International Journal of Analysis and Applications
Volume 16, Number 3 (2018), 368-373
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-16-2018-368

SOME REMARKS ON THE ZEROS OF TRIBONACCI POLYNOMIALS

ÖZNUR ÖZTUNÇ KAYMAK*

Balıkesir University, 10145 Balıkesir, Turkey

* *Corresponding author:* oztunc@balikesir.edu.tr

Abstract. In this paper, the zeros of Tribonacci polynomials are studied. The bound containing the zeros of Tribonacci polynomials has been numerically examined with comparisons. On the other hand, a new algorithm is given so that it can be used in other boundary problems.

1. Introduction

The features of polynomials have played an important role in many scientific areas such as control theory, signal processing, crytography and mathematical biology. If we have an accurate estimate of the region containing all the zeros of a polynomial, then the amount of work needed to find exact zeros can be considerably reduced in comparison with using the classical methods, and for this reason there is always a need for better and better estimates for the region containing all the zeros of a polynomial. So, there have been a number of theorems on computations for the roots of polynomials. We consider the generalization of well-known Fibonacci polynomials called $R-$Bonacci polynomials. The polynomial $R_n(x)$ is defined by the following recursive equation in [6] for any integer n and $r \geq 2$:

$$R_n(x) = \sum_{j=0}^{\left[\frac{(r-1)(n-1)}{r}\right]} \binom{n-j-1}{j}_r x^{(r-1)(n-1)-rj}, \tag{1.1}$$

Received 2017-12-19; accepted 2018-02-12; published 2018-05-02.

2010 *Mathematics Subject Classification.* 11B39, 12E10, 30C15.

Key words and phrases. Tribonacci polynomials; R-Bonacci polynomials; Fibonacci Numbers.

where $a_n = \begin{pmatrix} n \\ r \end{pmatrix}_r$ is the $r-$nomial coefficient. Recently, some results have been given on the zeros of $R-$Bonacci polynomials $R_n(x)$ when $r = 2, 3$ in [5]. M. X. He, P. E. Ricci and D. Simon found interesting curves composing of the zeros of $R-$Bonacci polynomials in [12]. (If we set $r = 2, 3$, then the classical Fibonacci polynomials and Tribonacci polynomials are obtained respectively.) Explicit forms have been given for the roots Fibonacci polynomials in [7]. Since these forms have been used in many practical areas eg. Graph Energy, Stable Polynomials, Hand Printed Characters in [15], [16], [18] and [19], it is important to find the zeros of these polynomials. On the other hand, Moore and Prodinger examined the asymptotic behaviour of the maximal roots of Fibonacci-type polynomials in [2] and [9], respectively. Furthermore, the absolute values of complex zeros of Fibonacci-like polynomials have been investigated by Matyas in [8]. Unlike Fibonacci polynomials, explicit forms for the zeros of Tribonacci polynomials have not been found yet. Instead, zero attractors of these polynomials for $r = 3$ are determined by W. Goh, M. X. He and P. E. Ricci in [11]. The symmetric polynomials of the zeros of Tribonacci polynomials are found by M. X. He, D. Simon and P. E. Ricci in [3]. These results are generalized for the zeros of $R-$Bonacci polynomials and their derivatives in [17].

The aim of this paper is to determine the desired region on the complex plane containing the zeros of Tribonacci polynomials. To support this region, we present numerical results for these polynomials to compare the results with the known regions. Afterwards, we develop a new algorithm to use in numerical calculations.

2. Bounds For Polynomials

There have been a number of theorems on computations for the roots of polynomials. These studies date back to the work of Cauchy [1]. Let

$$P(z) = z^n + \sum_{k=0}^{n-1} a_k \ z^k, \ a_k \ \in \mathbb{C} \tag{2.1}$$

be a complex polynomial of degree n. All the zeros of the polynomial $P(z)$ lie in the region

$$B = \{z : |z| < 1 + \gamma\}, \tag{2.2}$$

where $\gamma = \max_{0 \le k \le n-1} |a_k|$.

Recently, Diaz-Barrero have improved this region by describing two annuli containing all the zeros of a polynomial where the inner and outer radii are stated in terms of the well-known Fibonacci numbers, respectively [13].

Theorem 2.1. *[13] Let* $P(z)=\sum\limits_{k=0}^{n}a_k\ z^k\ \ (a_k\neq 0,1\leq k\leq n)$ *be a non-consant polynomial with complex coefficients. Then, all its zeros lie in the ring shaped region*

$$\Re_1=\left\{z\in\mathbb{C}:r_1\leq |z|\leq r_2\right\}, \tag{2.3}$$

where

$$r_1=\frac{3}{2}\min_{1\leq k\leq n}\left\{\frac{2^nF_k\ C(n,k)}{F_{4n}}\left|\frac{a_0}{a_k}\right|\right\}^{\frac{1}{k}} \tag{2.4}$$

and

$$r_2=\frac{2}{3}\max_{1\leq k\leq n}\left\{\frac{F_{4n}}{2^nF_k\ C(n,k)}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{\frac{1}{k}}. \tag{2.5}$$

For another annulus, they considered Lucas numbers which are defined by $L_n=L_{n-1}+L_{n-2}$ for $n\geq 0$ with the initial conditions $L_0=2,\ L_1=1$ (see [5] and [10] for the basic properties of these number sequences).

Theorem 2.2. *[4] Let* $P(z)=\sum\limits_{k=0}^{n}a_k\ z^k\ \ (a_k\neq 0,0\leq k\leq n)$ *be a non-consant polynomial with complex coefficients. Then its all zeros of* $P(z)$ *lie in the annulus*

$$\Re_2=\left\{z\in\mathbb{C}:r_1\leq |z|\leq r_2\right\}, \tag{2.6}$$

where

$$r_1=\min_{1\leq k\leq n}\left\{\frac{L_k}{L_{n+2}-3}\left|\frac{a_0}{a_k}\right|\right\}^{\frac{1}{k}} \tag{2.7}$$

and

$$r_2=\max_{1\leq k\leq n}\left\{\frac{L_{n+2}-3}{L_k}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{\frac{1}{k}}. \tag{2.8}$$

3. Tribonacci polynomials

For $r=3$ in (1.1), $R-$Bonacci polynomials are named Tribonacci polynomials, defined by the recurrence relation

$$T_{n+3}(x)=x^2T_{n+2}(x)+xT_{n+1}(x)+T_n(x) \tag{3.1}$$

with $T_0(x)=0,\ T_1(x)=1$ and $T_2(x)=x^2$. Despite its complicate form, the coefficients of these polynomials have many interesting properties. It is known that Tribonacci zeros constitute 3-stars as shown below in Figure 1.

As seen in the above Figure 1, the zeros of these polynomial being invariant with rotation $2\pi/3$ are separated into 3 sets. So, it may be interesting to find a smallest disc that contains the zeros of these polynomials since the explicit expressions for the zeros of these polynomials are not known. In order to examine a new disc containing the zeros of a given polynomial in (3.1), we have used the following identity

$$\sum_{i=1}^{n}F_i=F_{n+2}-1, \tag{3.2}$$

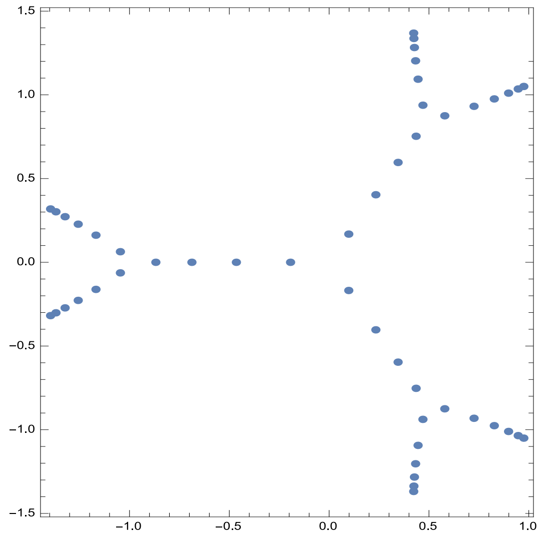

FIGURE 1. The Zeros of $T_{25}(x)$

where F_n is the $n-$th Fibonacci number in [14]. Recall that Fibonacci numbers are defined by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial conditions $F_0 = 0$, $F_1 = 1$. This identity (3.2) can be found in [5]. Now, we consider the following theorem.

Theorem 3.1. *[14] Let* $P(z) = \sum_{k=0}^{n} a_k\ z^k$ $(a_k \neq 0, 0 \leq k \leq n)$ *be a non-consant complex polynomial. Then all its zeros lie in the annulus*

$$\Re_3 = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}, \tag{3.3}$$

where

$$r_1 = \min_{1 \leq k \leq n} \left\{ \frac{F_k}{F_{n+2} - 1} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \tag{3.4}$$

and

$$r_2 = \max_{1 \leq k \leq n} \left\{ \frac{F_{n+2} - 1}{F_k} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}. \tag{3.5}$$

Now we give the following theorem.

Theorem 3.2. *Let* $T_n(x)$ *be a Tribonacci polynomial with* $n \equiv k (\bmod 3)$. *Then all its zeros lie*

$$\Re_3 = \left\{ z \in \mathbb{C} : r_1 \leq |z| \leq \left\{ \frac{F_{n+2} - 1}{F_3} |a_{n-3}| \right\}^{\frac{1}{3}} \right\}, \tag{3.6}$$

where $r_1 = \left\{ \frac{F_3}{F_{n+2}-1} \left| \frac{1}{a_3} \right| \right\}^{\frac{1}{3}}$ *for* $k = 1$ *and* $r_1 = 0$ *for* $k = 0, 2$.

Proof. The result is obviously obtained when the r-nomial coefficients are written instead □

It will be immediately noticed that the areas containing the zeros of Tribonacci polynomials will be a disc or an annulus. After reaching this result for the zeros of these polynomials, it is numerically examined by comparing with the existing boundaries. When comparisons are made, one of the oldest boundaries and the latest bounds are selected. Below is a table giving inner, outer radius and the area for the zeros of these polynomials.

Polynomials	$\Re_1 - Area$	$\Re_2 - Area$	Cauchy′s Bound	$\Re_3 - Area$
$T_3(x)$	$\|z\| \leq 1.31 - 5.39$	$\|z\| \leq 1.55 - 7.54$	$\|z\| \leq 2 - 12.56$	$\|z\| \leq 1.51 - 7.16$
$T_4(x)$	$0.57 \leq \|z\| \leq 2.20 - 14.18$	$0.35 \leq \|z\| \leq 2.80 - 24.24$	$\|z\| \leq 3 - 28.27$	$0.36 \leq \|z\| \leq 2.71 - 22.66$
$T_5(x)$	$\|z\| \leq 4.07 - 52.04$	$\|z\| \leq 4.48 - 63.05$	$\|z\| \leq 4 - 50.26$	$\|z\| \leq 4.32 - 58.62$
$T_6(x)$	$\|z\| \leq 7.90 - 196.06$	$\|z\| \leq 6.83$ / 146.55	$\|z\| \leq 7 - 153.93$	$\|z\| \leq 6.58 - 136.02$
$T_7(x)$	$0.10 \leq \|z\| \leq 15.81 - 785.22$	$0.09 \leq \|z\| \leq 10.16 - 324.26$	$\|z\| \leq 11 - 380.13$	$0.10 \leq \|z\| \leq 9.79 - 301.07$
$T_8(x)$	$\|z\| \leq 32.28 - 3273.53$	$\|z\| \leq 14.89 - 696.52$	$\|z\| \leq 17 - 907.92$	$\|z\| \leq 14.35 - 646.92$
$T_9(x)$	$\|z\| \leq 66.92 - 14069$	$\|z\| \leq 21.62 - 1468.46$	$\|z\| \leq 31 - 3019.07$	$\|z\| \leq 20.83 - 1363.1$
$T_{10}(x)$	$0.14 \leq \|z\| \leq 140.30 - 61839.3$	$0.03 \leq \|z\| \leq 31.15 - 3048.36$	$\|z\| \leq 51 - 8171.28$	$0.03 \leq \|z\| \leq 30.02 - 2831.2$

Above, the upper bounds containing the zeros of the first ten Tribonacci polynomials are shown numerically. (Since the zero of $T_0(x), T_2(x)$ is only 0 and $T_1(x)$ is a fixed polynomial, they are not included in this table.) At first glance, it seems that Diaz-Barrero's boundary($\Re_1$) is the most useful among others. But, it is clearly observed that actual area is gradually growing except for first few Tribonacci polynomials. And this shows that this bound is unusable for these polynomials. On the other hand, it is clear that our boundary($\Re_3$) gives better results in terms of both r_1, r_2 and area, when compared to the bound of Cauchy, one of the oldest boundaries and the bound of Dalal and Govil($\Re_2$) which have been recently obtained. Although not included in the table, these calculations have been numerically verified for the first 20 Tribonacci polynomials. Hence, the disk or annulus obtained by our boundary and containing the zeros of each Tribonacci polynomial will remain the smallest. This is the desired ideal situation.

4. Application

In this section, we develop an algorithm. At the beginning of the algorithm, all Tribonacci polynomials are generated and then the coefficients are taken to be used at any boundary. The algorithm makes it easy for us to make this computation for the Tribonacci Polynomial as many as we would like to. Furthermore, this algorithm can be developed and adapted for other boundary value problems. And, thanks to this algorithm, testing of the usability of any boundary can be done without needing long and complicated operations.

Table 1. The Algorithm

```
t[0]=0;
t[1]=1;
t[2] = x^2;
For[i=2,i¡16,i++,
t[n](x_):=t(i+1)=x^2 t(i) + xt(i − 1) + t(i − 2);
a[n]:=Coefficient[t(n)(x),x,2 i];
a[n-3]:=Coefficient[t(n)(x),x,2 i-3];
s[n]:=Coefficient[t(n)(x),x,0];
e[n]:=Exponent[t(n)(x),x];
```

$$\text{Print}\left[\max\left(\left\{N\left[\sqrt[3]{\frac{\left|\frac{a(n-3)}{a(n)}\right|\left(F_{e(n)+2}-1\right)}{F_3}}\right]\right\}\right)\right]$$

References

[1] A. L. Cauchy, Exercises de Mathématiques IV, Anné de Bure Fréres, Paris, 1829.

[2] G. Moore, The limit of the golden numbers is 3/2, Fibonacci Quart. 32 (1994), 211-217.

[3] M. X. He, D. Simon, P. E. Ricci, Dynamics of the zeros of Fibonacci polynomials, Fibonacci Quart. 35 (1997), 160-168.

[4] A. Dalal, N. K. Govil, On Region Containing all the Zeros of a Polynomial, Appl. Math. Comput. 219 (2013), 9609-9614.

[5] T. Koshy, Fibonacci and Lucas Numbers with Applications, Wiley-Interscience, 2001.

[6] V. E. Hoggat, M. Bicknell, Generalized Fibonacci polynomials, Fibonacci Quart. 11 (1973), 457-465.

[7] V. E. Hoggat, M. Bicknell, Roots of Fibonacci polynomials, Fibonacci Quart. 11 (1973), 271-274.

[8] F. Matyas, Bounds for the zeros of Fibonacci-Like Polynomials, Acta Acad. Paed. Agriensis, Sec. Math. 25 (1998), 15-20.

[9] H. Prodinger, The asymptotic behavior of the golden numbers, Fibonacci Quart. 34 (1996), 224-225.

[10] J. H. Conway, R. Guy, The Book of Numbers, Copernicus, New York, 1996.

[11] W. Goh, M. X. He, P. E. Ricci, On the universal zero attractor of the Tribonacci-related polynomials, Calcolo 46 (2009), 95-129.

[12] M. X. He, P. E. Ricci, D. Simon, Numerical results on the zeros of generalized Fibonacci polynomials, Calcolo 34 (1997), 25-40.

[13] J.L. Diaz-Barrero, An annulus for the zeros of polynomials, J. Math. Anal. Appl. 273 (2002), 349-352.

[14] Ö. Öztunç, $R-$Bonacci Polynomials and Their Derivatives, Ph. D. Thesis, Balıkesir University, (2014).

[15] X. Li, Y. Shi, I. Gutman, Graph Energy, Springer, New York, 2012.

[16] S. Fisk, Polynomials, roots, and interlacing, ArXiv mat/0612833v2, (2006).

[17] N. Y. Özgür, Ö. Öztunç, On the Zeros of $R-$Bonaci Polynomials, ArXiv:1711.01150, (2017).

[18] P. G. Anderson, R. S. Gaborski, The Polynomial Method Augmented by Supervised Training for Hand-Printed Character Recognition, Artificial Neural Networks and Genetic Algorithms, Proceedings of the International Conference, (1993).

[19] S. Ucar, N. Y. Ozgur, Right Circulant Matrices with Generalized Fibonacci and Lucas Polynomials and Coding Theory, ArXiv:1801.01766 [math.CO].