International Journal of Analysis and Applications
ISSN 2291-8639
Volume 4, Number 1 (2014), 68-77
http://www.etamaths.com

COEFFICIENT ESTIMATES OF MEROMORPHIC BI- STARLIKE FUNCTIONS OF COMPLEX ORDER

T. JANANI AND G. MURUGUSUNDARAMOORTHY*

ABSTRACT. In the present investigation, we define a new subclass of meromorphic bi-univalent functions class Σ' of complex order $\gamma \in \mathbb{C}\backslash\{0\}$, and obtain the estimates for the coefficients $|b_0|$ and $|b_1|$. Further we pointed out several new or known consequences of our result.

1. Introduction and Definitions

Denote by $\mathcal{A}$ the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are univalent in the open unit disc

$$\Delta = \{z : |z| < 1\}.$$

Also denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent and normalized by the conditions

$$f(0) = 0 = f'(0) - 1$$

in Δ. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ includes the class $\mathcal{S}^*(\alpha)(0 \leq \alpha < 1)$ of starlike functions of order α in Δ and the class $\mathcal{K}(\alpha)(0 \leq \alpha < 1)$ of convex functions of order α

$$\Re\left(\frac{z\, f'(z)}{f(z)}\right) > \alpha \quad \text{or} \quad \Re\left(1 + \frac{z\, f''(z)}{f'(z)}\right) > \alpha, (z \in \Delta)$$

respectively.Further a function $f(z) \in \mathcal{A}$ is said to be in the class $S(\gamma)$ of univalent function of complex order $\gamma(\gamma \in \mathbb{C} \setminus \{0\})$ if and only if

$$\frac{f(z)}{z} \neq 0 \ \ and \ \Re\left(1 + \frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)} - 1\right]\right) > 0, z \in \Delta.$$

By taking $\gamma = (1-\alpha)cos\beta\ e^{-i\beta}$, $|\beta| < \frac{\pi}{2}$ and $0 \leq \alpha < 1$, the class $\mathcal{S}((1-\alpha)cos\beta\ e^{-i\beta}) \equiv \mathcal{S}(\alpha,\beta)$ called the generalized class of β-spiral-like functions of order $\alpha(0 \leq \alpha < 1)$.

An analytic function φ is subordinate to an analytic function ψ, written by

$$\varphi(z) \prec \psi(z),$$

2000 *Mathematics Subject Classification.* 30C45 , 30C50.

Key words and phrases. Analytic functions; Univalent functions; Meromorphic functions; Bi-univalent functions; Bi-starlike and bi-convex functions of complex order; Coefficient bounds.

provided there is an analytic function ω defined on Δ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1$$

satisfying

$$\varphi(z) = \psi(\omega(z)).$$

Ma and Minda [9] unified various subclasses of starlike and convex functions for which either of the quantity

$$\frac{z\,f'(z)}{f(z)} \quad \text{or} \quad 1 + \frac{z\,f''(z)}{f'(z)}$$

is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk $\Delta, \phi(0) = 1, \phi'(0) > 0$ and ϕ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{z\,f'(z)}{f(z)} \prec \phi(z).$$

Similarly, the class of Ma-Minda convex functions consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$1 + \frac{z\,f''(z)}{f'(z)} \prec \phi(z).$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, \quad (z \in \Delta)$$
$$\text{and} \quad f(f^{-1}(w)) = w, \quad (|w| < r_0(f); r_0(f) \geq 1/4)$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathcal{A}$ given by (1.1), is said to be bi-univalent in Δ if both $f(z)$ and $f^{-1}(z)$ are univalent in Δ, these classes are denoted by Σ. Earlier, Brannan and Taha [2] introduced certain subclasses of bi-univalent function class Σ, namely bi-starlike functions $\mathcal{S}^*_\Sigma(\alpha)$ and bi-convex function $\mathcal{K}_\Sigma(\alpha)$ of order α corresponding to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively. For each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were found [2, 17]. But the coefficient problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1, 2\};\ \ \mathbb{N} := \{1, 2, 3, \cdots\})$$

is still an open problem(see[1, 2, 8, 10, 17]). Recently several interesting subclass-es of the bi-univalent function class Σ have been introduced and studied in the literature(see[15, 18, 19]).

A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\phi)$ and $\mathcal{K}_\Sigma(\phi)$.In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk Δ, satisfying

$\phi(0) = 1, \phi'(0) > 0$ and $\phi(\Delta)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{1.3}$$

Let Σ' denote the class of meromorphic univalent functions g of the form

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \tag{1.4}$$

defined on the domain $\Delta^* = \{z : 1 < |z| < \infty\}$. Since $g \in \Sigma'$ is univalent, it has an inverse $g^{-1} = h$ that satisfy

$$g^{-1}(g(z)) = z, \ (z \in \Delta^*)$$

and

$$g(g^{-1}(w)) = w, (M < |w| < \infty, M > 0)$$

where

$$g^{-1}(w) = h(w) = w + \sum_{n=0}^{\infty} \frac{C_n}{w^n}, \quad (M < |w| < \infty). \tag{1.5}$$

Analogous to the bi-univalent analytic functions, a function $g \in \Sigma'$ is said to be meromorphic bi-univalent if $g^{-1} \in \Sigma'$. We denote the class of all meromorphic bi-univalent functions by $\mathcal{M}_{\Sigma'}$. Estimates on the coefficients of meromorphic univalent functions were widely investigated in the literature, for example, Schiffer[13] obtained the estimate $|b_2| \leq \frac{2}{3}$ for meromorphic univalent functions $g \in \Sigma'$ with $b_0 = 0$ and Duren [3] gave an elementary proof of the inequality $|b_n| \leq \frac{2}{(n+1)}$ on the coefficient of meromorphic univalent functions $g \in \Sigma'$ with $b_k = 0$ for $1 \leq k < \frac{n}{2}$. For the coefficient of the inverse of meromorphic univalent functions $h \in \mathcal{M}_{\Sigma'}$, Springer [14] proved that $|C_3| \leq 1$ and $|C_3 + \frac{1}{2}C_1^2| \leq \frac{1}{2}$ and conjectured that $|C_{2n-1}| \leq \frac{(2n-1)!}{n!(n-1)!}$, $(n = 1, 2, ...)$.

In 1977, Kubota [7] has proved that the Springer conjecture is true for $n = 3, 4, 5$ and subsequently Schober [12] obtained a sharp bounds for the coefficients $C_{2n-1}, 1 \leq n \leq 7$ of the inverse of meromorphic univalent functions in Δ^*. Recently, Kapoor and Mishra [6] (see [16]) found the coefficient estimates for a class consisting of inverses of meromorphic starlike univalent functions of order α in Δ^*.

Motivated by the earlier work of [4, 5, 6, 20], in the present investigation, a new subclass of meromorphic bi-univalent functions class Σ' of complex order $\gamma \in \mathbb{C}\backslash\{0\}$, is introduced and estimates for the coefficients $|b_0|$ and $|b_1|$ of functions in the newly introduced subclass are obtained. Several new consequences of the results are also pointed out.

Definition 1.1. *For $0 \leq \lambda \leq 1, \mu \geq 0, \mu > \lambda$ a function $g(z) \in \Sigma'$ given by (1.4) is said to be in the class $\mathcal{M}_{\Sigma'}^{\gamma}(\lambda, \mu, \phi)$ if the following conditions are satisfied:*

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu} + \lambda g'(z)\left(\frac{g(z)}{z}\right)^{\mu-1} - 1\right] \prec \phi(z) \tag{1.6}$$

and

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu} + \lambda h'(w)\left(\frac{h(w)}{w}\right)^{\mu-1} - 1\right] \prec \phi(w) \tag{1.7}$$

where $z, w \in \Delta^*$, $\gamma \in \mathbb{C}\backslash\{0\}$ and the function h is given by (1.5).

By suitably specializing the parameters λ and μ, we state the new subclasses of the class meromorphic bi-univalent functions of complex order $\mathcal{M}_{\Sigma'}^{\gamma}(\lambda,\mu,\phi)$ as illustrated in the following Examples.

Example 1.1. *For $0 \leq \lambda < 1, \mu = 1$ a function $g \in \Sigma'$ given by (1.4) is said to be in the class $\mathcal{M}_{\Sigma'}^{\gamma}(\lambda,1,\phi) \equiv \mathcal{F}_{\Sigma'}^{\gamma}(\lambda,\phi)$ if it satisfies the following conditions respectively:*

$$1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(z)}{z}\right)+\lambda g'(z)-1\right] \prec \phi(z)$$

and

$$1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{h(w)}{w}\right)+\lambda h'(w)-1\right] \prec \phi(w)$$

where $z,w \in \Delta^*$, $\gamma \in \mathbb{C}\backslash\{0\}$ and the function h is given by (1.5).

Example 1.2. *For $\lambda = 1, 0 \leq \mu < 1$ a function $g \in \Sigma'$ given by (1.4) is said to be in the class $\mathcal{M}_{\Sigma'}^{\gamma}(1,\mu,\phi) \equiv \mathcal{B}_{\Sigma'}^{\gamma}(\mu,\phi)$ if it satisfies the following conditions respectively:*

$$1+\frac{1}{\gamma}\left[g'(z)\left(\frac{g(z)}{z}\right)^{\mu-1}-1\right] \prec \phi(z)$$

and

$$1+\frac{1}{\gamma}\left[h'(w)\left(\frac{h(w)}{w}\right)^{\mu-1}-1\right] \prec \phi(w)$$

where $z,w \in \Delta^*$, $\gamma \in \mathbb{C}\backslash\{0\}$ and the function h is given by (1.5).

Example 1.3. *For $\lambda = 1, \mu = 0,$ a function $g \in \Sigma'$ given by (1.4) is said to be in the class $\mathcal{M}_{\Sigma'}^{\gamma}(1,0,\phi) \equiv \mathcal{S}_{\Sigma'}^{\gamma}(\phi)$ if it satisfies the following conditions respectively:*

$$1+\frac{1}{\gamma}\left(\frac{zg'(z)}{g(z)}-1\right) \prec \phi(z)$$

and

$$1+\frac{1}{\gamma}\left(\frac{wh'(w)}{h(w)}-1\right) \prec \phi(w)$$

where $z,w \in \Delta^*$, $\gamma \in \mathbb{C}\backslash\{0\}$ and the function h is given by (1.5).

2. Coefficient estimates for the function class $\mathcal{M}_{\Sigma'}^{\gamma}(\lambda,\mu,\phi)$

In this section we obtain the coefficients $|b_0|$ and $|b_1|$ for $g \in \mathcal{M}_{\Sigma'}^{\gamma}(\lambda,\mu,\phi)$ associating the given functions with the functions having positive real part. In order to prove our result we recall the following lemma.

Lemma 2.1. [11] *If $\Phi \in \mathcal{P}$, the class of all functions with $\Re(\Phi(z)) > 0, (z \in \Delta)$ then*

$$|c_k| \leq 2, \text{ for each } k,$$

where

$$\Phi(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad \textit{for } z \in \Delta.$$

Define the functions p and q in $\mathcal{P}$ given by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \cdots$$

and

$$q(z) = \frac{1+v(z)}{1-v(z)} = 1 + \frac{q_1}{z} + \frac{q_2}{z^2} + \cdots .$$

It follows that

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[\frac{p_1}{z} + \left(p_2 - \frac{p_1^2}{2}\right)\frac{1}{z^2} + \cdots\right]$$

and

$$v(z) = \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[\frac{q_1}{z} + \left(q_2 - \frac{q_1^2}{2}\right)\frac{1}{z^2} + \cdots\right].$$

Note that for the functions $p(z), q(z) \in \mathcal{P}$, we have

$$|p_i| \leq 2 \text{ and } |q_i| \leq 2 \text{ for each } i.$$

Theorem 2.1. *Let g is given by (1.4) be in the class $\mathcal{M}^{\gamma}_{\Sigma'}(\lambda, \mu, \phi)$. Then*

$$|b_0| \leq \left|\frac{\gamma B_1}{\mu - \lambda}\right| \tag{2.1}$$

and

$$|b_1| \leq \left| \gamma\sqrt{\left(\frac{(\mu-1)\gamma B_1^2}{2(\mu-\lambda)^2}\right)^2 + \left(\frac{B_2}{\mu - 2\lambda}\right)^2} \right| \tag{2.2}$$

where $\gamma \in \mathbb{C}\backslash\{0\}, 0 \leq \lambda \leq 1, \mu \geq 0, \mu > \lambda$ and $z, w \in \Delta^*$.

Proof. It follows from (1.6) and (1.7) that

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu} + \lambda g'(z)\left(\frac{g(z)}{z}\right)^{\mu-1} - 1\right] = \phi(u(z)) \tag{2.3}$$

and

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu} + \lambda h'(w)\left(\frac{h(w)}{w}\right)^{\mu-1} - 1\right] = \phi(v(w)). \tag{2.4}$$

In light of (1.4), (1.5), (1.6) and (1.7), we have

$$\begin{aligned} 1 + \frac{1}{\gamma}&\left[(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu} + \lambda g'(z)\left(\frac{g(z)}{z}\right)^{\mu-1} - 1\right] \\ &= 1 + \frac{1}{\gamma}\left[(\mu-\lambda)\frac{b_0}{z} + (\mu - 2\lambda)[\frac{(\mu-1)}{2}b_0^2 + b_1]\frac{1}{z^2} + ...\right] \\ &= 1 + B_1 p_1 \frac{1}{2z} + \left[\frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2\right]\frac{1}{z^2} + ... \end{aligned} \tag{2.5}$$

and

$$1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu}+\lambda h'(w)\left(\frac{h(w)}{w}\right)^{\mu-1}-1\right] \tag{2.6}$$
$$=1+\frac{1}{\gamma}\left[-(\mu-\lambda)\frac{b_0}{z}+(\mu-2\lambda)[\frac{(\mu-1)}{2}b_0^2-b_1]\frac{1}{z^2}+...\right]$$
$$=1+B_1q_1\frac{1}{2w}+\left[\frac{1}{2}B_1(q_2-\frac{q_1^2}{2})+\frac{1}{4}B_2q_1^2\right]\frac{1}{w^2}+...$$

Now, equating the coefficients in (2.5) and (2.6), we get

$$\frac{1}{\gamma}(\mu-\lambda)b_0=\frac{1}{2}B_1p_1, \tag{2.7}$$

$$\frac{1}{\gamma}(\mu-2\lambda)\left[(\mu-1)\frac{b_0^2}{2}+b_1\right]=\frac{1}{2}B_1(p_2-\frac{p_1^2}{2})+\frac{1}{4}B_2p_1^2, \tag{2.8}$$

$$-\frac{1}{\gamma}(\mu-\lambda)b_0=\frac{1}{2}B_1q_1, \tag{2.9}$$

and

$$\frac{1}{\gamma}(\mu-2\lambda)\left[(\mu-1)\frac{b_0^2}{2}-b_1\right]=\frac{1}{2}B_1(q_2-\frac{q_1^2}{2})+\frac{1}{4}B_2q_1^2. \tag{2.10}$$

From (2.7) and (2.9), we get

$$p_1=-q_1 \tag{2.11}$$

and

$$8(\mu-\lambda)^2b_0^2=\gamma^2B_1^2(p_1^2+q_1^2).$$

Hence,

$$b_0^2=\frac{\gamma^2B_1^2(p_1^2+q_1^2)}{8(\mu-\lambda)^2}. \tag{2.12}$$

Applying Lemma (2.1) for the coefficients p_1 and q_1, we have

$$|b_0|\leq\left|\frac{\gamma B_1}{\mu-\lambda}\right|.$$

Next, in order to find the bound on $|b_1|$ from (2.8), (2.10) and (2.11), we obtain

$$(\mu-2\lambda)^2\ b_1^2=(\mu-2\lambda)^2(\mu-1)^2\frac{b_0^4}{4} \tag{2.13}$$
$$-\gamma^2\ \left(\frac{B_1^2}{4}p_2q_2+(B_2-B_1)B_1(p_2+q_2)\frac{p_1^2}{8}+(B_1-B_2)^2\ \frac{p_1^4}{16}\right).$$

Using (2.12) and applying Lemma (2.1) once again for the coefficients p_1,p_2 and q_2, we get

$$|b_1|\leq\ \left|\ \gamma\sqrt{\left(\frac{(\mu-1)\gamma B_1^2}{2(\mu-\lambda)^2}\right)^2+\left(\frac{B_2}{\mu-2\lambda}\right)^2}\ \right|.$$

□

Corollary 2.1. *Let $g(z)$ is given by (1.4) be in the class $\mathcal{F}^{\gamma}_{\Sigma'}(\lambda,\phi)$. Then*

$$|b_0| \leq \left|\frac{\gamma B_1}{1-\lambda}\right| \tag{2.14}$$

and

$$|b_1| \leq \left|\frac{\gamma B_2}{2\lambda-1}\right| \tag{2.15}$$

where $\gamma \in \mathbb{C}\backslash\{0\}, 0 \leq \lambda < 1$ and $z, w \in \Delta^*$.

Corollary 2.2. *Let $g(z)$ is given by (1.4) be in the class $\mathcal{B}^{\gamma}_{\Sigma'}(\mu,\phi)$. Then*

$$|b_0| \leq \left|\frac{\gamma B_1}{\mu-1}\right| \tag{2.16}$$

and

$$|b_1| \leq \left|\ \gamma\ \sqrt{\left(\frac{\gamma B_1^2}{2(\mu-1)}\right)^2 + \left(\frac{B_2}{\mu-2}\right)^2}\ \right| \tag{2.17}$$

where $\gamma \in \mathbb{C}\backslash\{0\}, 0 \leq \mu < 1$ and $z, w \in \Delta^*$.

Corollary 2.3. *Let $g(z)$ is given by (1.4) be in the class $\mathcal{S}^{\gamma}_{\Sigma'}(\phi)$. Then*

$$|b_0| \leq |\gamma\ B_1| \tag{2.18}$$

and

$$|b_1| \leq \left|\frac{\gamma}{2}\ \sqrt{\gamma^2 B_1^4 + B_2^2}\right| \tag{2.19}$$

where $\gamma \in \mathbb{C}\backslash\{0\}$ and $z, w \in \Delta^*$.

3. Corollaries and concluding Remarks

Analogous to (1.3), by setting $\phi(z)$ as given below:

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1), \tag{3.1}$$

we have

$$B_1 = 2\alpha,\ B_2 = 2\alpha^2.$$

For $\gamma = 1$ and $\phi(z)$is given by (3.1) we state the following corollaries:

Corollary 3.1. *Let g is given by (1.4) be in the class $\mathcal{M}^{1}_{\Sigma'}(\lambda,\mu,\left(\frac{1+z}{1-z}\right)^{\alpha}) \equiv \mathcal{M}_{\Sigma'}(\lambda,\alpha)$. Then*

$$|b_0| \leq \frac{2\alpha}{|\mu-\lambda|}$$

and

$$|b_1| \leq \left|\ 2\alpha^2\sqrt{\frac{(\mu-1)^2}{(\mu-\lambda)^4} + \frac{1}{(\mu-2\lambda)^2}}\ \right|$$

where $0 < \lambda \leq 1, \mu \geq 0, \mu > \lambda$ and $z, w \in \Delta^*$.

Corollary 3.2. *Let $g(z)$ is given by (1.4) be in the class* $\mathcal{F}^{1}_{\Sigma'}(\lambda, \left(\frac{1+z}{1-z}\right)^{\alpha}) \equiv \mathcal{F}_{\Sigma'}(\lambda, \alpha)$, *then*

$$|b_0| \leq \frac{2\alpha}{|1-\lambda|}$$

and

$$|b_1| \leq \frac{2\alpha^2}{|1-2\lambda|}$$

where $0 \leq \lambda < 1$ *and* $z, w \in \Delta^*$.

Corollary 3.3. *Let $g(z)$ is given by (1.4) be in the class* $\mathcal{B}^{1}_{\Sigma'}(\lambda, \left(\frac{1+z}{1-z}\right)^{\alpha}) \equiv \mathcal{B}_{\Sigma'}(\mu, \alpha)$, *then*

$$|b_0| \leq \frac{2\alpha}{|\mu-1|}$$

and

$$|b_1| \leq \left| 2\alpha^2\sqrt{\frac{1}{(\mu-1)^2} + \frac{1}{(\mu-2)^2}} \right|$$

where $0 \leq \mu < 1$ *and* $z, w \in \Delta^*$.

Corollary 3.4. *Let $g(z)$ is given by (1.4) be in the class* $\mathcal{S}^{1}_{\Sigma'}\left(\left[\frac{1+z}{1-z}\right]^{\alpha}\right) \equiv \mathcal{S}_{\Sigma'}(\alpha)$ *then*

$$|b_0| \leq 2\alpha$$

and

$$|b_1| \leq \alpha^2\sqrt{5}$$

where $z, w \in \Delta^*$.

On the other hand if we take

$$\phi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \leq \beta < 1), \tag{3.2}$$

then

$$B_1 = B_2 = 2(1-\beta).$$

For $\gamma = 1$ and $\phi(z)$is given by (3.2) we state the following corollaries:

Corollary 3.5. *Let g is given by (1.4) be in the class* $\mathcal{M}^{1}_{\Sigma'}\left(\lambda, \mu, \frac{1+(1-2\beta)z}{1-z}\right) \equiv \mathcal{M}_{\Sigma'}(\lambda, \mu, \beta)$. *Then*

$$|b_0| \leq \frac{2(1-\beta)}{|\mu-\lambda|}$$

and

$$|b_1| \leq \left| 2(1-\beta)\sqrt{\frac{(\mu-1)^2(1-\beta)^2}{(\mu-\lambda)^4} + \frac{1}{(\mu-2\lambda)^2}} \right|$$

where $0 \leq \lambda \leq 1, \mu \geq 0, \mu > \lambda$ *and* $z, w \in \Delta^*$.

Remark 3.1. *We obtain the estimates $|b_0|$ and $|b_1|$ as obtained in the Corollaries 3.2 to 3.4 for function g given by (1.4) are in the subclasses defined in Examples 1.1 to 1.3.*

Concluding Remarks: Let a function $g \in \Sigma'$ given by (1.4). By taking $\gamma = (1-\alpha)cos\beta\ e^{-i\beta}$, $|\beta| < \frac{\pi}{2}$, $0 \leq \alpha < 1$ the class $\mathcal{M}^{\gamma}_{\Sigma'}(\lambda, \mu, \phi) \equiv \mathcal{M}^{\beta}_{\Sigma'}(\alpha, \lambda, \mu, \phi)$ called the generalized class of $\beta-$ bi spiral-like functions of order $\alpha(0 \leq \alpha < 1)$ satisfying the following conditions.

$$e^{i\beta}\left[(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu} + \lambda g'(z)\left(\frac{g(z)}{z}\right)^{\mu-1}\right] \prec [\phi(z)(1-\alpha)+\alpha]cos\ \beta + isin\beta$$

and

$$e^{i\beta}\left[(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu} + \lambda h'(w)\left(\frac{h(w)}{w}\right)^{\mu-1}\right] \prec [\phi(w)(1-\alpha)+\alpha]cos\ \beta + isin\beta$$

where $0 \leq \lambda \leq 1, \mu \geq 0$ and $z, w \in \Delta^*$ and the function h is given by (1.5).

For function $g \in \mathcal{M}^{\beta}_{\Sigma'}(\alpha, \lambda, \mu, \phi)$ given by (1.4),by choosing $\phi(z) = (\frac{1+z}{1-z})$, (or $\phi(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$), we obtain the estimates $|b_0|$ and $|b_1|$ by routine procedure (as in Theorem2.1) and so we omit the details.

References

[1] D.A. Brannan, J.G. Clunie (Eds.), Aspects of Contemporary Complex Analysis (Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1–20, 1979), Academic Press, New York and London, 1980.

[2] D.A. Brannan, T.S. Taha, On some classes of bi-univalent functions, in: S.M. Mazhar, A. Hamoui, N.S. Faour (Eds.), Mathematical Analysis and its Applications, Kuwait; February 18–21, 1985, in: KFAS Proceedings Series, vol. 3, Pergamon Press, Elsevier Science Limited, Oxford, 1988, 53–60. see also Studia Univ. Babeś-Bolyai Math. 31 (2) (1986) 70–77.

[3] P. L. Duren, Coefficients of meromorphic schlicht functions, Proceedings of the American Mathematical Society, vol. 28, 169–172, 1971.

[4] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, Journal of Classical Analysis **2**(1) (2013), 49–60.

[5] S. G. Hamidi,S. A. Halim ,Jay M. Jahangiri,Coefficent estimates for bi-univalent strongly starlike and Bazilevic functions, International Journal of Mathematics Research. Vol. 5, No. 1 (2013), 87–96.

[6] G. P. Kapoor and A. K. Mishra, Coefficient estimates for inverses of starlike functions of positive order,Journal of Mathematical Analysis and Applications, vol. 329, no. 2, 922–934, 2007.

[7] Y. Kubota, Coefficients of meromorphic univalent functions, Kōdai Math. Sem. Rep. 28 (1976/77), no. 2-3, 253–261.

[8] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. 18 (1967) 63–68.

[9] W.C. Ma, D. Minda, A unified treatment of some special classes of functions, *in: Proceedings of the Conference on Complex Analysis, Tianjin,* 1992, 157 – 169, Conf. Proc.Lecture Notes Anal. 1. Int. Press, Cambridge, MA, 1994.

[10] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Rational Mech. Anal. 32 (1969) 100–112.

[11] C. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

[12] G. Schober, Coefficients of inverses of meromorphic univalent functions, Proc. Amer. Math. Soc. 67 (1977), no. 1, 111–116.

[13] M. Schiffer, Sur un probléme dextrémum de la représentation conforme, Bull. Soc. Math. France 66 (1938), 48–55.

[14] G. Springer, The coefficient problem for schlicht mappings of the exterior of the unit circle, Trans. Amer. Math. Soc. 70 (1951), 421–450.

[15] H.M. Srivastava, A.K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. 23 (2010) 1188–1192.

[16] H. M. Srivastava, A. K. Mishra, and S. N. Kund, Coefficient estimates for the inverses of starlike functions represented by symmetric gap series, Panamerican Mathematical Journal, vol. 21, no. 4, 105–123, 2011.
[17] T.S. Taha, Topics in univalent function theory, Ph.D. Thesis, University of London, 1981.
[18] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett. 25 (2012), 990–994.
[19] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, Appl. Math. Comput. 218 (2012), 11461–11465.
[20] Q-H. Xu , Chun-Bo Lv , H.M. Srivastava Coefficient estimates for the inverses of a certain general class of spirallike functions Appl. Math. Comput. 219 (2013) 7000–7011.

School of Advanced Sciences, VIT University, Vellore - 632 014, India

*Corresponding author