International Journal of Analysis and Applications
ISSN 2291-8639
Volume 4, Number 1 (2014), 36-44
http://www.etamaths.com

ON THE BEHAVIOR NEAR THE ORIGIN OF A SINE SERIES WITH COEFFICIENTS OF MONOTONE TYPE

XHEVAT Z. KRASNIQI

Abstract. In this paper we have obtained some asymptotic equalities of the sum function of a trigonometric sine series expressed in terms of its special type of coefficients.

1. Introduction

Let us consider the sine series

$$\sum_{m=1}^{\infty} a_m \sin mx \tag{1.1}$$

with coefficients tending to zero and such that the sequence $\{a_m\}$ satifsies condition $\triangle a_m = a_m - a_{m+1} \geq 0$ or $\triangle^2 a_m = \triangle a_m - \triangle a_{m+1} \geq 0$ for all m. It is a well-known fact that under such conditions the series (1.1) converges for all x (see [12], page 95). We denote by $g(x)$ its sum.

As usually we write $g(u) \sim h(u), u \to 0$ if there exist absolute positive constants A and B such that $Ah(u) \leq g(u) \leq Bh(u)$ is in a neighborhood of the point $u = 0$, and write $g(u) \approx h(u)$ if $\lim_{u\to 0} \frac{g(u)}{h(u)} = 1$. Likewise, throughout this paper the constants in the $\mathcal{O}$-expression denote positive absolute constants and they may be different in different relations.

Several authors have investigated the behavior of the sum $g(x)$ near the origin expressed in terms of the coefficients a_m. Seemingly, the first was Young [11] who consider this problem, and he was concerned solely about estimates of $|g(x)|$ from above. Then Salem ([3], [4], Theorem 1) proved that if the sequence $\{ma_m\}$ is monotone decreasing, then the following order equality holds

$$g(x) \sim \sum_{m=1}^{\ell} m a_m x,$$

where $x \in I_\ell := \left(\frac{\pi}{\ell+1}, \frac{\pi}{\ell}\right], \ell = 1, 2, \ldots, \quad x \to 0.$

Later on, Aljančić, Bojanić and Tomić ([5], Theorem 2) give asymptotic expression for $g(x)$ as $x \to 0$, when the coefficients a_m are convex ($\triangle^2 a_m \geq 0$) and can be represent as the values $A(m)$ of a slowly varying (in Karamata's sense) function

2010 *Mathematics Subject Classification.* 42A20, 42A32.

Key words and phrases. Sine series, (k, s)-monotone, convex sequence, asymptotic equality.

$A(z)$, i.e. for each $t > 0$

$$\lim_{z\to\infty} \frac{A(tz)}{A(z)} = 1. \tag{1.2}$$

Their result is equivalent to the following statement which can be deduce from one result given by Telyakovskiĭ ([6], Theorem 2) and it is formulated as a corollary in this form:

Corollary 1.1. *Suppose that the coefficients a_m of the series (1.1) are convex and that $a_m = A(m)$, for a slowly varying function $A(z)$. Then the following asymptotic equality holds true:*

$$g(x) \approx a_\ell \frac{1}{x}, \quad x \in I_\ell, \quad x \to 0.$$

Telyakovskiĭ deduced this result after the proof, in the same paper, of the following two theorems:

Theorem 1.1. *Assume that $a_m \downarrow 0$. Then for $x \in I_\ell$ the following estimate is valid*

$$g(x) = \sum_{m=1}^{\ell} m a_m x + \mathcal{O}\left(\frac{1}{\ell^3}\sum_{m=1}^{\ell} m^3 a_m\right).$$

Theorem 1.2. *Let $a_m \to 0$ and let the sequence $\{a_m\}$ be convex. If $x \in I_\ell$, where $\ell \geq 11$, then the following estimate holds true*

$$\frac{a_\ell}{2}\cot\frac{x}{2} + \frac{1}{2\ell}\sum_{m=1}^{\ell-1} m^2 \triangle a_m \leq g(x) \leq \frac{a_\ell}{2}\cot\frac{x}{2} + \frac{6}{\ell}\sum_{m=1}^{\ell-1} m^2 \triangle a_m.$$

Note also that the above theorems as well as some of [1] are generalized and extended in [7]-[10].

For an integer $k \geq 0$ and a real sequence $\{a_m\}_{m=0}^{\infty}$ denote

$$\begin{aligned} \triangle_k a_m &= \sum_{i=0}^{k} (-1)^i C_k^i a_{m+i} \qquad (\triangle_0 a_m = a_m), \\ \{\triangle\}_k a_m &= \sum_{i=0}^{k} C_k^i a_{m+i} \qquad (\{\triangle\}_0 a_m = a_m). \end{aligned}$$

Definition 1.1 ([2]). *A sequence $\{a_m\}_{m=0}^{\infty}$ is said to be (k,s)-monotone if $a_m \to 0$ as $m \to \infty$ and $\triangle_k(\{\triangle\}_s a_m) \geq 0$, for some $k \geq 0, s \geq 0$ and all m.*

It is easy to see that that if a sequence $\{a_m\}$ ($a_m \to 0$ as $m \to \infty$) is non-increasing, then it is $(1,s)$-monotone for all $s = 0,1,2,\ldots$. The converse statement is not always true. For example, if we consider the sequence $\{a_m\}$ such that $a_m \to 0$ as $m \to \infty$ and $a_{2m} = 0$, $a_{2m+1} \geq a_{2m+3}$ for $m = 0,1,2,\ldots$, then this sequence is not non-increasing but it is $(1,1)$-monotone.

Chronologically this definition arises the following question: What is the behavior near the origin of the series (1.1) with (k,s)-monotone coefficients? The answer to this question is the main goal of this paper. Precisely, we shall answer to this question only considering the cases when the series (1.1) has: $(1,1)$-monotone, or $(1,2)$-monotone, or $(2,1)$-monotone, or $(2,2)$-monotone coefficients.

For the proof of our results we need the following two lemmas proved in [2].

Lemma 1.1. *Let $\{a_m\}_{m=0}^{\infty}$ be a sequence such that $a_m \to 0$ as $m \to \infty$ and $\triangle^k a_m \geq 0$ for some $k \geq 1$ and all m. Then for all $r = 0, 1, \ldots, k-1$ and all m the following inequality $\triangle^r a_m \geq 0$ holds.*

Lemma 1.2. *Let $\{a_m\}_{m=0}^{\infty}$ be a (k, s)-monotone sequence. If $k = 1, s = 1$ or $s = 2$, then*

$$g(x) = \frac{a_0}{2}\left(1 - \tan\frac{x}{2}\right) + \frac{1}{\left(2\cos\frac{x}{2}\right)^s}\sum_{m=1}^{\infty}\{\triangle\}_s a_{m-1}\sin\left(ms - 2 + s\right)\frac{x}{2},$$

allmost everywhere.

Lemma 1.3. *Let $\overline{B}_m(x) = \sum_{i=0}^{m}\sin\left(i-1\right)\frac{x}{2}$. Then the following estimates hold:*

$$\left|\overline{B}_m(x)\right| \leq \frac{2\pi}{x}, \quad 0 < x \leq \pi.$$

Proof. After some elementary calculation we have

$$\begin{aligned}\left|\overline{B}_m(x)\right| &= \left|\frac{1}{2\sin\frac{x}{2}}\sum_{i=0}^{m}\left[\cos\left(i-2\right)\frac{x}{2} - \cos\frac{ix}{2}\right]\right| \\ &= \left|\frac{\cos\frac{x}{2} + \cos x - \cos\left(m-1\right)\frac{x}{2} - \cos\frac{mx}{2}}{2\sin\frac{x}{2}}\right| \\ &\leq \frac{2}{\left|\sin\frac{x}{2}\right|} \leq \frac{2\pi}{x}, \quad 0 < x \leq \pi.\end{aligned}$$

□

2. Main Results

The following theorem considers sine series with $(1, 1)$-monotone sequence.

Theorem 2.1. *Assume that $\{a_m\}_{m=1}^{\infty}$ is a $(1, 1)$-monotone sequence. Then for $x \in I_\ell$ the following estimate is valid*

$$g(x) = \frac{1}{2\cos\frac{x}{2}}\left\{\frac{1}{2}\sum_{m=1}^{\ell} m\{\triangle\}_1 a_m x + \mathcal{O}\left(\frac{1}{\ell^3}\sum_{m=1}^{\ell} m^3\{\triangle\}_1 a_m\right)\right\}. \tag{2.1}$$

Proof. By the Lemma 1.2 ($a_0 = 0$) we have

$$g(x) = \frac{1}{2\cos\frac{x}{2}}\sum_{m=1}^{\infty}\{\triangle\}_1 a_{m-1}\sin\left(m-1\right)\frac{x}{2}. \tag{2.2}$$

Then the use of Abel's transformation gives

$$\begin{aligned}H(x) &= \lim_{p\to\infty}\left\{\sum_{m=1}^{p-1}\triangle\left(\{\triangle\}_1 a_{m-1}\right)\overline{B}_m(x) + \{\triangle\}_1 a_{p-1}\overline{B}_p(x) + \{\triangle\}_1 a_0\sin\frac{x}{2}\right\} \\ &= \sum_{m=1}^{\infty}\triangle\left(\{\triangle\}_1 a_{m-1}\right)\overline{B}_m(x) + \{\triangle\}_1 a_0\sin\frac{x}{2} := H_\ell^{(1)}(x) + H_\ell^{(2)}(x), \quad (2.3)\end{aligned}$$

where

$$H_\ell^{(1)}(x) = \sum_{m=1}^{\ell+1}\triangle\left(\{\triangle\}_1 a_{m-1}\right)\overline{B}_m(x) + \{\triangle\}_1 a_0\sin\frac{x}{2},$$

and

$$H_\ell^{(2)}(x)=\sum_{m=\ell+2}^{\infty}\triangle\left(\{\triangle\}_1 a_{m-1}\right)\overline{B}_m(x).$$

Let us estimate first $H_\ell^{(1)}(x)$. Based on Lemma 1.3, our assumption $\triangle\left(\{\triangle\}_1 a_m\right)\geq 0$ for all m, the well-known relation $\sin t=t+\mathcal{O}(t^3)$, as $t\to 0$, and $x\in I_\ell$ we have

$$\begin{aligned}
H_\ell^{(1)}(x) &= \sum_{m=1}^{\ell+1}\left(\{\triangle\}_1 a_{m-1}-\{\triangle\}_1 a_m\right)\overline{B}_m(x)+\{\triangle\}_1 a_0\sin\frac{x}{2}\\
&= \sum_{m=0}^{\ell}\{\triangle\}_1 a_m\left[\overline{B}_{m+1}(x)-\overline{B}_m(x)\right]-\{\triangle\}_1 a_{\ell+1}\overline{B}_{\ell+1}(x)\\
&= \sum_{m=1}^{\ell}\{\triangle\}_1 a_m\sin\frac{mx}{2}+\frac{2\pi}{x}\{\triangle\}_1 a_{\ell+1}\\
&= \frac{1}{2}\sum_{m=1}^{\ell}m\{\triangle\}_1 a_m x+\mathcal{O}\left(\frac{1}{\ell^3}\sum_{m=1}^{\ell}m^3\{\triangle\}_1 a_m\right)+\mathcal{O}\left(\ell\{\triangle\}_1 a_\ell\right).
\end{aligned}$$

By virtue of monotonicity of $\{\triangle\}_1 a_m$ we obtain

$$\ell\{\triangle\}_1 a_\ell\leq\frac{4}{\ell^3}\left\{\frac{\ell(\ell+1)}{2}\right\}^2\{\triangle\}_1 a_\ell\leq\frac{4}{\ell^3}\sum_{m=1}^{\ell}m^3\{\triangle\}_1 a_m.$$

Thus,

$$H_\ell^{(1)}(x)=\frac{1}{2}\sum_{m=1}^{\ell}m\{\triangle\}_1 a_m x+\mathcal{O}\left(\frac{1}{\ell^3}\sum_{m=1}^{\ell}m^3\{\triangle\}_1 a_m\right). \tag{2.4}$$

Furthermore, since $x\in I_\ell$ and $|\overline{B}_m(x)|=\mathcal{O}\left(\frac{1}{x}\right)$ by the Lemma 1.2, we notice that

$$\begin{aligned}
H_\ell^{(2)}(x) &= \mathcal{O}\left(\frac{1}{x}\sum_{m=\ell+2}^{\infty}\left(\{\triangle\}_1 a_{m-1}-\{\triangle\}_1 a_m\right)\right)\\
&= \mathcal{O}\left((\ell+1)\{\triangle\}_1 a_{\ell+1}\right)=\mathcal{O}\left(\ell\{\triangle\}_1 a_\ell\right)\\
&= \mathcal{O}\left(\frac{1}{\ell^3}\sum_{m=1}^{\ell}m^3\{\triangle\}_1 a_m\right). \qquad (2.5)
\end{aligned}$$

Finally, relations (2.2)-(2.5) prove completely estimation (2.1). □

Corollary 2.1. *Let* $\{a_m\}_{m=1}^{\infty}$ *be a* $(1,1)$*-monotone sequence and the series*

$$\sum_{m=1}^{\infty}m\left(a_m+a_{m+1}\right)$$

converges. Then the following asymptotic equality

$$\lim_{x\to 0}\frac{g(x)}{x}=\frac{1}{4}\sum_{m=1}^{\infty}m\left(a_m+a_{m+1}\right)$$

holds true.

Proof. In accordance with Theorem 2.1 it is enough to prove that

$$\frac{1}{\ell^2}\sum_{m=1}^{\ell} m^3\{\triangle\}_1 a_m \to 0, \quad \text{as} \quad \ell\to\infty.$$

Indeed, for an arbitrary natural number M we can write

$$\frac{1}{\ell^2}\sum_{m=1}^{\ell} m^3\{\triangle\}_1 a_m \le \frac{1}{\ell^2}\sum_{m=1}^{M} m^3\{\triangle\}_1 a_m + \sum_{m=M+1}^{\infty} m\{\triangle\}_1 a_m.$$

If a number $\varepsilon > 0$ be chosen, then by hypotesis a number $M = M(\varepsilon)$ exists, such that

$$\sum_{m=M+1}^{\infty} m\{\triangle\}_1 a_m < \frac{\varepsilon}{2}.$$

Likewise, for all sufficiently large ℓ

$$\frac{1}{\ell^2}\sum_{m=1}^{M} m^3\{\triangle\}_1 a_m < \frac{\varepsilon}{2}.$$

Then obviously, for such ℓ we have

$$\frac{1}{\ell^2}\sum_{m=1}^{\ell} m^3\{\triangle\}_1 a_m < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

□

The following statements can be proved similarly therefore we will skip their proofs.

Theorem 2.2. *Assume that* $\{a_m\}_{m=1}^{\infty}$ *is a* $(1,2)$*-monotone sequence. Then for* $x \in I_\ell$ *the following estimate is valid*

$$g(x) = \frac{1}{\left(2\cos\frac{x}{2}\right)^2}\left\{\sum_{m=0}^{\ell}(m+1)\{\triangle\}_2 a_m x + \mathcal{O}\left(\frac{1}{\ell^3}\sum_{m=0}^{\ell}(m+1)^3\{\triangle\}_2 a_m\right)\right\}.$$

Corollary 2.2. *Suppose that* $\{a_m\}_{m=1}^{\infty}$ *is a* $(1,2)$*-monotone sequence and the series*

$$\sum_{m=0}^{\infty}(m+1)\left(a_m + 2a_{m+1} + a_{m+2}\right)$$

converges. Then the following asymptotic equality

$$\lim_{x\to 0}\frac{g(x)}{x} = \frac{1}{4}\sum_{m=0}^{\infty}(m+1)\left(a_m + 2a_{m+1} + a_{m+2}\right)$$

holds true.

The proof of the next statement is more complicated and that is why we will sketch it in more details.

Theorem 2.3. *Assume that* $\{a_m\}_{m=1}^\infty$ *is a* $(2,2)$*-monotone sequence. Then for* $x \in I_\ell$, $\ell \geq 11$ *the following estimate is valid*

$$\frac{\{\triangle\}_2 a_{\ell-1}}{2}\cot\frac{x}{2} + \frac{1}{2\ell}\sum_{m=1}^{\ell-1} m^2 \triangle\left(\{\triangle\}_2 a_{m-1}\right) \leq g(x)\left(2\cos\frac{x}{2}\right)^2 \leq \frac{\{\triangle\}_2 a_{\ell-1}}{2}\cot\frac{x}{2} + \frac{6}{\ell}\sum_{m=1}^{\ell-1} m^2 \triangle\left(\{\triangle\}_2 a_{m-1}\right).$$

Proof. By the Lemma 1.1 the condition $\triangle_2\left(\{\triangle\}_2 a_m\right) \geq 0$ implies $\triangle\left(\{\triangle\}_2 a_m\right) \geq 0$. Therefore by the Lemma 1.2 we have

$$g(x) = \frac{1}{\left(2\cos\frac{x}{2}\right)^2}\sum_{m=1}^{\infty}\{\triangle\}_2 a_{m-1}\sin mx.$$

Applying Abel's transformation we obtain

$$g(x) = \frac{1}{\left(2\cos\frac{x}{2}\right)^2}\sum_{m=1}^{\infty}\triangle\left(\{\triangle\}_2 a_{m-1}\right)\widetilde{D}_m(x), \tag{2.6}$$

where $\widetilde{D}_m(x) = \sum_{i=1}^m \sin ix$ is the conjugate Dirichlet kernel.

For $x \in (0,\pi]$ and $m = 0,1,2,\ldots,$ introduce the functions

$$\varphi_m(x) := -\frac{\cos\left(m+1/2\right)x}{2\sin x/2}$$

and

$$\psi_m(x) := \sum_{i=0}^{m}\varphi_i(x) = -\frac{\sin\left(m+1\right)x}{4\sin^2(x/2)}.$$

Denoting $H(x) := \sum_{m=1}^\infty \triangle\left(\{\triangle\}_2 a_{m-1}\right)\widetilde{D}_m(x)$ one can write

$$\begin{aligned} H(x) &= \sum_{m=1}^{\ell-1}\triangle\left(\{\triangle\}_2 a_{m-1}\right)\widetilde{D}_m(x) \\ &\quad + \sum_{m=\ell}^{\infty}\triangle\left(\{\triangle\}_2 a_{m-1}\right)\left(\frac{1}{2}\cot\frac{x}{2} + \varphi_m(x)\right) \\ &= \frac{\{\triangle\}_2 a_\ell}{2}\cot\frac{x}{2} + \sum_{m=1}^{\ell-1}\triangle\left(\{\triangle\}_2 a_{m-1}\right)\widetilde{D}_m(x) + \sum_{m=\ell}^{\infty}\triangle\left(\{\triangle\}_2 a_{m-1}\right)\varphi_m(x) \\ &= \frac{a_{\ell-1}+2a_\ell+a_{\ell+1}}{2}\cot\frac{x}{2} + E_\ell(x) + F_\ell(x). \qquad (2.7)\end{aligned}$$

We shall make use of the representation (2.7) for $x \in I_\ell$, and from now and till the end of the proof of our theorem we supose that $x \in I_\ell$ but we shall not remind of it.

The following estimate is true in view of the monotonous decay of $\triangle\left(\{\triangle\}_2 a_{m-1}\right)$ and the positivity of $\widetilde{D}_m(x)$ for $m \leq \ell$:

$$\begin{aligned} E_\ell(x) &\geq \triangle\left(\{\triangle\}_2 a_{\ell-1}\right)\sum_{m=1}^{\ell-1}\left(\frac{1}{2}\cot\frac{x}{2} + \varphi_m(x)\right) \\ &= \triangle\left(\{\triangle\}_2 a_{\ell-1}\right)\left(\frac{\ell}{2}\cot\frac{x}{2} + \psi_{\ell-1}(x)\right) = \frac{\triangle\left(\{\triangle\}_2 a_{\ell-1}\right)}{4\sin^2(x/2)}\left(\ell\sin x - \sin \ell x\right). \qquad (2.8)\end{aligned}$$

Let us estimate $F_\ell(x)$ from above. Applying Abel's transformation we have

$$\begin{aligned} |F_\ell(x)| &= \left| \lim_{n\to\infty} \left\{ \sum_{m=\ell}^{n-1} \triangle_2 \left(\{\triangle\}_2 a_{m-1}\right) \psi_m(x) \right. \right. \\ & \left. \left. +\triangle\left(\{\triangle\}_2 a_{n-1}\right)\psi_n(x) - \triangle\left(\{\triangle\}_2 a_{\ell-1}\right)\psi_{\ell-1}(x) \right\} \right| \\ &\leq \sum_{m=\ell}^{\infty} \triangle_2\left(\{\triangle\}_2 a_{m-1}\right) |\psi_m(x) - \psi_{\ell-1}(x)| \\ &\leq \frac{\triangle\left(\{\triangle\}_2 a_{\ell-1}\right)}{4\sin^2(x/2)} (1+\sin \ell x)\,. \end{aligned} \tag{2.9}$$

From (2.8) and (2.9), in a similiar way as Telyakovskiĭ did [6], for $\ell \geq 11$ we can show that

$$\frac{1}{2}E_\ell(x) + F_\ell(x) > 0.$$

Further, if $m < \ell$, then

$$\widetilde{D}_m(x) \geq \sum_{i=1}^{m} \frac{2}{\pi} ix \geq \frac{m(m+1)}{\ell+1} > \frac{m^2}{\ell}.$$

Therefore,

$$\frac{1}{2}E_\ell(x) \geq \frac{1}{2\ell}\sum_{m=1}^{\ell-1} m^2 \triangle\left(\{\triangle\}_2 a_{m-1}\right). \tag{2.10}$$

From (2.10), (2.7), and (2.6) we obtain the estimate of $g(x)$ from below

$$g(x) \geq \frac{1}{\left(2\cos\frac{x}{2}\right)^2}\left(\frac{a_{\ell-1}+2a_\ell+a_{\ell+1}}{2}\cot\frac{x}{2} + \frac{1}{2\ell}\sum_{m=1}^{\ell-1} m^2\triangle\left(\{\triangle\}_2 a_{m-1}\right)\right).$$

Since

$$\widetilde{D}_m(x) \leq m^2 x \leq \frac{\pi m^2}{\ell},$$

then

$$E_\ell(x) \leq \frac{\pi}{\ell}\sum_{m=1}^{\ell-1} m^2 \triangle\left(\{\triangle\}_2 a_{m-1}\right). \tag{2.11}$$

For the estimate (2.9) we can write

$$|F_\ell(x)| \leq \frac{\triangle\left(\{\triangle\}_2 a_{\ell-1}\right)}{2\sin^2(x/2)} \leq \triangle\left(\{\triangle\}_2 a_{\ell-1}\right)\frac{\pi^2}{2x^2} \leq \frac{(\ell+1)^2}{2}\triangle\left(\{\triangle\}_2 a_{\ell-1}\right),$$

and for $\ell \geq 11$

$$\frac{(\ell+1)^2}{2} < \frac{2,4}{\ell}\sum_{m=1}^{\ell-1} m^2,$$

hence, by reason of the monotonicity of $\triangle\left(\{\triangle\}_2 a_{\ell-1}\right)$ we get

$$|F_\ell(x)| \leq \frac{2,4}{\ell}\sum_{m=1}^{\ell-1} m^2\triangle\left(\{\triangle\}_2 a_{m-1}\right). \tag{2.12}$$

Estimates (2.12), (2.13), and (2.7) give the estimate of $g(x)$ from above

$$g(x) \leq \frac{1}{\left(2\cos\frac{x}{2}\right)^2}\left(\frac{a_{\ell-1}+2a_\ell+a_{\ell+1}}{2}\cot\frac{x}{2}+\frac{6}{\ell}\sum_{m=1}^{\ell-1}m^2\triangle\left(\{\triangle\}_2 a_{m-1}\right)\right).$$

The proof is completed. □

It follows from Theorem 2.3 that for $x \in I_\ell$ in a sufficiently small neighbourhood of the origin we have

$$g(x) = \frac{1}{2\,(1+\cos x)}\left(\frac{\{\triangle\}_2 a_{\ell-1}}{2}\cot\frac{x}{2}+O\left(\frac{1}{\ell}\sum_{m=1}^{\ell-1}m^2\triangle\left(\{\triangle\}_2 a_{m-1}\right)\right)\right). \tag{2.13}$$

Corollary 2.3. *Assume that* $\{a_m\}_{m=1}^{\infty}$ *is a* $(2,2)$*-monotone sequence. Then the following order equality is true*

$$g(x) \sim (\ell-1)\{\triangle\}_2 a_{\ell-1} + \frac{1}{\ell}\sum_{m=1}^{\ell-1} m\{\triangle\}_2 a_{m-1}.$$

Proof. Since $\lim_{x\to 0} x\cot x = 1$, then it is enough to prove that

$$\frac{1}{\ell}\sum_{m=1}^{\ell-1}(2m-1)\{\triangle\}_2 a_{m-1} - (\ell-1)\{\triangle\}_2 a_{\ell-1} \leq \frac{1}{\ell}\sum_{m=1}^{\ell-1} m^2\triangle\left(\{\triangle\}_2 a_{m-1}\right)$$

and

$$\frac{1}{\ell}\sum_{m=1}^{\ell-1} m^2\triangle\left(\{\triangle\}_2 a_{m-1}\right) \leq \frac{1}{\ell}\sum_{m=1}^{\ell-1}(2m-1)\{\triangle\}_2 a_{m-1}.$$

Indeed, putting $\{\triangle\}_2 a_{m-1} := b_{m-1}$, we can write

$$\begin{aligned}\frac{1}{\ell}\sum_{m=1}^{\ell-1} m^2\triangle b_{m-1} &= \frac{1}{\ell}\left[b_0+3b_1+5b_2+\cdots+(2\ell-3)b_{\ell-2}-(\ell-1)^2 b_{\ell-1}\right]\\ &\leq \frac{1}{\ell}\sum_{m=1}^{\ell-1}(2m-1)b_{m-1} \leq \frac{1}{\ell}\sum_{m=1}^{\ell-1}(2m-1)\{\triangle\}_2 a_{m-1},\end{aligned} \tag{2.14}$$

because by the Lemma 1.1, $b_{m-1} \geq 0$ holds true.

On the other hand we get

$$(\ell-1)^2 b_{\ell-1} \leq \ell(\ell-1)b_{\ell-1},$$

therefore the proof of the corollary is completed. □

Remark 2.1. *Similar statement with Theorem 2.3 holds true for the series (1.1) with* $(2,1)$*-monotone coefficients.*

References

[1] A. Yu. Popov, Estimates for the sums of sine series with monotone coefficients of certain classes, (Russian) *Mat. Zametki* **74** (2003), no. 6, 877–888; translation in *Math. Notes* 74 (2003), no. 5-6, 829–840.

[2] B. V. Simonov, On trigonometric series with (k,s)-monotone coefficients in weighted spaces, *Izv. Vyssh. Uchebn. Zaved., Mat.* 2003, no. **5**, 42-54.

[3] R. Salem, Détermination de l'ordre de grandeur á l'origine de certains séries trigonometriques, C. R. Acad. Sci. Paris **186** (1928), 1804-1806.

[4] R. Salem, Essais sur les series trigonometriques, Paris, 1940.

[5] S. Aljančić, R. Bojanić and M. Tomić, Sur le comportement asymtotique au voisinage de zéro des séries trigonométriques de sinus á coefficients monotones, Publ. Inst. Math. Acad. Serie Sci., **10** (1956), 101-120.

[6] S. A. Telyakovskiĭ, On the behavior near the origin of the sine series with convex coefficients, Publ. Inst. Math. Nouvelle serie, **58** (72) (1995), 43-50.

[7] Xh. Z. Krasniqi, Certain estimates for double sine series with multiple–monotone coefficients, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **27** (2011), no. 2, 233–243.

[8] Xh. Z. Krasniqi, On the behavior near the origin of double sine series with monotone coefficients. *Math. Bohem.* **134** (2009), no. 3, 255–273.

[9] Xh. Z. Krasniqi, Some estimates of r-th derivative of the sums of sine series with monotone coefficients of higher order near the origin. *Int. J. Math. Anal. (Ruse)* **3** (2009), no. 1-4, 59–69.

[10] Xh. Z. Krasniqi, N. L. Braha, On the behavior of r-derivative near the origin of sine series with convex coefficients. *JIPAM. J. Inequal. Pure Appl. Math.* **8** (2007), no. 1, Article 22, 6 pp. (electronic).

[11] W. H. Young, On the mode of oscillation of a Fourier series and of its allied series, Proc. London Math. Soc. **12** (1913), 433-452.

[12] N. Bary, Trigonometric series, Moscow, 1961 (in Russian).

University of Prishtina "Hasan Prishtina", Faculty of Education, Department of Mathematics and Informatics, Avenue "Mother Theresa" 5, 10000 Prishtina, Kosova