International Journal of Analysis and Applications
Volume 16, Number 3 (2018), 317-327
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-16-2018-317

# LACUNARY $\mathcal{I}_2$-INVARIANT CONVERGENCE AND SOME PROPERTIES

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Abstract. In this paper, the concept of lacunary invariant uniform density of any subset $A$ of the set $\mathbb{N}\times\mathbb{N}$ is defined. Associate with this, the concept of lacunary $\mathcal{I}_2$-invariant convergence for double sequences is given. Also, we examine relationships between this new type convergence concept and the concepts of lacunary invariant convergence and $p$-strongly lacunary invariant convergence of double sequences. Finally, introducing lacunary $\mathcal{I}_2^*$-invariant convergence concept and lacunary $\mathcal{I}_2$-invariant Cauchy concepts, we give the relationships among these concepts and relationships with lacunary $\mathcal{I}_2$-invariant convergence concept.

## 1. Introduction

Several authors have studied invariant convergent sequences (see, [8–10, 13, 15–17, 19]).

Let $\sigma$ be a mapping of the positive integers into themselves. A continuous linear functional $\phi$ on $\ell_\infty$, the space of real bounded sequences, is said to be an invariant mean or a $\sigma$-mean if it satisfies following conditions:

(1) $\phi(x)\geq 0$, when the sequence $x=(x_n)$ has $x_n\geq 0$ for all $n$,
(2) $\phi(e)=1$, where $e=(1,1,1,...)$ and
(3) $\phi(x_{\sigma(n)})=\phi(x_n)$ for all $x\in\ell_\infty$.

The mappings $\sigma$ are assumed to be one-to-one and such that $\sigma^m(n)\neq n$ for all positive integers $n$ and $m$, where $\sigma^m(n)$ denotes the $m$ th iterate of the mapping $\sigma$ at $n$. Thus, $\phi$ extends the limit functional on $c$,

2010 *Mathematics Subject Classification.* 40A05, 40A35.

*Key words and phrases.* double sequence; $\mathcal{I}$-convergence; lacunary sequence; invariant convergence; $\mathcal{I}$-Cauchy sequence.

the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$.

In the case $\sigma$ is translation mappings $\sigma(n) = n + 1$, the $\sigma$-mean is often called a Banach limit.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ is denoted by $I_r = (k_{r-1}, k_r]$ (see, [4]).

The concept of lacunary strongly $\sigma$-convergence was introduced by Savaş [17] as below:

$$L_\theta = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_{\sigma^k(m)} - L| = 0, \text{ uniformly in } m \right\}.$$

Pancaroğlu and Nuray [13] defined the concept of lacunary invariant summability and the space $[V_{\sigma\theta}]_q$ as follows:

A sequence $x = (x_k)$ is said to be lacunary invariant summable to $L$ if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{m\in I_r} x_{\sigma^m(n)} = L,$$

uniformly in $n$.

A sequence $x = (x_k)$ is said to be strongly lacunary $q$-invariant convergent $(0 < q < \infty)$ to $L$ if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{m\in I_r} |x_{\sigma^m(n)} - L|^q = 0,$$

uniformly in $n$ and it is denoted by $x_k \to L\big([V_{\sigma\theta}]_q\big)$.

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [5] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers $\mathbb{N}$.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if $(i)$ $\emptyset \in \mathcal{I}$, $(ii)$ For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, $(iii)$ For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if $(i)$ $\emptyset \notin \mathcal{F}$, $(ii)$ For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, $(iii)$ For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

For any ideal there is a filter $\mathcal{F}(\mathcal{I})$ corresponding with $\mathcal{I}$, given by

$$\mathcal{F}(\mathcal{I}) = \big\{ M \subset \mathbb{N} : (\exists A \in \mathcal{I})(M = \mathbb{N} \backslash A) \big\}.$$

Recently, the concepts of lacunary $\sigma$-uniform density of the set $A \subseteq \mathbb{N}$, lacunary $\mathcal{I}_\sigma$-convergence, lacunary $\mathcal{I}_\sigma^*$-convergence, lacunary $\mathcal{I}_\sigma$-Cauchy and $\mathcal{I}_\sigma^*$-Cauchy sequences of real numbers were defined by Ulusu and

Nuray [20] and similar concepts can be seen in [12].

Let $\theta = \{k_r\}$ be a lacunary sequence, $A \subseteq \mathbb{N}$ and

$$s_r := \min_n \Big\{ \big|A \cap \{\sigma^m(n) : m \in I_r\}\,\big| \Big\}$$

and

$$S_r := \max_n \Big\{ \big|A \cap \{\sigma^m(n) : m \in I_r\}\,\big| \Big\}.$$

If the following limits exist

$$\underline{V_\theta}(A) := \lim_{r\to\infty} \frac{s_r}{h_r}, \qquad \overline{V_\theta}(A) := \lim_{r\to\infty} \frac{S_r}{h_r}$$

then they are called a lower lacunary $\sigma$-uniform (lower $\sigma\theta$-uniform) density and an upper lacunary $\sigma$-uniform (upper $\sigma\theta$-uniform) density of the set $A$, respectively. If $\underline{V_\theta}(A) = \overline{V_\theta}(A)$, then $V_\theta(A) = \underline{V_\theta}(A) = \overline{V_\theta}(A)$ is called the lacunary $\sigma$-uniform density or $\sigma\theta$-uniform density of $A$.

Denote by $\mathcal{I}_{\sigma\theta}$ the class of all $A \subseteq \mathbb{N}$ with $V_\theta(A) = 0$.

Let $\mathcal{I}_{\sigma\theta} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $(x_k)$ is said to be lacunary $\mathcal{I}_\sigma$-convergent or $\mathcal{I}_{\sigma\theta}$-convergent to the number $L$ if for every $\varepsilon > 0$

$$A_\varepsilon := \big\{k : |x_k - L| \geq \varepsilon\big\}$$

belongs to $\mathcal{I}_{\sigma\theta}$; i.e., $V_\theta(A_\varepsilon) = 0$. In this case we write $\mathcal{I}_{\sigma\theta} - \lim x_k = L$.

The set of all $\mathcal{I}_{\sigma\theta}$-convergent sequences will be denoted by $\mathfrak{I}_{\sigma\theta}$.

Let $\mathcal{I}_{\sigma\theta} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be $\mathcal{I}^*_{\sigma\theta}$-convergent to the number $L$ if there exists a set $M = \{m_1 < m_2 < ...\} \in \mathcal{F}(\mathcal{I}_{\sigma\theta})$ such that $\lim\limits_{k\to\infty} x_{m_k} = L$. In this case we write $\mathcal{I}^*_{\sigma\theta} - \lim x_k = L$.

A sequence $(x_k)$ is said to be lacunary $\mathcal{I}_\sigma$-Cauchy sequence or $\mathcal{I}_{\sigma\theta}$-Cauchy sequence if for every $\varepsilon > 0$, there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon) = \big\{k : |x_k - x_N| \geq \varepsilon\big\}$$

belongs to $\mathcal{I}_{\sigma\theta}$; i.e., $V_\theta\big(A(\varepsilon)\big) = 0$.

A sequence $x = (x_k)$ is said to be $\mathcal{I}^*_{\sigma\theta}$-Cauchy sequences if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(\mathcal{I}_{\sigma\theta})$ such that

$$\lim_{k,p\to\infty} |x_{m_k} - x_{m_p}| = 0.$$

Convergence and $\mathcal{I}$-convergence of double sequences in a metric space and some properties of this convergence, and similar concepts which are noted following can be seen in [1, 2, 6, 7, 14, 18].

A double sequence $x = (x_{kj})_{k,j\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, whenever $k, j > N_\varepsilon$. In this case, we write $P - \lim_{k,j\to\infty} x_{kj} = L$ *or* $\lim_{k,j\to\infty} x_{kj} = L$.

A double sequence $x = (x_{kj})$ is said to be bounded if $\sup_{k,j} x_{kj} < \infty$. The set of all bounded double sequences of sets will be denoted by $\ell^2_\infty$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible ideal if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Throughout the paper we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$.

$\mathcal{I}_2^0 = \{A \subset \mathbb{N}\times\mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

An admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{E_1, E_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{F_1, F_2, ...\}$ such that $E_j \Delta F_j \in \mathcal{I}_2^0$, i.e., $E_j \Delta F_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}_2$ (hence $F_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Let $(X, \rho)$ be a metric space. A sequence $x = (x_{mn})$ in $X$ is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$

$$A(\varepsilon) = \{(m, n) \in \mathbb{N}\times\mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case, we write $\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L$.

The double sequence $\theta = \{(k_r, j_u)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty \quad and \quad j_0 = 0,\ \bar{h}_u = j_u - j_{u-1} \to \infty \quad as \quad r, u \to \infty.$$

We use the following notations in the sequel:

$$k_{ru} = k_r j_u,\ \ h_{ru} = h_r \bar{h}_u,\ \ I_{ru} = \{(k, j) : k_{r-1} < k \leq k_r \ \ and \ \ j_{u-1} < j \leq j_u\},$$

$$q_r = \frac{k_r}{k_{r-1}} \ \ and \ \ q_u = \frac{j_u}{j_{u-1}}.$$

Also, the idea of $\mathcal{I}_2$-invariant convergence concepts and $\mathcal{I}_2$-invariant Cauchy concepts of double sequences were defined by Dündar and Ulusu (see [3]).

## 2. Lacunary $\mathcal{I}_2$-Invariant Convergence

In this section, firstly, the concepts of lacunary invariant convergence of double sequence and lacunary invariant uniform density of any subset $A$ of the set $\mathbb{N}\times\mathbb{N}$ are defined. Associate with this uniform density, the concept of lacunary $\mathcal{I}_2$-invariant convergence for double sequences is given. Also, we examine relationships between this new type convergence concept and the concepts of lacunary invariant convergence, $p$-strongly lacunary invariant convergence for double sequences.

**Definition 2.1.** *A double sequence $x = (x_{kj})$ is said to be lacunary invariant convergent to $L$ if*

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{k,j\in I_{ru}} x_{\sigma^k(m),\sigma^j(n)} = L,$$

*uniformly in $m,n$ and it is denoted by $x_{kj} \to L\big(V_2^{\sigma\theta}\big)$.*

**Definition 2.2.** *Let $\theta = \{(k_r, j_u)\}$ be a double lacunary sequence, $A \subseteq \mathbb{N}\times\mathbb{N}$ and*

$$s_{ru} := \min_{m,n} \Big| A \cap \big\{ \big(\sigma^k(m), \sigma^j(n)\big) : (k,j) \in I_{ru} \big\} \Big|$$

*and*

$$S_{ru} := \max_{m,n} \Big| A \cap \big\{ \big(\sigma^k(m), \sigma^j(n)\big) : (k,j) \in I_{ru} \big\} \Big|.$$

*If the following limits exist*

$$\underline{V_2^\theta}(A) := \lim_{r,u\to\infty} \frac{s_{ru}}{h_{ru}}, \qquad \overline{V_2^\theta}(A) := \lim_{r,u\to\infty} \frac{S_{ru}}{h_{ru}},$$

*then they are called a lower lacunary $\sigma$-uniform density and an upper lacunary $\sigma$-uniform density of the set $A$, respectively. If $\underline{V_2^\theta}(A) = \overline{V_2^\theta}(A)$, then $V_2^\theta(A) = \underline{V_2^\theta}(A) = \overline{V_2^\theta}(A)$ is called the lacunary $\sigma$-uniform density of $A$.*

Denote by $\mathcal{I}_2^{\sigma\theta}$ the class of all $A \subseteq \mathbb{N}\times\mathbb{N}$ with $V_2^\theta(A) = 0$.

Throughout the paper we take $\mathcal{I}_2^{\sigma\theta}$ as a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$.

**Definition 2.3.** *A double sequence $x = (x_{kj})$ is said to be lacunary $\mathcal{I}_2$-invariant convergent or $\mathcal{I}_2^{\sigma\theta}$-convergent to the $L$ if for every $\varepsilon > 0$, the set*

$$A_\varepsilon := \big\{ (k,j) \in I_{ru} : |x_{kj} - L| \geq \varepsilon \big\}$$

*belongs to $\mathcal{I}_2^{\sigma\theta}$; i.e., $V_2^\theta(A_\varepsilon) = 0$. In this case, we write*

$$\mathcal{I}_2^{\sigma\theta} - \lim x_{kj} = L \quad or \quad x_{kj} \to L\big(\mathcal{I}_2^{\sigma\theta}\big).$$

The set of all $\mathcal{I}_2^{\sigma\theta}$-convergent sequences will be denoted by $\mathfrak{I}_2^{\sigma\theta}$.

**Theorem 2.1.** *If* $\mathcal{I}_2^{\sigma\theta} - \lim x_{kj} = L_1$ *and* $\mathcal{I}_2^{\sigma\theta} - \lim y_{kj} = L_2$, *then*

(i) $\mathcal{I}_2^{\sigma\theta} - \lim(x_{kj} + y_{kj}) = L_1 + L_2$

(ii) $\mathcal{I}_2^{\sigma\theta} - \lim \alpha x_{kj} = \alpha L_1$ *($\alpha$ is a constant).*

*Proof.* The proof is clear so we omit it. □

**Theorem 2.2.** *Suppose that $x = (x_{kj})$ is a bounded double sequence. If $(x_{kj})$ is lacunary $\mathcal{I}_2$-invariant convergent to $L$, then $(x_{kj})$ is lacunary invariant convergent to $L$.*

*Proof.* Let $\theta = \{(k_r, j_u)\}$ be a double lacunary sequence, $m, n \in \mathbb{N}$ be an arbitrary and $\varepsilon > 0$. Now, we calculate

$$t(k,j,r,u) := \left| \frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} x_{\sigma^k(m),\sigma^j(n)} - L \right|.$$

We have

$$t(k,j,r,u) \leq t^{(1)}(k,j,r,u) + t^{(2)}(k,j,r,u),$$

where

$$t^{(1)}(k,j,r,u) := \frac{1}{h_{ru}} \sum_{\substack{k,j \in I_{ru} \\ |x_{\sigma^k(m),\sigma^j(n)} - L| \geq \varepsilon}} |x_{\sigma^k(m),\sigma^j(n)} - L|$$

and

$$t^{(2)}(k,j,r,u) := \frac{1}{h_{ru}} \sum_{\substack{k,j \in I_{ru} \\ |x_{\sigma^k(m),\sigma^j(n)} - L| < \varepsilon}} |x_{\sigma^k(m),\sigma^j(n)} - L|.$$

We get $t^{(2)}(k,j,r,u) < \varepsilon$, for every $m, n = 1, 2, \ldots$. The boundedness of $x = (x_{kj})$ implies that there exists a $K > 0$ such that

$$|x_{\sigma^k(m),\sigma^j(n)} - L| \leq K, \quad ((k,j) \in I_{ru}; m, n = 1, 2, \ldots).$$

Then, this implies that

$$\begin{aligned} t^{(1)}(k,j,r,u) &\leq \frac{K}{h_{ru}} \left| \left\{ (k,j) \in I_{ru} : |x_{\sigma^k(m),\sigma^j(n)} - L| \geq \varepsilon \right\} \right| \\ &\leq K \frac{\max_{m,n} \left| \left\{ (k,j) \in I_{ru} : |x_{\sigma^k(m),\sigma^j(n)} - L| \geq \varepsilon \right\} \right|}{h_{ru}} = K \frac{S_{ru}}{h_{ru}}, \end{aligned}$$

hence $(x_{kj})$ is lacunary invariant summable to $L$. □

The converse of Theorem 2.2 does not hold. For example, $x = (x_{kj})$ is the double sequence defined by following;

$$x_{kj} := \begin{cases} 1 \quad , \quad \text{if} \quad \begin{array}{l} k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \\ j_{r-1} < j < j_{r-1} + [\sqrt{\bar{h}_u}], \end{array} \quad \text{and} \quad k+j \quad \text{is an even integer.} \\ \\ 0 \quad , \quad \text{if} \quad \begin{array}{l} k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \\ j_{r-1} < j < j_{r-1} + [\sqrt{\bar{h}_u}], \end{array} \quad \text{and} \quad k+j \quad \text{is an odd integer.} \end{cases}$$

When $\sigma(m) = m + 1$ and $\sigma(n) = n + 1$, this sequence is lacunary invariant convergent to $\frac{1}{2}$ but it is not lacunary $\mathcal{I}_2$-invariant convergent.

In [20], Ulusu and Nuray gave some inclusion relations between $[V_{\sigma\theta}]_q$-convergence and lacunary $\mathcal{I}$-invariant convergence, and showed that these are equivalent for bounded sequences. Now, we shall give analogous theorems which states inclusion relations between $[V_2^{\sigma\theta}]_p$-convergence and lacunary $\mathcal{I}_2$-invariant convergence, and show that these are equivalent for bounded double sequences.

**Definition 2.4.** *A double sequence $x = (x_{kj})$ is said to be strongly lacunary invariant convergent to $L$ if*

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{k,j\in I_{ru}} |x_{\sigma^k(m),\sigma^j(n)} - L|,$$

*uniformly in $m, n$ and it is denoted by $x_{kj} \to L\big([V_2^{\sigma\theta}]\big)$.*

**Definition 2.5.** *A double sequence $x = (x_{kj})$ is said to be $p$-strongly lacunary invariant convergent $(0 < p < \infty)$ to $L$ if*

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{k,j\in I_{ru}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p = 0,$$

*uniformly in $m, n$ and it is denoted by $x_{kj} \to L\big([V_2^{\sigma\theta}]_p\big)$.*

**Theorem 2.3.** *If a double sequence $x = (x_{kj})$ is $p$-strongly lacunary invariant convergent to $L$, then $(x_{kj})$ is lacunary $\mathcal{I}_2$-invariant convergent to $L$.*

*Proof.* Assume that $x_{kj} \to L\big([V_2^{\sigma\theta}]_p\big)$ and given $\varepsilon > 0$. Then, for every double lacunary sequence $\theta = \{(k_r, j_u)\}$ and for every $m, n \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{k,j\in I_{ru}} \big|x_{\sigma^k(m),\sigma^j(n)} - L\big|^p &\geq \sum_{\substack{(k,j)\in I_{ru}\\ |x_{\sigma^k(m),\sigma^j(n)}-L|\geq\varepsilon}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p \\ &\geq \varepsilon^p\big|\big\{(k,j)\in I_{ru} : |x_{\sigma^k(m),\sigma^j(n)} - L| \geq \varepsilon\big\}\big| \\ &\geq \varepsilon^p \max_{m,n}\big|\big\{(k,j)\in I_{ru} : |x_{\sigma^k(m),\sigma^j(n)} - L| \geq \varepsilon\big\}\big|\end{aligned}$$

and

$$\begin{aligned}\frac{1}{h_{ru}}\sum_{k,j\in I_{ru}} \big|x_{\sigma^k(m),\sigma^j(n)} - L\big|^p &\geq \varepsilon^p \frac{\max_{m,n}\big|\big\{(k,j)\in I_{ru} : |x_{\sigma^k(m),\sigma^j(n)} - L| \geq \varepsilon\big\}\big|}{h_{ru}} \\ &= \varepsilon^p \frac{S_{ru}}{h_{ru}}.\end{aligned}$$

This implies

$$\lim_{r,u\to\infty} \frac{S_{ru}}{h_{ru}} = 0$$

and so $(x_{kj})$ is $\mathcal{I}_2^{\sigma\theta}$-convergent to $L$. □

**Theorem 2.4.** *If a double sequence $x = (x_{kj}) \in \ell_\infty^2$ and $(x_{kj})$ is lacunary $\mathcal{I}_2$-invariant convergent to $L$, then $(x_{kj})$ is $p$-strongly lacunary invariant convergent to $L$ $(0 < p < \infty)$.*

*Proof.* Suppose that $x = (x_{kj}) \in \ell_\infty^2$ and $x_{kj} \to L\big(\mathcal{I}_2^{\sigma\theta}\big)$. Let $0 < p < \infty$ and $\varepsilon > 0$. By assumption we have $V_2^\theta\big(A_\varepsilon\big) = 0$. The boundedness of $(x_{kj})$ implies that there exists $K > 0$ such that

$$|x_{\sigma^k(m),\sigma^j(n)} - L| \leq K, \quad ((k,j)\in I_{r,u};\, m,n = 1,2,\dots).$$

Observe that, for every $m, n \in \mathbb{N}$ we have

$$\begin{aligned}\frac{1}{h_{ru}}\sum_{k,j\in I_{ru}} \big|x_{\sigma^k(m),\sigma^j(n)} - L\big|^p &= \frac{1}{h_{ru}} \sum_{\substack{k,j\in I_{ru}\\ |x_{\sigma^k(m),\sigma^j(n)}-L|\geq\varepsilon}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p \\ &\quad + \frac{1}{h_{ru}} \sum_{\substack{k,j\in I_{ru}\\ |x_{\sigma^k(m),\sigma^j(n)}-L|<\varepsilon}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p \\ &\leq K \frac{\max_{m,n}\big|\big\{(k,j)\in I_{ru} : |x_{\sigma^k(m),\sigma^j(n)} - L| \geq \varepsilon\big\}\big|}{h_{ru}} + \varepsilon^p \\ &\leq K\frac{S_{ru}}{h_{ru}} + \varepsilon^p.\end{aligned}$$

Hence, we obtain

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{k,j\in I_{ru}} \left|x_{\sigma^k(m),\sigma^j(n)} - L\right|^p = 0,$$

uniformly in $m, n$. □

**Theorem 2.5.** *A double sequence $x = (x_{kj}) \in \ell_\infty^2$ and $(x_{kj})$ is lacunary $\mathcal{I}_2$-invariant convergent to $L$ if and only if $(x_{kj})$ is $p$-strongly lacunary invariant convergent to $L$ $(0 < p < \infty.)$*

*Proof.* This is an immediate consequence of Theorem 2.3 and Theorem 2.4. □

Now, introducing lacunary $\mathcal{I}_2^*$-invariant convergence concept, lacunary $\mathcal{I}_2^\sigma$-Cauchy double sequence and $\mathcal{I}_{2^*}^{\sigma\theta}$-Cauchy double sequence concepts, we give the relationships among these concepts and relationships with lacunary $\mathcal{I}_2$-invariant convergence concept.

**Definition 2.6.** *A double sequence $x = (x_{kj})$ is lacunary $\mathcal{I}_2^*$-invariant convergent or $\mathcal{I}_{2^*}^{\sigma\theta}$-convergent to $L$ if and only if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^{\sigma\theta})$ $(\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2^{\sigma\theta})$ such that*

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M_2}} x_{kj} = L. \tag{2.1}$$

*In this case, we write $\mathcal{I}_{2^*}^{\sigma\theta} - \lim x_{kj} = L$ or $x_{kj} \to L(\mathcal{I}_{2^*}^{\sigma\theta})$.*

**Theorem 2.6.** *If a double sequence $x = (x_{kj})$ is lacunary $\mathcal{I}_2^*$-invariant convergent to $L$, then this sequence is lacunary $\mathcal{I}_2$-invariant convergent to $L$.*

*Proof.* Since $\mathcal{I}_{2^*}^{\sigma\theta} - \lim_{k,j\to\infty} x_{kj} = L$, there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^{\sigma\theta})$ $(\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2^{\sigma\theta})$ such that

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M_2}} x_{kj} = L.$$

Given $\varepsilon > 0$. By (2.1), there exists $k_0, j_0 \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, for all $(k,j) \in M_2$ and $k \geq k_0, j \geq j_0$. Hence, for every $\varepsilon > 0$, we have

$$\begin{aligned} T(\varepsilon) &= \left\{(k,j)\in\mathbb{N}\times\mathbb{N} : |x_{kj} - L| \geq \varepsilon\right\} \\ &\subset H \cup \Big(M_2 \cap \big((\{1,2,...,(k_0-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,...,(k_0-1)\})\big)\Big). \end{aligned}$$

Since $\mathcal{I}_2^{\sigma\theta} \subset 2^{\mathbb{N}\times\mathbb{N}}$ is a strongly admissible ideal,

$$H \cup \Big(M_2 \cap \big((\{1,2,...,(k_0-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,...,(k_0-1)\})\big)\Big) \in \mathcal{I}_2^{\sigma\theta},$$

so we have $T(\varepsilon) \in \mathcal{I}_2^{\sigma\theta}$ that is $V_2^\theta\big(T(\varepsilon)\big) = 0$. Hence, $\mathcal{I}_2^{\sigma\theta} - \lim_{k,j\to\infty} x_{kj} = L$. □

The converse of Theorem 2.6, which it's proof is similar to the proof of Theorems in [1–3], holds if $\mathcal{I}_2^{\sigma\theta}$ has property $(AP2)$.

**Theorem 2.7.** *Let $\mathcal{I}_2^{\sigma\theta}$ has property $(AP2)$. If a double sequence $x = (x_{kj})$ is lacunary $\mathcal{I}_2$-invariant convergent to $L$, then this sequence is lacunary $\mathcal{I}_2^*$-invariant convergent to $L$.*

Finally, we define the concepts of lacunary $\mathcal{I}_2$-invariant Cauchy and lacunary $\mathcal{I}_2^*$-invariant Cauchy double sequences.

**Definition 2.7.** *A double sequence $(x_{kj})$ is said to be lacunary $\mathcal{I}_2$-invariant Cauchy sequence or $\mathcal{I}_2^{\sigma\theta}$-Cauchy sequence, if for every $\varepsilon > 0$, there exist numbers $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that*

$$A(\varepsilon) = \big\{(k,j),(s,t) \in I_{ru} : |x_{kj} - x_{st}| \geq \varepsilon\big\} \in \mathcal{I}_2^{\sigma\theta},$$

*that is, $V_2^{\theta}\big(A(\varepsilon)\big) = 0$.*

**Definition 2.8.** *A double sequence $(x_{kj})$ is lacunary $\mathcal{I}_2^*$-invariant Cauchy sequence or $\mathcal{I}_{2^*}^{\sigma\theta}$-Cauchy sequence if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^{\sigma\theta})$ (i.e., $\mathbb{N} \times \mathbb{N} \backslash M_2 = H \in \mathcal{I}_2^{\sigma\theta}$) such that for every $(k,j),(s,t) \in M_2$*

$$\lim_{k,j,s,t\to\infty} |x_{kj} - x_{st}| = 0.$$

The proof of the following theorems are similar to the proof of Theorems in [2,3,11], so we omit them.

**Theorem 2.8.** *If a double sequence $x = (x_{kj})$ is $\mathcal{I}_2^{\sigma\theta}$-convergent, then $(x_{kj})$ is an $\mathcal{I}_2^{\sigma\theta}$-Cauchy sequence.*

**Theorem 2.9.** *If a double sequence $x = (x_{kj})$ is $\mathcal{I}_{2^*}^{\sigma\theta}$-Cauchy sequence, then $(x_{kj})$ is $\mathcal{I}_2^{\sigma\theta}$-Cauchy sequence.*

**Theorem 2.10.** *Let $\mathcal{I}_2^{\sigma\theta}$ has property $(AP2)$. If a double sequence $x = (x_{kj})$ is $\mathcal{I}_2^{\sigma\theta}$-Cauchy sequence then, $(x_{kj})$ is $\mathcal{I}_{2^*}^{\sigma\theta}$-Cauchy sequence.*

***Acknowledgements:*** This study supported by Afyon Kocatepe University Scientific Research Coordination Unit with the project number 17.Kariyer.21.

## References

[1] P. Das, P. Kostyrko, W. Wilczyński and P. Malik, $\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences, Math. Slovaca, 58(5) (2008), 605-620.

[2] E. Dündar and B. Altay, $\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy of double sequences, Acta Math. Sci., 34B(2) (2014), 343-353.

[3] E. Dündar, U. Ulusu and F. Nuray, On ideal invariant convergence of double sequences and some properties, Creat. Math. Inf., 27(2) (2018), (in press).

[4] J. A. Fridy and C. Orhan, Lacunary statistical convergence, Pacific J. Math., 160(1) (1993), 43–51.

[5] P. Kostyrko, T. Šalát and W. Wilczyński, $\mathcal{I}$-Convergence, Real Anal. Exchange, 26(2) (2000), 669–686.

[6] V. Kumar, On $\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences, Math. Commun. 12 (2007), 171–181.

[7] S. A. Mohiuddine and E. Savaş, Lacunary statistically convergent double sequences in probabilistic normed spaces, Ann Univ. Ferrara, 58 (2012), 331–339.

[8] M. Mursaleen, Matrix transformation between some new sequence spaces, Houston J. Math., 9 (1983), 505–509.

[9] M. Mursaleen, On finite matrices and invariant means, Indian J. Pure Appl. Math., 10 (1979), 457–460.

[10] M. Mursaleen and O. H. H. Edely, On the invariant mean and statistical convergence, Appl. Math. Lett., 22(11) (2009), 1700–1704.

[11] A. Nabiev, S. Pehlivan and M. Gürdal, On $\mathcal{I}$-Cauchy sequences, Taiwanese J. Math., 11(2) (2007), 569–576.

[12] F. Nuray, H. Gök and U. Ulusu, $\mathcal{I}_\sigma$-convergence, Math. Commun. 16 (2011) 531–538.

[13] N. Pancaroğlu and F. Nuray, Statistical lacunary invariant summability, Theor. Math. Appl., 3(2) (2013), 71–78.

[14] A. Pringsheim, Zur theorie der zweifach unendlichen Zahlenfolgen, Math. Ann., 53 (1900), 289?21.

[15] R. A. Raimi, Invariant means and invariant matrix methods of summability, Duke Math. J., 30(1) (1963), 81–94.

[16] E. Savaş, Some sequence spaces involving invariant means, Indian J. Math., 31 (1989), 1–8.

[17] E. Savaş, Strongly $\sigma$-convergent sequences, Bull. Calcutta Math., 81 (1989), 295–300.

[18] E. Savaş and R. Patterson, Double $\sigma$-convergence lacunary statistical sequences, J. Comput. Anal. Appl., 11(4) (2009).

[19] P. Schaefer, Infinite matrices and invariant means, Proc. Amer. Math. Soc., 36 (1972), 104–110.

[20] U. Ulusu and F. Nuray, Lacunary $\mathcal{I}_\sigma$-convergence, (under review).