**(1) (1997) 97–112.
[17] Z. Xu, Harmonic analysis on Chébli-Trimèche hypergroups, Ph.D. thesis, Murdoch University, Perth, Western Australia, 1994.
[18] H. Zeuner, The central limit theorem for Chébli-Trimèche hypergroups, J. Theoret. Probab. **2**(1) (1989) 51–63.

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