



SOLVABILITY OF MULTI-POINT VALUE PROBLEMS WITH INTEGRAL CONDITION AT RESONANCE

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ABSTRACT. In this paper, we study a boundary value problem at resonance with a multi-integral boundary conditions. By constructing suitable operators, we establish an existence theorem upon the coincidence degree theory of Mawhin. An example is given to show the effectiveness of our results.

1. INTRODUCTION

Boundary value problem involves ordinary differential equation with non local condition appears in physical science and applied mathematics. Moreover the theory of boundary value problems with integral condition is found in deferent areas like applied mathematics and applied physics for example plasma physics, heat conduction, thermo-elasticity, underground water flew. In recent years, the boundary value problem at resonance for ordinary differential equations have been extensively studied and many results have been obtained, we refer to [1], [2], [5]- [7], [10]- [12] and the references therein. Moreover, lots of works on multi-point boundary value problems have appeared, for examples, see [3]- [10].

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The goal of this paper is to provide sufficient conditions that ensure the existence of solutions for the following multi-point boundary value problem (BVP)

$$x''(t) = f(t, x(t), x'(t)), t \in (0, 1) \quad (1.1)$$

$$x(0) = 0, x(1) = \sum_{k=1}^m \lambda_k \int_0^{\eta_k} x(t) dt, \quad (1.2)$$

where $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is caratheodory function, $\eta_k \in (0, 1)$ and $\sum_{k=1}^m \lambda_k \eta_k^2 = 2$.

We say that the (BVP) (1.1)-(1.2) is a resonance problem if the dimension of the kernel of the linear operator $Lx = x''$ is not less than one under the boundary conditions 1.2. Otherwise, we call them a problem at nonresonance.

In the present work, if $\sum_{k=1}^m \lambda_k \eta_k^2 = 2$, then, (BVP) (1.1)-(1.2) is at resonance, since equation

$$x''(t) = 0, t \in (0, 1)$$

with boundary condition 1.2 has nontrivial solutions $x(t) = ct, c \in \mathbb{R}, t \in [0, 1]$.

This paper is organized as follows, in section 2 we stated some definitions and lemmas needed in our proofs. In section 3 we treated the existence of solution by using the coincidence degree theory of Mawhin [9], [10]. we ended by giving an example illustrating the previous results.

2. PRELIMINARIES

For the convenience of the reader to understand the coincidence degree theory, we briefly recall some notations, definitions and theorems which will be used later.

Definition 2.1. Let X, Y , be real Banach spaces, the linear operator $L : \text{dom} L \subset X \rightarrow Y$ is said to be a Fredholm map of index zero provided that $\ker L$, the kernel of L , is of the same finite dimension as the $Y/\text{Im} L$, where $\text{Im} L$ is the image of L .

Let L be a Fredholm map of index zero, and $P : X \rightarrow X, Q : Y \rightarrow Y$ be continuous projectors, such that $\text{Im} P = \ker L, \text{Ker} Q = \text{Im} L$. Then $X = \ker L \oplus \ker P, Y = \text{Im} L \oplus \text{Im} Q$, thus $L|_{\text{dom} L \cap \ker P} : \text{dom} L \cap \ker P \rightarrow \text{Im} L$ is invertible, denote its inverse by K_P .

Definition 2.2. Let L be a Fredholm map of index zero and Ω be an open bounded subset of X , such that $\text{dom} L \cap \Omega \neq \phi$, the map $N : X \rightarrow Y$ is said to be L -compact on $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \rightarrow X$ is compact.

Theorem 2.1. ([13]) Let L be a Fredholm operator of index zero and let N be L -compact on $\bar{\Omega}$. Assume that the following conditions are satisfied.

(i) $Lx \neq \lambda Nx$, for every $(x, \lambda) \in [(domL \setminus KerL) \cap \partial\Omega] \times (0, 1)$.

(ii) $Nx \notin Im L$, for every $x \in KerL \cap \partial\Omega$.

(iii) $deg(QN|_{ker L}, ker L \cap \Omega, 0) \neq 0$, where $Q : Y \rightarrow Y$ is a projection as above with $Im L = ker Q$.

Then, the equation $Lx = Nx$ has at least one solution in $domL \cap \bar{\Omega}$.

In the following, we shall use the classical Banach spaces $X = C^1[0, 1]$ and $Y = L^1[0, 1]$ equipped respectively with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, $\|x\|_\infty = \max_{t \in [0, 1]} |x(t)|$ and $\|y\|_1 = \int_0^1 |y(t)| dt$.

We will use the space $AC^2[a, b] = \{u \in C^1[a, b], u' \in AC[a, b]\}$, where $AC[a, b]$ is the space of absolutely continuous functions on $[a, b]$.

3. EXISTENCE OF SOLUTIONS

Define the operator $L : dom L \subset X \rightarrow Y$ by $Lx = x''$, where

$$domL = \left\{ x \in W^{2,1}(0, 1) : x(0) = 0, x(1) = \sum_{k=1}^m \lambda_k \int_0^{\eta_k} x(t) dt; \eta_k \in (0, 1), \sum_{k=1}^m \lambda_k \eta_k^2 = 2 \right\}.$$

Let $N : X \rightarrow Y$ the operator

$$Nx(t) = f(t, x(t), x'(t)), t \in (0, 1).$$

Then, the equation 1.1, can be written as $Lx = Nx$.

Next, in order to apply Theorem 2.1, we need the following Lemma.

Lemma 3.1. (i) $ker L = \{x \in domL : x = ct, c \in \mathbb{R}, t \in [0, 1]\}$;

(ii) $Im L = \{y \in Y : \int_0^1 (1-s)y(s) ds - \frac{1}{2} \sum_{k=1}^m \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 y(s) ds = 0\}$;

(iii) $L : domL \subset X \rightarrow Y$ is a Fredholm operator of index zero, and the linear continuous projector operator $Q : Y \rightarrow Y$ can be defined as

$$Qy(t) = k \cdot (Ry) \cdot t$$

where

$$Ry = \int_0^1 (1-s)y(s) ds - \frac{1}{2} \sum_{k=1}^m \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 y(s) ds$$

and

$$k^{-1} = \frac{1}{6} - \frac{1}{24} \sum_{k=1}^m \lambda_k \eta_k^4.$$

(iv) The linear operator $K_p : Im L \rightarrow domL \cap ker P$ can be written as

$$K_p y = \int_0^t (t-s)y(s) ds,$$

Moreover, for all $y \in \text{Im } L$ we have

$$\|K_p y\| \leq \|y\|_1. \tag{3.1}$$

Proof. (i) For $x \in \ker L$, we have $x''(t) = 0$. Then, we obtain $x(t) = a + bt$, where $a, b \in \mathbb{R}$.

From $x(0) = 0$, we have $a = 0$.

Again, since $\sum_{k=1}^m \lambda_k \eta_k^2 = 2$, then from $x(1) = \sum_{k=1}^{k=m} \lambda_k \int_0^{\eta_k} x(t) dt$, we get

$$\ker L = \{x \in \text{dom } L : x = ct, c \in \mathbb{R}, t \in [0, 1]\}$$

(ii) To prove that

$$\text{Im } L = \{y \in Y : \int_0^1 (1-s)y(s)ds - \frac{1}{2} \sum_{k=1}^{k=m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 y(s)ds = 0\},$$

we show that, the linear equation

$$x'' = y, \tag{3.2}$$

has a solution $x(t)$ satisfied, $x(0) = 0$, $x(1) = \sum_{k=1}^m \lambda_k \int_0^{\eta_k} x(t) dt$, $\sum_{k=1}^m \lambda_k \eta_k^2 = 2$, if and only if

$$\int_0^1 (1-s)y(s)ds - \frac{1}{2} \sum_{k=1}^m \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 y(s)ds = 0.$$

In fact, by integrating equation 3.2 and tacking and account that $x(0) = 0$, we get

$$x(t) = x'(0)t + \int_0^t (t-s)y(s)ds$$

again from $x(1) = \sum_{k=1}^{k=m} \lambda_k \int_0^{\eta_k} x(t) dt$, we obtain

$$\begin{aligned} x'(0) + \int_0^1 (1-s)y(s)ds &= \sum_{k=1}^m \lambda_k \int_0^{\eta_k} \left[x'(0)t + \int_0^t (t-s)y(s)ds \right] dt \\ &= x'(0) + \sum_{k=1}^m \lambda_k \left[\int_0^{\eta_k} \int_0^t (t-s)y(s)ds dt \right] \\ &= x'(0) + \frac{1}{2} \sum_{k=1}^m \lambda_k \left[\int_0^{\eta_k} (\eta_k - s)^2 y(s)ds \right] \end{aligned}$$

which implies

$$\int_0^1 (1-s)y(s)ds - \frac{1}{2} \sum_{k=1}^m \lambda_k \left[\int_0^{\eta_k} (\eta_k - s)^2 y(s)ds \right] = 0$$

(iii) For $y \in Y$, we take the projector Q as

$$Qy = k \left(\int_0^1 (1-s)y(s)ds - \frac{1}{2} \sum_{k=1}^m \lambda_k \left[\int_0^{\eta_k} (\eta_k - s)^2 y(s)ds \right] \right) t,$$

where

$$k^{-1} = \int_0^1 (1-s)s ds - \frac{1}{2} \sum_{k=1}^m \lambda_k \left[\int_0^{\eta_k} (\eta_k - s)^2 s ds \right] = \frac{1}{6} - \frac{1}{24} \sum_{k=1}^m \lambda_k \eta_k^4.$$

It follows from $\ker Q = \text{Im } L$ that $Y = \text{Im } L \oplus \text{Im } Q$, thus, $\text{co dim } L = \dim \ker L = 1$.

Hence, L is a Fredholm operator of index zero.

(iv) Taking $P : X \rightarrow X$ as follows,

$$Px = x'(0)t$$

then, the generalized inverse $K_p : \text{Im } L \rightarrow \text{dom } L \cap \ker P$ of L can be written as

$$K_p y = \int_0^t (t-s)y(s) ds$$

In fact, for $y \in \text{Im } L$, we have

$$(LK_p)y(t) = y(t)$$

and, for $x \in \text{dom } L \cap \ker P$, we know

$$(K_p L)x(t) = \int_0^t (t-s)x''(s) ds = x(t)$$

This shows that $K_p = (L|_{\text{dom } L \cap \ker P})^{-1}$.

Finally from the definition of K_p , we get

$$\|K_p y\|_\infty \leq \int_0^1 (1-s)|y(s)| ds \leq \int_0^1 |y(s)| ds = \|y\|_1.$$

□

Now, we give the result on the existence of a solution for the problem (1.1)-(1.2).

Theorem 3.1. *Assume that the following conditions are satisfied :*

(H₁) *There exists nonnegative functions $\alpha, \beta, \gamma \in L^1[0, 1]$, such that, for all $(x, y) \in \mathbb{R}^2, t \in [0, 1]$, satisfying the following inequalities:*

$$|f(t, x, y)| \leq \alpha(t)|x| + \beta(t)|y| + \gamma(t) \tag{3.3}$$

(H₂) *There exists a constant $M > 0$, such that, for $x \in \text{dom } L$, if $|x'(t)| > M$, for all $t \in [0, 1]$, then,*

$$\int_0^1 (1-s)f(s, x(s), x'(s)) ds - \frac{1}{2} \sum_{k=1}^m \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s, x(s), x'(s)) ds \neq 0$$

(H₃) *There exists a constant $M^* > 0$, such that, for any $x(t) = ct \in \ker L$ with $|c| > M^*$ either*

$$c \left[\int_0^1 (1-s)f(s, x(s), c)ds - \frac{1}{2} \sum_{k=1}^m \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s, x(s), c)ds \right] < 0$$

or

$$c \left[\int_0^1 (1-s)f(s, x(s), c)ds - \frac{1}{2} \sum_{k=1}^{k=m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s, x(s), c)ds \right] > 0$$

then BVP (1.1)-(1.2) has at least one solution in $C^1 [0, 1]$, provided $\|\alpha(t)\| + \|\beta(t)\| \leq \frac{1}{2}$.

Next, in order to prove Theorem 3.1, we need the following Lemma.

Lemma 3.2. *Suppose that Ω is an open bounded subset of X such that $\text{dom } L \cap \bar{\Omega} \neq \emptyset$. Then N is L -compact on $\bar{\Omega}$.*

Proof. Suppose that $\Omega \subset X$ is a bounded set. Without loss of generality, we may assume that $\Omega = B(0, r)$, then for any $x \in \bar{\Omega}$, $\|x\| \leq r$. For $x \in \bar{\Omega}$, and by condition 3.3, we obtain

$$\begin{aligned} |QNx| &\leq k \int_0^1 |f(s, x(s), x'(s))| ds + \frac{k}{2} \sum_{k=1}^m \lambda_k \int_0^{\eta_k} |f(s, x(s), x'(s))| ds \\ &\leq k \int_0^1 |\alpha(s)| |x(s)| + |\beta(s)| |x'(s)| + |\gamma(t)| ds + \frac{k}{2} \sum_{k=1}^m \lambda_k \int_0^{\eta_k} |\alpha(s)| |x(s)| + |\beta(s)| |x'(s)| + |\gamma(t)| ds \\ &\leq \left(k + \frac{k}{2} \sum_{k=1}^m \lambda_k \right) [r (\|\alpha(t)\|_1 + \|\beta(t)\|_1) + \|\gamma(t)\|_1] \end{aligned}$$

thus,

$$\|QNx\|_1 \leq \left(k + \frac{k}{2} \sum_{k=1}^m \lambda_k \right) [r (\|\alpha(t)\|_1 + \|\beta(t)\|_1) + \|\gamma(t)\|_1], \tag{3.4}$$

which implies that $QN(\bar{\Omega})$ is bounded. Next, we show that $K_P(I - Q)N(\bar{\Omega})$ is compact. For $x \in \bar{\Omega}$, by condition 3.3 we have

$$\|Nx\|_1 = \int_0^1 |f(t, x(s), x'(s))| ds \leq r (\|\alpha(t)\|_1 + \|\beta(t)\|_1) + \|\gamma(t)\|_1. \tag{3.5}$$

On the other hand, from the definition of K_P and together with (3.1), (3.4) and (3.5) one gets

$$\begin{aligned} \|K_P(I - Q)Nx\| &\leq \|(I - Q)Nx\|_1 \leq \|Nx\|_1 + \|QNx\|_1 \\ &\leq \left(1 + k + \frac{k}{2} \sum_{k=1}^m \lambda_k \right) (r (\|\alpha(t)\|_1 + \|\beta(t)\|_1) + \|\gamma(t)\|_1). \end{aligned}$$

It follows that $K_P(I - Q)N(\bar{\Omega})$ is uniformly bounded.

Let us prove that $K_P(I - Q)N(\bar{\Omega})$ is equicontinuous. For any $x \in \bar{\Omega}$, and any $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, we have

$$\begin{aligned} (K_P(I - Q)Nx)(t_1) - (K_P(I - Q)Nx)(t_2) &= \\ &= \left| \int_0^{t_1} (t_1 - s)(I - Q)Nx(s) ds \right. \end{aligned}$$

$$\begin{aligned} & \left| - \int_0^{t_2} (t_2 - s)(I - Q)Nx(s) ds \right| \\ & \leq \left[\int_0^{t_1} (t_2 - t_1)|(I - Q)Nx(s)| ds \right. \\ & \quad \left. + \int_{t_1}^{t_2} (t_2 - s)|(I - Q)Nx(s)| ds \right] \rightarrow 0, \end{aligned}$$

as $t_1 \rightarrow t_2$. On the other hand we have

$$\begin{aligned} & |(K_P(I - Q)Nx)'(t_1) - (K_P(I - Q)Nx)'(t_2)| \\ & = \left| \int_0^{t_1} (I - Q)Nx(s) ds - \int_0^{t_2} (I - Q)Nx(s) ds \right| \rightarrow 0, \end{aligned}$$

as $t_1 \rightarrow t_2$. So $K_P(I - Q)N(\overline{\Omega})$ is equicontinuous. so, the Ascoli-Arzela theorem ensure that $K_P(I - Q)N : \overline{\Omega} \rightarrow X$ is compact . The proof is complete □

Now we give the proof of Theorem 3.1

Proof. Firstly, we need to construct the set Ω satisfying all the conditions in Theorem 2.1, which is separated into the following three steps.

Step 1. First we show that the following set

$$\Omega_1 = \{x \in \text{dom}L \setminus \ker L : Lx = \lambda Nx, \text{ for some } \lambda \in [0, 1]\},$$

is bounded.

In fact, let $x \in \Omega_1$, we have $Lx = \lambda Nx$ and $Lx \neq 0$, so $\lambda \neq 0$, thus $QNx = 0$, from which it yields

$$\int_0^1 (1 - s)f(s, x(s), x'(s))ds - \frac{1}{2} \sum_{k=1}^{k=m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s, x(s), x'(s))ds = 0,$$

thus, from condition (H_2) of Theorem 3.1, there exists $t_0 \in [0, 1]$, such that $|x'(t_0)| \leq M$.

In view of

$$x'(0) = x'(t_0) - \int_0^{t_0} x''(t)dt$$

then,

$$\|Px\| = |x'(0)| \leq M + \|x''\|_1 \leq M + \|Nx\|_1 \tag{3.6}$$

Again for $x \in \Omega_1$, $x \in \text{dom}L \setminus \ker L$, then $(I - P)x \in \text{dom}L \cap \ker P$ and $LPx = 0$,

thus from Lemma 3.1 , we know

$$\|(I - P)x\| = \|K_P L(I - P)x\| \leq \|L(I - P)x\|_1 = \|Lx\|_1 \leq \|Nx\|_1 \tag{3.7}$$

From (3.6) and (3.7), we have

$$\|x\| \leq \|Px\| + \|(I - P)x\| \leq M + 2\|Nx\|_1 \tag{3.8}$$

If condition 3.3 holds, then from (3.8) , we obtain

$$\|x\| \leq 2 \left[\|\alpha\|_1 \|x\|_\infty + \|\beta\|_1 \|x'\|_\infty + \|\gamma\|_1 + \frac{M}{2} \right]. \tag{3.9}$$

Since $\|x\|_\infty \leq \|x\|$, then from (3.9) it yields

$$\|x\|_\infty \leq \frac{2}{1 - 2\|\alpha\|_1} \left[\|\beta\|_1 \|x'\|_\infty + \|\gamma\|_1 + \frac{M}{2} \right] \tag{3.10}$$

By using $\|x'\|_\infty \leq \|x\|$, (3.9) and (3.10) one has

$$\|x'\|_\infty \left[1 - \frac{2\|\beta\|_1}{1 - 2\|\alpha\|_1} \right] \leq \frac{2}{1 - 2\|\alpha\|_1} \left[\|\gamma\|_1 + \frac{M}{2} \right]$$

therefore,

$$\|x'\|_\infty \left[\frac{1 - 2\|\alpha\|_1 - 2\|\beta\|_1}{1 - 2\|\alpha\|_1} \right] \leq \frac{1}{1 - 2\|\alpha\|_1} [2\|\gamma\|_1 + M]$$

i.e.

$$\|x'\|_\infty \leq \frac{2 \left[\|\gamma\|_1 + \frac{M}{2} \right]}{1 - 2\|\alpha\|_1 - 2\|\beta\|_1}, \tag{3.11}$$

thus, from (3.10) and (3.11), there exist $M_1 > 0$, such that

$$\|x\| \leq M_1$$

which proves that Ω_1 is bounded.

Step 2. We will show that the set

$$\Omega_2 = \{x \in \ker L : Nx \in \text{Im } L\}$$

is bounded.

Let $x \in \Omega_2$, then $x(t) = bt, b \in \mathbb{R}, t \in [0, 1]$, and $Q Nx = 0$, therefor

$$\int_0^1 (1 - s)f(s, bs, b)ds - \frac{1}{2} \sum_{k=1}^{k=m} \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s, bs, b)ds = 0$$

In view of condition (H_2) of Theorem 3.1, there exists $t_0 \in [0, 1]$, such that $|x'(t_0)| \leq M$ that is $|b| \leq M$,

so

$$\|x'\|_\infty = |b| \leq M,$$

from which, we get

$$\|x\| = \max \{ \|x\|_\infty, \|x'\|_\infty \} = |b| \leq M,$$

which implies that Ω_2 is bounded.

Step 3. In the next, we show the boundedness of the following set

$$\Omega_3 = \{x \in \ker L : -\lambda Jx + (1 - \lambda)QNx = 0, \lambda \in [0, 1]\},$$

where, $J : \ker L \rightarrow \text{Im } Q$ is the linear isomorphism given by $J(ct) = ct, \forall c \in \mathbb{R}, t \in [0, 1]$.

If the first part of condition (H_3) of Theorem 3.1 holds, that is, there exists $M^* > 0$, such that, for any $c \in \mathbb{R}$, if $|c| > M^*$, then

$$c \left[\int_0^1 (1 - s)f(s, x(s), c)ds - \frac{1}{2} \sum_{k=1}^m \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s, x(s), c)ds \right] < 0 \tag{3.12}$$

Let $x \in \Omega_3$, then $x(t) = ct$ and $\lambda Jx = (1 - \lambda)QNx$, that is equivalently written as

$$\lambda c = (1 - \lambda)k \left[\int_0^1 (1 - s)f(s, x(s), c)ds - \frac{1}{2} \sum_{k=1}^m \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s, x(s), c)ds \right],$$

if $\lambda = 1$, then $c = 0$. Otherwise, if $|c| > M^*$, then in view of (3.12) and together with the fact that $k > 0$, we get

$$\lambda c^2 = (1 - \lambda)kc \left[\int_0^1 (1 - s)f(s, x(s), c)ds - \frac{1}{2} \sum_{k=1}^m \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s, x(s), c)ds \right] < 0$$

which contradicts $\lambda c^2 \geq 0$. Thus, $\Omega_3 \subset \{x \in \text{Ker } L : \|x\| \leq M^*\}$ is bounded.

On the other hand, If the second part of condition (H_3) of Theorem 3.1 holds, by applying similar reasoning as above, we can prove that Ω_3 is bounded.

Finally, we shall prove that all conditions of Theorem 2.1 are satisfied.

Let Ω be a bounded open subset of X , such that $\cup_{i=1}^3 \bar{\Omega}_i \subset \Omega$. It follows from Lemma 3.1 that L is a Fredholm operator of index zero. By Lemma 3.2, we have N is L -compact on $\bar{\Omega}$. By virtue of the definition of Ω , the assumptions (i) and (ii) are satisfied.

Now we prove that condition (iii) of Theorem 2.1 is satisfied. Let $H(x, \lambda) = \pm \lambda Jx + (1 - \lambda)QNx$. Since $\bar{\Omega}_3 \subset \Omega$, then $H(x, \lambda) \neq 0$ for every $x \in \ker L \cap \partial\Omega$. By the homotopy property of degree, we get

$$\begin{aligned} \deg(QN|_{\ker L}, \Omega \cap \ker L, 0) &= \deg(H(\cdot, 0), \Omega \cap \ker L, 0) \\ &= \deg(H(\cdot, 1), \Omega \cap \ker L, 0) \\ &= \deg(\pm J, \Omega \cap \ker L, 0) \neq 0. \end{aligned}$$

Hence, the assumption (iii) of Theorem 2.1 holds.

Since all hypothesis of Theorem 2.1 are satisfied. Therefore, equation $Lx = Nx$ has at least one solution in $\text{dom } L \cap \bar{\Omega}$; i.e. boundary value problem (1.1)-(1.2) has at least one solution in X . The proof is completed. \square

4. AN ILLUSTRATIVE EXAMPLE

In this section we give an example to illustrate the usefulness of our main results.

Consider the multi-point boundary value problem

$$(P) \begin{cases} x''(t) = f(t, x(t), x'(t)), t \in (0, 1), \\ x(0) = 0, x(1) = 4 \int_0^{\frac{1}{2}} x(t)dt + \frac{16}{9} \int_0^{\frac{3}{4}} x(t)dt \end{cases}$$

with

$$f(t, x, y) = \frac{t}{4}x + \left(\frac{1-t^2}{4}\right)y + t.$$

Since $\sum_{k=1}^2 \lambda_k \eta_k^2 = 4 \left(\frac{1}{2}\right)^2 + \frac{16}{9} \left(\frac{3}{4}\right)^2 = 2$, the problem (P) is at resonance.

We have

$$|f(t, x, y)| \leq \alpha(t)|x| + \beta(t)|y| + \gamma(t),$$

where $\alpha(t) = \frac{t}{4}, \beta(t) = \frac{1-t^2}{4}$ and $\gamma(t) = t$, then α, β are nonnegative and belong to $L^1[0, 1]$, so, hypothesis (H_1) of Theorem 3.1 is satisfied.

We claim that condition (H_2) of Theorem 3.1 is satisfied, indeed, for $M = 1.8214 > 0$ and $x \in \text{dom}L$, $x(t) = ct$, if $|x'(t)| > M$, for all $t \in [0, 1]$, then,

$$\begin{aligned} & \int_0^1 (1-s)f(s, x(s), x'(s))ds - \frac{1}{2} \sum_{k=1}^2 \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s, x(s), x'(s))ds \\ &= \int_0^1 (1-s)\left(\frac{c}{4} + s\right)ds - \frac{1}{2} \left(4 \int_0^{\frac{1}{2}} \left(\frac{1}{2} - s\right)^2 \left(\frac{c}{4} + s\right)ds + \frac{16}{9} \int_0^{\frac{3}{4}} \left(\frac{3}{4} - s\right)^2 \left(\frac{c}{4} + s\right)ds \right) \\ &= \frac{7}{96}c + \frac{17}{128} \neq 0. \end{aligned}$$

Now, for $M^* = 2 > 0$ and any $x(t) = ct \in \ker L$ with $|c| > M^*$, we have

$$\begin{aligned} & c \left[\int_0^1 (1-s)f(s, x(s), x'(s))ds - \frac{1}{2} \sum_{k=1}^2 \lambda_k \int_0^{\eta_k} (\eta_k - s)^2 f(s, x(s), x'(s))ds \right] \\ &= \frac{7}{96}c^2 + \frac{17}{128}c > 0, \end{aligned}$$

consequently, condition (H_3) of Theorem 3.1 is satisfied.

Finally, a simple calculus gives $\|\alpha\|_1 + \|\beta\|_1 = \frac{1}{8} + \frac{1}{6} \leq \frac{1}{2}$. We conclude from Theorem 3.1 that the problem (P) has at least one solution in $C^1[0, 1]$.

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