International Journal of Analysis and Applications
Volume 16, Number 3 (2018), 400-413
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-16-2018-400

ON WEAKLY 2-ABSORBING SEMI-PRIMARY SUBMODULES OF MODULES OVER COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with identity and let M be a unitary R-module. We say that a proper submodule N of M is a weakly 2-absorbing semi-primary submodule if $a_1, a_2 \in R, m \in N$ with $0 \neq a_1a_2m \in N$, then $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n. In this paper, we study weakly 2-absorbing semi-primary submodules and we prove some basic properties of these submodules. Also, we give a characterization of weakly 2-absorbing semi-primary submodules and we investigate weakly 2-absorbing semi-primary submodules of some well-known modules.

1. Introduction

Throughout this paper, we assume that all rings are commutative with $1 \neq 0$. Let R be a commutative ring and let M be an R-module. We will denote by $(N : M)$ a residual of N by M, that is, the set of all $r \in R$ such that $rM \subseteq N$. Clearly, $\sqrt{I} = \{r \in R : r^n \in I$ for some positive integer $n\}$ denotes the radical ideal of R.

In 2003, Anderson and Smith [1] introduced the concept of a weakly prime ideal of a commutative ring. They said that a proper ideal P of the commutative ring R is weakly prime if $a, b \in R$ and $0 \neq ab \in P$, then $a \in P$ or $b \in P$. A weakly primary ideals were first introduced and studied by Atani and Farzalipour in [2]. Recall that a proper ideal P of R is called a weakly primary ideal of R as in [2] if for $a, b \in R$ with $0 \neq ab \in P$, then $a \in P$ or $b^n \in P$ for some positive integer n. Clearly, a weakly prime ideal of R is also a

2010 *Mathematics Subject Classification.* 13A15,13F05.
Key words and phrases. weakly 2-absorbing semi-primary submodule; 2-absorbing semi-primary submodule; absorbing semi-primary triple-zero; weakly 2-absorbing primary ideal.

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weakly primary ideal of R. The concept of weakly 2-absorbing ideals, which is a generalization of 2-absorbing ideals, was introduced by Badawi and Darani in [3]. Recall from [3] that a proper ideal I of R is said to be a weakly 2-absorbing ideal of R if whenever $a, b, c \in R$ with $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In [4], Badawi et. al. defined a proper ideal I of a commutative ring R to be a weakly 2-absorbing primary ideal if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

The concept of weakly prime submodule was introduced and studied by Behboodi and Koohi [5]. We recall that a proper submodule N of M is called a weakly prime submodule, if $0 \neq rm \in N$, where $r \in R, m \in M$, then $m \in N$ or $r \in (N : M)$. The idea of decomposition of submodules into weakly primary submodules were introduced by Atani and Farzalipour in [2]. A weakly primary submodule N of M to be a proper submodule of M and if $r \in R, m \in M$ and $0 \neq rm \in N$, then $m \in N$ or $r^n \in (N : M)$ for some positive integer n. Clearly, every primary submodule of a module is a weakly primary submodule. In [6], the concept of weakly 2-absorbing submodule generalized to 2-absorbing submodule of a module over a commutative ring. A proper submodule N of M is called a weakly 2-absorbing submodule, if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $ab \in (N : M)$ or $am \in N$ or $bm \in N$. In 2016, Mostafanasab et al. [11] introduced the concept of weakly 2-absorbing primary submodules of modules over commutative rings with identities. Recall that a proper submodule N of M is called a weakly 2-absorbing primary submodule of M as in [11] if whenever $0 \neq abm \in N$ for some $a, b \in R$ and $m \in M$, then $ab \in (N : M)$ or $am \in M - rad(N)$ or $bm \in M - rad(N)$. The concept of weakly classical prime submodule, which is a generalization of classical prime submodule, was introduced by Mostafanasab et al. in [10]. Recall from [10] that a proper submodule N of M is said to be a weakly classical prime submodule of M if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$. The concept of weakly classical primary submodule, a generalization of primary submodules was introduced and investigated in [9]. He weakly classical primary submodule N of M to be a proper submoduleof R and if $a, b \in R$ and $0 \neq abm \in N$, then $am \in N$ or $mb^n \in N$ for some positive integer n.

Motivated and inspired by the above works, the purposes of this paper are to introduce generalizations of weakly 2-absorbing primary submodule to the context of weakly 2-absorbing semi-primary submodule. A proper submodule N of M to be a weakly 2-absorbing semi-primary submodule of M if whenever $0 \neq a_1a_2m \in N$ for $a_1, a_2 \in R, m \in M$, then $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n. Some characterizations of weakly 2-absorbing semi-primary submodules are obtained. Moreover, we investigate relationships between 2-absorbing semi-primary and weakly 2-absorbing semi-primary submodules of modules over commutative rings.

2. Properties of weakly 2-Absorbing Semiprimary Submodules

The results of the following theorems seem to play an important role to study weakly 2-absorbing semi-primary submodules of modules over commutative rings; these facts will be used frequently and normally we shall make no reference to this definition.

Definition 2.1. *A proper submodule N of an R-module M is called a weakly 2-absorbing semi-primary (2-absorbing semi-primary) submodule, if for each $m \in M$ and $a_1, a_2 \in R$, $0 \neq a_1a_2m \in N (a_1a_2m \in N)$, then $a_1a_2 \in \sqrt{(N:M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n.*

Remark 2.1. *It is easy to see that every weakly 2-absorbing primary submodule (2-absorbing semi-primary) submodule is weakly 2-absorbing semi-primary submodule.*

The following example shows that the converse of Definition 2.1 is not true.

Example 2.1. *Let $R = \boldsymbol{Z}$ and $M = \boldsymbol{Z}$. Consider the submodule $N = 12\boldsymbol{Z}$ of M. It is easy to see that N is a 2-absorbing semi-primary submodule of M. Notice that $2 \cdot 2 \cdot 3 \in N$, but $2 \cdot 3 \notin N$ and $(2 \cdot 2)^n \notin (N : M)$ for all positive integer n. Therefore N is not a 2-absorbing primary submodule of M.*

Example 2.2. *Let $R = \boldsymbol{Z}$ and $M = \boldsymbol{Z}_{30}$. Consider the submodule $N = \{[0]\}$ of M. It is easy to see that N is a weakly 2-absorbing semi-primary submodule of M. Notice that $(2\cdot3)[5] \in \{[0]\}$, but $2\cdot3 \notin \sqrt{(N:M)}, 2[5] \notin \{[0]\}$ and $3^n[5] \notin \{[0]\}$ for all positive integer n. Therefore N is not a 2-absorbing semi-primary submodule of M.*

Theorem 2.1. *Let N be a proper submodule of an R-module M. Then the following statements hold:*

1. *If N is a weakly 2-absorbing semi-primary submodule of M, then $(N : m)$ is a weakly 2-absorbing primary ideal of R for every $m \in M - N$.*
2. *For every $m \in M - N$ if $(N : m)$ is a weakly primary ideal of R, then N is a weakly 2-absorbing semi-primary submodule of M.*

Proof. 1. Let $a_1, a_2, a_3 \in R$ such that $0 \neq a_1a_2a_3 \in (N : m)$. Clearly, $0 \neq a_1a_3(a_2m) \in N$. By Definition 2.1, $a_1a_3 \in \sqrt{(N:M)} \subseteq \sqrt{(N:m)}$ or $a_1a_2m \in N$ or $a_3^n a_2m \in N$ for some positive integer n. Therefore $a_1a_2 \in (N : m)$ or $a_2a_3 \in \sqrt{(N:m)}$ or $a_1a_3 \in \sqrt{(N:m)}$. Hence $(N : m)$ is a weakly 2-absorbing primary ideal of R.

2. Let $a_1, a_2 \in R$ such that $0 \neq a_1a_2m \in N$. Then $0 \neq a_1a_2 \in (N : m)$. By assumption, $a_1 \in (N : m)$ or $a_2^n \in (N : m)$ for some positive integer n. Therefore $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n. Hence N is a weakly 2-absorbing semi-primary submodule of M. □

But the converse of the above theorem is not true. For every $m \in M - N$, if $(N : m)$ is weakly 2-absorbing primary ideal, then N may not be weakly 2-absorbing semi-primary. Let $M = \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$ be an

Z-module. Consider the submodule $N = \{0\} \times 6\mathbf{Z} \times \mathbf{Z}$ of M. Clearly, $(N : (m_1, m_2, m_3)) = \{0\}$ is a weakly 2-absorbing primary ideal of R, where $(m_1, m_2, m_3) \in M - N$. Notice that $(0, 0, 0) \neq (2 \cdot 3)(0, 1, 1) \in N$, but $2 \cdot 3 \notin \sqrt{(N : M)}, 2(0, 1, 1) \notin N$ and $3^n(0, 1, 1) \notin N$ for all positive integer n. Therefore N is not a weakly 2-absorbing semi-primary submodule of M.

Theorem 2.2. *If N is a weakly 2-absorbing semi-primary submodule of an R-module M, then $(N : r)$ is a weakly 2-absorbing semi-primary submodule of M containing N for every $r \in R - (N : M)$.*

Proof. Let $a_1, a_2 \in R$ and $m \in M$ such that $0 \neq a_1a_2m \in (N : r)$. Then $0 \neq a_1a_2(rm) = ra_1a_2m \in N$. By Definition 2.1, $a_1a_2 \in \sqrt{(N : M)}$ or $a_1rm \in N$ or $a_2^n rm \in N$ for some positive integer n. Therefore $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in (N : r)$ or $a_2^n \in (N : r)$ for some positive integer n. Hence $(N : r)$ is a weakly 2-absorbing semi-primary submodule of M. □

Theorem 2.3. *Let $\{0\}$ be a 2-absorbing semi-primary submodule of an R-module M. Then N is a weakly 2-absorbing semi-primary submodule of M if and only if N is a 2-absorbing semi-primary submodule of M.*

Proof. Suppose that N is a 2-absorbing semi-primary submodule of M. Clearly, N is a weakly 2-absorbing semi-primary submodule of M.

Conversely, assume that N is a weakly 2-absorbing semi-primary submodule of M. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2m \in N$. If $a_1a_2m \notin \{0\}$, then $0 \neq a_1a_2m \in N$. By Definition 2.1, $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n. Now if $a_1a_2m \in \{0\}$, then $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n. Hence N is a 2-absorbing semi-primary submodule of M. □

Theorem 2.4. *Let M and $\acute{M}$ be two R-modules and $f : M \to \acute{M}$ be an epimorphism of an R-module. If N is a weakly 2-absorbing semi-primary submodule of M such that $ker f \subseteq N$, then $f(N)$ is a weakly 2-absorbing semi-primary submodule of $\acute{M}$.*

Proof. Let $a_1, a_2 \in R$ and $\acute{m} \in \acute{M}$ such that $0 \neq a_1a_2\acute{m} \in f(N)$. Thus $0 \neq a_1a_2\acute{m} = \acute{m}_0$ for some $\acute{m}_0 \in f(N)$. Since f is an epimorphism, there exist $m \in M$ and $m_0 \in N$ such that $\acute{m} = f(m)$ and $\acute{m}_0 = f(m_0)$. This implies that $0 \neq a_1a_2f(m) = f(m_0)$. Therefore $f(a_1a_2m - m_0) = 0$ and so $a_1a_2m - m_0 \in ker f \subseteq N$. Also, $0 \neq a_1a_2m \in N$, because if $a_1a_2m = 0$, then $m_0 \in ker f$. It follows that $f(m_0) = 0$, a contradiction. Now, since N is a weakly 2-absorbing semi-primary, we have $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n. Therefore $a_1a_2 \in \sqrt{(f(N) : \acute{M})}$ or $a_1\acute{m} \in f(N)$ or $a_2^n\acute{m} \in f(N)$ for some positive integer n. Hence $f(N)$ is a 2-absorbing semi-primary submodule of $\acute{M}$. □

Theorem 2.5. *Let M be an R-module and $N \subseteq K$ be two submodules of M. If K is a weakly 2-absorbing semi-primary submodule of M, then K/N is a weakly 2-absorbing semi-primary submodule of M/N.*

Proof. Let $a_1, a_2 \in R$ and $m \in M$ such that $N \neq a_1a_2(m + N) \in (K/N)$. Then $0 \neq a_1a_2m \in K$. By Definition 2.1, $a_1a_2 \in \sqrt{(K : M)}$ or $a_1m \in K$ or $a_2^n m \in K$ for some positive integer n. Therefore $a_1a_2 \in \sqrt{(K/N : M/N)}$ or $a_1(m + N) \in K/N$ or $a_2^n(m + N) \in K/N$ for some positive integer n. Hence K/N is a weakly 2-absorbing semi-primary submodule of M/N. □

Theorem 2.6. *Let M be an R-module and $N \subseteq K$ be two submodules of M. Suppose that N is a weakly 2-absorbing semi-primary submodule of M. If K/N is a weakly 2-absorbing semi-primary submodule of M/N, then K is a weakly 2-absorbing semi-primary submodule of M.*

Proof. Let $a_1, a_2 \in R$ and $m \in M$ such that $0 \neq a_1a_2m \in K$. If $a_1a_2m \in N$, then $0 \neq a_1a_2m \in N$. By Definition 2.1, $a_1a_2 \in \sqrt{(N : M)} \subseteq \sqrt{(K : M)}$ or $a_1m \in N \subseteq K$ or $a_2^n m \in N \subseteq K$ for some positive integer n. If $a_1a_2m \notin N$, then $N \neq a_1a_2(m + N) \in N$. Again, by Definition 2.1, $a_1a_2 \in \sqrt{(K/N : M/N)}$ or $a_1(m+N) \in K/N$ or $a_2^n(m+N) \in K/N$ for some positive integer n. Thus $a_1a_2 \in \sqrt{(K : M)}$ or $a_1m \in K$ or $a_2^n m \in K$ for some positive integer n. Hence K is a weakly 2-absorbing semi-primary submodule of M. □

Corollary 2.1. *Then N is a weakly 2-absorbing semi-primary submodule of an R-module M if and only if $N/\{0\}$ is a weakly 2-absorbing semi-primary submodule of an R-module $M/\{0\}$.*

Proof. It is straightforward by Theorem 2.5 and Theorem 2.6. □

Theorem 2.7. *Let N be a submodule of an R-module M and S be a multiplicative subset of R. If N is a weakly 2-absorbing semi-primary submodule of M such that $(N : M) \cap S = \emptyset$, then $S^{-1}N$ is a weakly 2-absorbing semi-primary submodule of $S^{-1}M$.*

Proof. Clearly, $S^{-1}N$ is a proper submodule of $S^{-1}M$. Let $a_1, a_2 \in R, s_1, s_2, s_3 \in S$ and $m \in M$ such that $0 \neq \frac{a_1}{s_1}\frac{a_2}{s_2}\frac{m}{s_3} \in S^{-1}N$. Then there exists $s \in S$ such that $sa_1a_2m \in N$. If $sa_1a_2m = 0$, then $\frac{a_1}{s_1}\frac{a_2}{s_2}\frac{m}{s_3} = \frac{sa_1}{ss_1}\frac{a_2}{s_2}\frac{m}{s_3} = \frac{0}{1}$, a contradiction. If $sa_1a_2m \neq 0$, then $0 \neq a_1a_2(sm) \in N$. By Definition 2.1, $a_1a_2 \in \sqrt{(N : M)}$ or $a_1sm \in N$ or $a_2^n sm \in N$ for some positive integer n. Thus $\frac{a_1}{s_1}\frac{a_2}{s_2} \in \sqrt{(S^{-1}N : S^{-1}M)}$ or $\frac{a_1}{s_1}\frac{m}{s_3} = \frac{a_1sm}{s_1s_3s} \in S^{-1}N$ or $(\frac{a_2}{s_2})^n\frac{m}{s_3} = \frac{a_2^n sm}{s_2^n s_3 s} \in S^{-1}N$ for some positive integer n. Hence $S^{-1}N$ is a weakly 2-absorbing semi-primary submodule of $S^{-1}M$. □

Theorem 2.8. *Let N be a submodule of an R-module M and S be a multiplicative subset of R. If $S^{-1}N$ is a weakly 2-absorbing semi-primary submodule of $S^{-1}M$ such that $S \cap Zd(N) = \emptyset$ and $S \cap Zd(M/N) = \emptyset$, then N is a weakly 2-absorbing semi-primary submodule of M.*

Proof. Let $a_1, a_2 \in R$ and $m \in M$ such that $0 \neq a_1a_2m \in N$. Then $\frac{a_1}{1}\frac{a_2}{1}\frac{m}{1} \in S^{-1}N$. If $\frac{a_1}{1}\frac{a_2}{1}\frac{m}{1} = \frac{0}{1}$, then there exists $s \in S$ such that $sa_1a_2m = 0$ which is a contradiction. If $\frac{a_1}{1}\frac{a_2}{1}\frac{m}{1} \neq \frac{0}{1}$, then $\frac{0}{1} \neq \frac{a_1}{1}\frac{a_2}{1}\frac{m}{1} \in S^{-1}N$. By Definition 2.1, $\frac{a_1}{1}\frac{a_2}{1} \in \sqrt{(S^{-1}N : S^{-1}M)}$ or $\frac{a_1}{1}\frac{m}{1} \in S^{-1}N$ or $(\frac{a_2}{1})^n\frac{m}{1} \in S^{-1}N$ for some positive integer

n. If $\frac{a_1}{1}\frac{a_2}{1} \in \sqrt{(S^{-1}N : S^{-1}M)}$, then $(\frac{a_1}{1}\frac{a_2}{1})^n \in (S^{-1}N : S^{-1}M)$ for some positive integer n. Thus there exists $s \in S$ such that $s(a_1a_2)^nM \subseteq N$ for some positive integer n. Since $S \cap Zd(M/N) = \emptyset$, we have $(a_1a_2)^nM \subseteq N$ so $a_1a_2 \in \sqrt{(N : M)}$. If $\frac{a_1}{1}\frac{m}{1} \in S^{-1}N$, there exists $s \in S$ such that $sa_1m \in N$. Thus $s(a_1m + N) = sa_1m + N = N$. But $S \cap Zd(M/N) = \emptyset$, $a_1m \in N$. If $(\frac{a_2}{1})^n\frac{a_m}{1} \in N$, there exists $s \in S$ such that such that $sa_1^nm \in N$ for some positive integer n. Thus $s(a_2^nm + N) = sa_2^nm + N = N$ for some positive integer n. Since $S \cap Zd(M/N) = \emptyset$, we have $a_2^nm \in N$ for some positive integer n. Therefore N is a weakly 2-absorbing semi-primary submodule of M. □

Theorem 2.9. *Let N be a proper submodule of an R-module M. The following conditions are equivalent:*

1. *N is a weakly 2-absorbing semi-primary submodule of M.*
2. *For every $a_1, a_2 \in R - (N : M)$ if $a_1a_2 \in R - \sqrt{(N : M)}$, then $(N : a_1a_2) \subseteq (0 : a_1a_2) \cup (N : a_1) \cup (N : a_2^n)$ for some positive integer n.*
3. *For every $a_1, a_2 \in R - (N : M)$ if R is a u-ring and $a_1a_2 \in R - \sqrt{(N : M)}$, then $(N : a_1a_2) \subseteq (0 : a_1a_2)$ or $(N : a_1a_2) \subseteq (N : a_1)$ or $(N : a_1a_2) \subseteq (N : a_2^n)$ for some positive integer n.*

Proof. $(1 \Rightarrow 2)$ Let $m \in (N : a_1a_2)$. Then $a_1a_2m \in N$. If $a_1a_2m = 0$, then $m \in (0 : a_1a_2) \subseteq (0 : a_1a_2) \cup (N : a_1) \cup (N : a_2^n)$ for some positive integer n. If $a_1a_2m \neq 0$, then $0 \neq a_1a_2m \in N$. By Definition 2.1, $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^nm \in N$ for some positive integer n. But $a_1a_2 \in R - \sqrt{(N : M)}$, $m \in (N : a_1)$ or $m \in (N : a_2^n)$ for some positive integer n. Therefore $m \in (N : a_1) \cup (N : a_2^n)$ for some positive integer n. Hence $(N : a_1a_2) = (0 : a_1a_2) \cup (N : a_1) \cup (N : a_2^n)$ for some positive integer n.

$(2 \Leftrightarrow 3)$ It is obvious.

$(2 \Rightarrow 1)$ Let $a_1, a_2 \in R$ such that $0 \neq a_1a_2m \in N$. Then $m \in (N : a_1a_2)$ and $m \notin (N : 0)$. By assumption, $m \in (0 : a_1a_2) \cup (N : a_1) \cup (N : a_2^n)$ for some positive integer n. Clearly, $a_1m \in N$ or $a_2^nm \in N$ for some positive integer n. Hence N is a weakly 2-absorbing semi-primary submodule of M. □

Corollary 2.2. *Let N be a proper submodule of an R-module M. The following conditions are equivalent:*

1. *N is a weakly 2-absorbing semi-primary submodule of M.*
2. *For every $a \in R - (N : M)$ and every ideal I of R such that $I \not\subseteq (N : M)$, if $aI \not\subseteq \sqrt{(N : M)}$, then $(N : aI) \subseteq (0 : aI) \cup (N : a) \cup (N : I^n)$ for some positive integer n.*
3. *For every $a \in R - (N : M)$ and every ideal I of R such that $I \not\subseteq (N : M)$, if R is a u-ring and $aI \not\subseteq \sqrt{(N : M)}$, then $(N : aI) \subseteq (0 : aI)$ or $(N : aI) \subseteq (N : a)$ or $(N : aI) \subseteq (N : I^n)$ for some positive integer n.*
4. *For every ideals I, J of R such that $I, J \not\subseteq (N : M)$, if $IJ \not\subseteq \sqrt{(N : M)}$, then $(N : IJ) \subseteq (0 : IJ) \cup (N : I) \cup (N : J^n)$ for some positive integer n.*
5. *For every ideals I, J of R such that $I, J \not\subseteq (N : M)$, if R is a u-ring and $IJ \not\subseteq \sqrt{(N : M)}$, then $(N : IJ) \subseteq (0 : IJ)$ or $(N : IJ) \subseteq (N : I)$ or $(N : IJ) \subseteq (N : J^n)$ for some positive integer n.*

Proof. It is clear from Theorem 2.9. □

Theorem 2.10. *Let N be a proper submodule of an R-module M. The following conditions are equivalent:*

(1) *N is a weakly 2-absorbing semi-primary submodule of M.*

(2) *For every $a \in R - (N : M)$ and $m \in M$, if $am \notin N$, then $(N : am) \subseteq (0 : am) \cup (\sqrt{((N : M)} : a) \cup \sqrt{(N : m)}$.*

Proof. $(1 \Rightarrow 2)$ Let $a \in R - (N : M)$ and $m \in M$ such that $am \notin N$. Assume that $r \in (N : am)$. Then $ram \in N$. If $ram \neq 0$, then $0 \neq ram \in N$. By Definition 2.1, $ar \in \sqrt{(N : M)}$ or $am \in N$ or $r^n m \in N$ for some positive integer n. Since $am \notin N$, we have $r \in (\sqrt{(N : M)} : a)$ or $r \in \sqrt{(N : m)}$. This implies that $r \in (\sqrt{(N : M)} : a) \cup \sqrt{(N : m)} \subseteq (0 : am) \cup (\sqrt{((N : M)} : a) \cup \sqrt{(N : m)}$. Thus $(N : am) \subseteq (0 : am) \cup (\sqrt{((N : M)} : a) \cup \sqrt{(N : m)}$. If $ram = 0$, then $r \in (0 : am) \subseteq (0 : am) \cup (\sqrt{((N : M)} : a) \cup \sqrt{(N : m)}$. Therefore $(N : am) \subseteq (0 : am) \cup (\sqrt{((N : M)} : a) \cup \sqrt{(N : m)}$.

$(2 \Rightarrow 1)$ It is clear. □

Corollary 2.3. *Let N be a proper submodule of an R-module M. The following conditions are equivalent:*

(1) *N is a weakly 2-absorbing semi-primary submodule of M.*

(2) *For every ideal I of R such that $I \subseteq R - (N : M)$ and $m \in M$, if $Im \nsubseteq N$, then $(N : Im) \subseteq (0 : Im) \cup (\sqrt{(N : M)} : I) \cup \sqrt{(N : m)}$.*

Proof. It is clear from Theorem 2.10. □

Definition 2.2. *Let N be a proper submodule of M. If N is a 2-absorbing semi-primary submodule and $a_1a_2m = 0, a_1a_2 \notin \sqrt{(N : M)}, a_1m \notin N$ and $a_2^n m \notin N$ for all positive integer n, then (a_1, a_2, m) is called a absorbing semi-primary triple-zero of N where $a_1, a_2 \in R, m \in M$.*

Theorem 2.11. *Let N be a weakly 2-absorbing semi-primary submodule of an R-module M. Suppose that K is a submodule of M and $a_1, a_2 \in R$ such that $N \subseteq K$ and $a_1a_2K \subseteq N$. If (a_1, a_2, m) is not a absorbing semi-primary triple-zero of N for every $m \in K$, then $a_1a_2 \in \sqrt{(K : M)}$ or $a_1K \subseteq N$ or $a_2^n K \subseteq N$ for some positive integer n.*

Proof. Assume that $a_1a_2 \notin \sqrt{(K : M)}, a_1K \nsubseteq N$ and $a_2^n K \nsubseteq N$ for all positive integer n. Then there are $k_1, k_2 \in K$ such that $a_1k_1 \notin N$ and $a_2^n k_2 \notin N$ for all positive integer n. If $a_1a_2k_1 \neq 0$, then $0 \neq a_1a_2k_1 \in N$. By Definition 2.1, $a_2^{n_1}k_1 \in N$ for some positive integer n_1. So let $a_1a_2k_1 = 0$. By Definition 2.2, $a_2^{n_2}k_1 \in N$ for some positive integer n_2. Now if $a_1a_2k_2 \neq 0$, then $0 \neq a_1a_2k_2 \in N$. Again, by Definition 2.1, $ak_2 \in N$. Next let $a_1a_2k_2 = 0$. Now by Definition 2.2, $a_1k_2 \in N$. Let $n_0 = max\{n_1, n_2\}$. Then $a_2^{n_0}k_1, a_1k_2 \in N$. Since $a_1a_2K \subseteq N$, we have $a_1a_2(k_1 + k_2) \in N$. If $a_1a_2(k_1 + k_2) \neq 0$, then $0 \neq a_1a_2(k_1 + k_2) \in N$. Thus by Definition 2.1, $a_1(k_1 + k_2) \in N$ or $a_2^{n_3}(k_1 + k_2) \in N$ for some positive integer n_3. This implies that $a_1k_1 \in N$

or $a_2^{n_4}k_2 \in N$ where $n_4 = max\{n_0, n_3\}$ and we get a contradiction. Assume that $a_1a_2(k_1 + k_2) = 0$. New since $(a_1, a_2, k_1 + k_2)$ is not a absorbing semi-primary triple-zero of N, we have $a_1(k_1 + k_2) \in N$ or $a_2^{n_5}(k_1 + k_2) \in N$ for some positive integer n_5. Clearly, $a_1k_1 \in N$ or $a_2^{n_6}k_2 \in N$, where $n_6 = max\{n_0, n_5\}$, which again is a contradiction. Hence $a_1a_2 \in \sqrt{(K : M)}$ or $a_1K \subseteq N$ or $a_2^nK \subseteq N$ for some positive integer n. □

Theorem 2.12. *Let N be a weakly 2-absorbing semi-primary submodule of an R-module M. Suppose that (a_1, a_2, m) is a absorbing semi-primary triple-zero of N for some $a_1, a_2 \in R$ and $m \in M$. Then*

(1) $a_1a_2N = \{0\}$;

(2) $a_1(N : M)m = \{0\}$;

(3) $(N : M)a_2m = \{0\}$;

(4) $(N : M)^2m = \{0\}$;

(5) $a_1(N : M)N = \{0\}$;

(6) $(N : M)a_2N = \{0\}$.

Proof. 1. Suppose that $a_1a_2N \neq \{0\}$. Then there exists $m_0 \in N$ such that $a_1a_2m_0 \notin \{0\}$. Thus $a_1a_2m + a_1a_2m_0 \neq 0$ so $0 \neq a_1a_2(m + m_0) \in N$. By Definition 2.1, $a_1a_2 \in \sqrt{(N : M)}$ or $a_1(m + m_0) \in N$ or $a_2^n(m+m_0) \in N$ for some positive integer n. Therefore $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^nm \in N$ for some positive integer n. This is a contradiction. Hence $a_1a_2N = \{0\}$.

2. Suppose that $a_1(N : M)m \neq \{0\}$. Then there exists $r \in (N : M)$ such that $a_1rm \neq 0$. Since $rm \in N$, we have $0 \neq a_1(a_2 + r)m \in N$. By Definition 2.1, $a_1(a_2 + r) \in \sqrt{(N : M)}$ or $a_1m \in N$ or $(a_2 + r)^nm \in N$ for some positive integer n. Thus $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n \in N$ for some positive integer n. This is a contradiction. Hence $a_1(N : M)m = \{0\}$.

3. The proof is similar to part 2.

4. Assume that $(N : M)^2m \neq \{0\}$. Then there exist $r, s \in (N : M)$ such that $rsm \neq 0$. Then by parts 1 and 2, $(a_1 + r)(a_2 + s)m \neq 0$. Clearly, $0 \neq (a_1 + r)(a_2 + s)m \in N$. By Definition 2.1, $(a_1 + r)(a_2 + s) \in \sqrt{(N : M)}$ or $(a_1 + r)m \in N$ or $(a_2 + s)^nm \in N$ for some positive integer n. Therefore $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n \in N$ for some positive integer n. This is a contradiction. Hence $(N : M)^2m = \{0\}$.

5. Suppose that $a_1(N : M)N \neq \{0\}$. Then there exist $r \in (N : M)$ and $m_0 \in N$ such that $a_1rm_0 \neq 0$. Therefore by parts 1 and 2 we conclude that $a_1(a_2 + r)(m + m_0) \neq 0$. Clearly, $0 \neq a_1(a_2 + r)(m + m_0) \in N$. By Definition 2.1, $a_1(a_2 + r) \in \sqrt{(N : M)}$ or $a_1(m + m_0) \in N$ or $(a_2 + r)^n(m + m_0) \in N$ for some positive integer n. Therefore $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^nm \in N$ for some positive integer n. This is a contradiction. Hence $a_1(N : M)N = \{0\}$.

6. The proof is similar to part 5. □

Theorem 2.13. *Let M be an R-module. If N is a weakly 2-absorbing semi-primary submodule of M that is not 2-absorbing semi-primary, then $(N:M)^2N=\{0\}$.*

Proof. Suppose that N is a weakly 2-absorbing semi-primary submodule of M that is not 2-absorbing semi-primary submodule. Then there exists a absorbing semi-primary triple-zero (a_1,a_2,m) of N for some $a_1,a_2\in R$ and $m\in M$. Assume that $(N:M)^2N\neq\{0\}$. Then there exist $r,s\in(N:M)$ and $m_0\in N$ such that $rsm_0\neq 0$. Since $(a_1+r)(a_2+s)(m+m_0)\neq 0$, we have $0\neq(a_1+r)(a_2+s)(m+m_0)\in N$. By Definition 2.1, $(a_1+r)(a_2+s)\in\sqrt{(N:M)}$ or $(a_1+r)(m+n)\in N$ or $(a_2+s)^n(m+n)\in N$ for some positive integer n. Therefore $a_1a_2\in\sqrt{(N:M)}$ or $a_1m\in N$ or $a_2^nm\in N$. This is a contradiction. Hence $(N:M)^2N=\{0\}$. □

Corollary 2.4. *Let M be a multiplication R-module. If N is a weakly 2-absorbing semi-primary submodule of M that is not 2-absorbing semi-primary submodule, then $N^3=\{0\}$.*

Proof. Suppose that N is a weakly 2-absorbing semi-primary submodule of M that is not 2-absorbing semi-primary submodule. By assumption, $N=(N:M)M$. Then by Theorem 2.13, $N^3=(N:M)^3M=(N:M)^2((N:M)M)=(N:M)^2N=\{0\}$. □

Lemma 2.1. *Suppose that N is a weakly 2-absorbing semi-primary submodule of an R-module M and $(0:m_2)$ is a 2-absorbing primary ideal of a ring R where $m_2\in M-N$. For all $m_1\in M$, if $rs\in(N:m_1)-\sqrt{(N:m_2)}$, then $(N:rsm_2)\subseteq(N:rm_2)\cup\sqrt{(N:s^nm_2)}$ for some positive integer n.*

Proof. Suppose that $rs\in(N:m_1)-(N:m_2)$ where $m_1\in M$ and $m_2\in M-N$. Let $a\in(N:rsm_2)$. Then $(ars)m_2=a(rsm_2)\in N$ so $ars\in(N:m_2)$. If $arsm_2\neq 0$, then $0\neq ars\in(N:m_2)$. By assumption, $ar\in(N:m_2)$ or $as\in\sqrt{(N:m_2)}$ or $rs\in\sqrt{(N:m_2)}$. By the assumption, $ar\in(N:m_2)$ or $as\in\sqrt{(N:m_2)}$. Thus $a\in(N:rm_2)$ or $a\in\sqrt{(N:s^nm_2)}$ for some positive integer n. This implies that $(N:rsm_2)\subseteq(N:rm_2)\cup\sqrt{(N:s^nm_2)}$ for some positive integer n. Now if $arsm_2=0$, then $ars\in(0:m_2)$. Thus $ar\in(0:m_2)$ or $as\in\sqrt{(N:m_2)}$ or $rs\in\sqrt{(N:m_2)}$. Therefore $(N:rsm_2)\subseteq(N:rm_2)\cup\sqrt{(N:s^nm_2)}$ for some positive integer n. □

Proposition 2.1. *Let N be an irreducible submodule of an R-module M. For all $r\in R$ if $(N:r)=(N:r^2)$, then N is a weakly 2-absorbing semi-primary submodule of M.*

Proof. Let $a_1,a_2\in R$ and $m\in M$ such that $0\neq a_1a_2m\in N$. Suppose that $a_1a_2\notin\sqrt{(N:M)}$, $a_1m\notin N$ and $a_2^nm\notin N$ for all positive integer n. Clearly, $N\subseteq(N+a_1a_2M)\cap(N+Ra_1m)\cap(N+Ra_2^nm)$ for all positive integer n. Let $m_0\in(N+a_1a_2M)\cap(N+Ra_1m)\cap(N+Ra_2^nm)$. This implies that $m_0\in N+a_1a_2M$, $m_0\in N+Ra_1m$ and $m_0\in N+Ra_2^nm$. Then there exist $r_1,r_2\in R$, $m_1\in M$ and $n_1,n_2\in N$ such that $n_1+a_1a_2m_1=m_0=n_2+r_1a_1m=m_0=n_3+b_2^nm$. Since $a_1n_1+a_1^2a_2m_1=a_1m_0=a_1n_2+r_1a_1^2m=a_1m_0=a_1n_3+a_1b_2^nm$, we have $a_1^2 r_1 m \in N$. It follows that $r_1 m \in (N : a_1^2)$. By the assumption, $r_1 m \in (N : a_1)$, so that $r_1 a_1 m \in N$. Thus $N = (N + a_1 a_2 M) \cap (N + Ra_1 m) \cap (N + Ra_2^n m)$. Now since N is an irreducible, we have $N + a_1 a_2 M \subseteq N$ or $a_1 m \in N + Ra_1 m \subseteq N$ or $a_2^n m \in N + Ra_2^n m \subseteq N$, a contradiction. Hence N is a weakly 2-absorbing semi-primary submodule of M. □

Theorem 2.14. *Let M_i be an R_i-module and N_i be a proper submodule of M_i, for $i = 1, 2$. If $N_1 \times M_2$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2$, then N_1 is a weakly 2-absorbing semi-primary submodule of M_1.*

Proof. Suppose that $N_1 \times M_2$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2$. Let $a_1, a_2 \in R_1$ and $m \in M_1$ such that $0 \neq a_1 a_2 m \in N_1$. Then $(0, 0) \neq (a_1, 0)(a_2, 0)(m, 0) = (a_1 a_2 m, 0) \in N_1 \times M_2$. By Definition 2.1, $(a_1 a_2, 0) = (a_1, 0)(a_2, 0) \in \sqrt{(N_1 \times M_2 : M_1 \times M_2)}$ or $(a_1 m, 0) = (a_1, 0)(m, 0) \in N_1 \times M_2$ or $(a_2^n m, 0) = (a_2, 0)^n (m, 0) \in N_1 \times M_2$ for some positive integer n. This implies that $a_1 a_2 \in \sqrt{(N_1 : M_1)}$ or $a_1 m \in N_1$ or $a_2^n m \in N_1$ for some positive integer n. Hence N_1 is a weakly 2-absorbing semi-primary submodule of M_1. □

Corollary 2.5. *Let M_i be an R_i-module and N_i be a proper submodule of M_i, for $i = 1, 2$. If $M_1 \times N_2$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2$, then N_2 is a weakly 2-absorbing semi-primary submodule of M_2.*

Proof. It is clear from Theorem 2.14. □

Corollary 2.6. *Let M_i be an R_i-module and N_i be a proper submodule of M_i, for $i = 1, 2, \ldots, k$. If $M_1 \times M_2 \times \ldots \times M_{j-1} \times N_j \times M_{j+1} \times \ldots \times M_k$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2 \times \ldots \times M_k$, then N_j is a weakly 2-absorbing semi-primary submodule of M_j.*

Proof. It is clear from Theorem 2.14 and Corollary 2.5. □

Theorem 2.15. *Let M_i be an R-module and let N_i be a proper submodule of M_i, for $i = 1, 2$. Then the following conditions are equivalent:*

(1) *$N_1 \times M_2$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2$.*

(2) (a) *N_1 is a weakly 2-absorbing semi-primary submodule of M_1.*

(b) *For each $a_1, a_2 \in R$ and $m \in M_1$ such that $a_1 a_2 m = 0$, if $a_1 a_2 \notin \sqrt{(N_1 : M_1)}$ and $a_1 m \notin N_1, a_2^n m \notin N_1$ for all positive integer n, then $a_1 a_2 \in (0 : M_2)$.*

Proof. $(1 \Rightarrow 2)$. (a). This follows from Theorem 2.14.

(b). Let $a_1 a_2 m = 0, a_1 m \notin N_1$ and $a_2^n m \notin N_1$ for all positive integer n, where $a_1, a_2 \in R$ and $m \in M_1$. Suppose that $a_1 a_2 \notin (0 : M_2)$. There exists $m_2 \in M_2$ such that $a_1 a_2 m_2 \neq 0$. Thus $(0, 0) \neq a_1 a_2 (m, m_2) =$

$(a_1a_2m, a_1a_2m_2) \in N_1 \times M_2$. By part 1, i.e., $a_1a_2 \in \sqrt{(N_1 \times M_2 : M_1 \times M_2)}$ or $a_1(m, m_2) \in N_1 \times M_2$ or $a_2^n(m, m_2) \in N_1 \times M_2$ for some positive integer n. Thus $a_1a_2 \in \sqrt{(N_1 : M_1)}$ or $a_1m \in N_1$ or $a_2^n m \in N_1$ which is a contradiction. Hence $a_1a_2 \in (0 : M_2)$.

$(2 \Rightarrow 1)$. Let $a_1, a_2 \in R$ and $(m_1, m_2) \in M_1 \times M_2$ such that $(0, 0) \neq (a_1a_2m_1, a_1a_2m_2) = a_1a_2(m_1, m_2) \in N_1 \times M_2$. If $a_1a_2m_1 \neq 0$, then $0 \neq a_1a_2m_1 \in N_1$. By part (a), $a_1a_2 \in \sqrt{(N_1 : M_1)}$ or $a_1m_1 \in N_1$ or $a_2^n m_1 \in N_1$ for some positive integer n. So $a_1a_2 \in \sqrt{(N_1 \times M_2 : M_1 \times M_2)}$ or $a_1(m_1, m_2) = (a_1m_1, a_1m_2) \in N_1 \times M_2$ or $a_2^n(m_1, m_2) = (a_2^n m_1, a_2^n m_2) \in N_1 \times M_2$, and thus we are done. If $a_1a_2m_1 = 0$, then $a_1a_2m_2 \neq 0$. Therefore $a_1a_2 \notin (0 : M_2)$. By part (b), $a_1a_2 \in \sqrt{(N_1 : M_1)}$ or $a_1m_1 \in N_1$ or $a_2^n m_1 \in N_1$ for some positive integer n. Thus $a_1a_2 \in \sqrt{(N_1 \times M_2 : M_1 \times M_2)}$ or $a_1(m_1, m_2) \in N_1 \times M_2$ or $a_2^n(m_1, m_2) \in N_1 \times M_2$. Hence $N_1 \times M_2$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2$. □

Corollary 2.7. *Let M_i be an R-module and let N_i be a proper submodule of M_i, for $i = 1, 2$. Then the following conditions are equivalent:*

(1) *$M_1 \times N_2$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2$.*

(2) (a) *N_2 is a weakly 2-absorbing semi-primary submodule of M_2.*

(b) *For each $a_1, a_2 \in R$ and $m \in M_2$ such that $a_1a_2m = 0$, if $a_1a_2 \notin \sqrt{(N_2 : M_2)}, a_1m \notin N_2$ and $a_2^n m \notin N_2$ for all positive integer n, then $a_1a_2 \in (0 : M_1)$.*

Proof. This follows from Theorem 2.15. □

Corollary 2.8. *Let M_i be an R-module and let N_i be a proper submodule of M_i, for $i = 1, 2, \ldots, k$. Then the following conditions are equivalent:*

(1) *$M_1 \times M_2 \times \ldots \times M_{i-1} \times N_i \times M_{i+1} \times M_k$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2 \times \ldots \times M_k$.*

(2) (a) *N_i is a weakly 2-absorbing semi-primary submodule of M_i.*

(b) *For each $a_1, a_2 \in R$ and $m \in M_2$ such that $a_1a_2m = 0$, if $a_1a_2 \notin \sqrt{(N_2 : M_2)}, a_1m \notin N_2$ and $a_2^n m \notin N_2$ for all positive integer n, then there exists $j \in \{1, 2, \ldots, k\}$ such that $a_1a_2 \in (0 : M_j)$.*

Proof. This follows from Theorem 2.15. □

Theorem 2.16. *Let N_i be a proper submodule of an R_i-module M_i, for $i = 1, 2$. Then the following conditions are equivalent:*

(1) *N_1 is a 2-absorbing semi-primary submodule of M_1.*

(2) *$N_1 \times M_2$ is a 2-absorbing semi-primary submodule of $M_1 \times M_2$.*

(3) *$N_1 \times M_2$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2$, where $M_2 \neq \{0\}$.*

Proof. $(1 \Rightarrow 2)$. This is clear, by Theorem 2.15.

$(2 \Rightarrow 3)$. The proof is clear.

$(3 \Rightarrow 1)$. Suppose that $N_1 \times M_2$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2$, where $M_2 \neq \{0\}$. Let $a_1, a_2 \in R_1$ and $m \in M_1$ such that $a_1a_2m \in N_1$. By assumption, there exists $m_2 \in M_2$ such that $m_2 \neq 0$. Since $(a_1, 1)(a_2, 1)(m, m_2) = (a_1a_2m, m_2) \neq (0, 0)$, we have $(0, 0) \neq (a_1, 1)(a_2, 1)(m, m_2) \in N_1 \times M_2$. By Definition 2.1, $(a_1, 1)(a_2, 1) \in \sqrt{(N_1 \times M_2 : M_1 \times M_2)}$ or $(a_1, 1)(m, m_2) \in N_1 \times M_2$ or $(a_2, 1)^n(m, m_2) \in N_1 \times M_2$ for some positive integer n. Therefore $a_1a_2 \in \sqrt{(N_1 : M_1)}$ or $a_1m \in N_1$ or $a_2^n m \in N_1$ for some positive integer n and hence N_1 is a 2-absorbing semi-primary submodule of M_1. □

Corollary 2.9. *Let N_i be a proper submodule of an R_i-module M_i, for $i = 1, 2$. Then the following conditions are equivalent:*

(1) *N_2 is a 2-absorbing semi-primary submodule of M_1.*

(2) *$M_1 \times N_2$ is a 2-absorbing semi-primary submodule of $M_1 \times M_2$.*

(3) *$M_1 \times N_2$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2$, where $M_1 \neq \{0\}$.*

Proof. This follows from Theorem 2.16. □

Corollary 2.10. *Let N_i be a proper submodule of an R_i-module M_i, for $i = 1, 2, \ldots, k$. Then the following conditions are equivalent:*

(1) *N_i is a 2-absorbing semi-primary submodule of M_1.*

(2) *$M_1 \times M_2 \times \ldots \times M_{i-1} \times N_i \times M_{i+1} \times M_k$ is a 2-absorbing semi-primary submodule of $M_1 \times M_2 \times \ldots \times M_k$.*

(3) *$M_1 \times M_2 \times \ldots \times M_{i-1} \times N_i \times M_{i+1} \times M_k$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2 \times \ldots \times M_k$, where $M_j \neq \{0\}$.*

Proof. This follows from Theorem 2.16 and Corollary 2.9. □

Theorem 2.17. *Let N_i be a proper submodule of an R_i-module M_i, for $i = 1, 2$. If $N_1 \times N_2$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2$, then*

(1) *N_1 is a weakly 2-absorbing semi-primary submodule of M_1.*

(2) *N_2 is a weakly 2-absorbing semi-primary submodule of M_2.*

Proof. (1). Suppose that $N_1 \times N_2$ is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2$. Let $a_1, a_2 \in R_1$ and $m \in M_1$ such that $0 \neq a_1a_2m \in N_1$. Clearly, $(0, 0) \neq (a_1, 1)(a_2, 1)(m, m_2) = (a_1a_2m, m_2) \in N_1 \times N_2$. By Definition 2.1, $(a_1a_2, 1) = (a_1, 1)(a_2, 1) \in \sqrt{(N_1 \times N_2 : M_1 \times M_2)}$ or $(a_1m, m_2) = (a_1, 1)(m, m_2) \in N_1 \times N_2$ or $(a_2^n m, m_2) = (a_2, 1)^n(m, m_2) \in N_1 \times N_2$ for some positive integer n. Therefore $a_1a_2 \in \sqrt{(N_1 : M_1)}$ or $a_1m \in N_1$ or $a_2^n m \in N_1$ for some positive integer n. Hence N_1 is a weakly 2-absorbing semi-primary submodule of M_1.

(2). This follows from part 1. □

Example 2.3. *Let $M = \boldsymbol{Z} \times \boldsymbol{Z}$ be an $\boldsymbol{Z}$-module. Consider the submodule $N = 5\boldsymbol{Z} \times 12\boldsymbol{Z}$ of M. It is easy to see that $5\boldsymbol{Z}$ and $12\boldsymbol{Z}$ are weakly 2-absorbing semi-primary submodule of M. Notice that $(0,0) \neq 2 \cdot 3(5,2) \in N$, but $2 \cdot 3 \notin \sqrt{(M : N)}, 2(5,2) \notin N$, and $(2 \cdot 3)^n \notin (N : M)$ for all positive integer n. Therefore N is not a weakly 2-absorbing semi-primary submodule of M. This example shows that the converse of Theorem 2.17 is not true.*

Theorem 2.18. *Let N_i be a submodule of an R_i-module M_i, for $i = 1, 2, 3$. If N is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2 \times M_3$, then $N = \{(0, 0, 0)\}$ or N is a 2-absorbing semi-primary submodule of $M_1 \times M_2 \times M_3$.*

Proof. Suppose that N is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2 \times M_3$ that is not 2-absorbing semi-primary. We will show that $N = \{(0, 0, 0)\}$. Now suppose that $N_1 \times N_2 \times N_3 = N \neq \{0\} \times \{0\} \times \{0\}$. Thus $N_i \neq \{0\}$, for some $i = 1, 2, 3$. We claim that $N_1 \neq \{0\}$. There exists $m_1 \in N_1$ such that $m_1 \neq 0$. To show that $N_2 = M_2$ or $N_3 = M_3$. Assume that $N_2 \neq M_2$ and $N_3 \neq M_3$. Thus there exist $m_2 \in M_2$ and $m_3 \in M_3$ such that $m_2 \notin N_2$ and $m_3 \notin N_3$. Since $(1, 0, 1)(1, 1, 0)(m_1, m_2, m_3) = (m_1, 0, 0) \neq (0, 0, 0)$, we have $(0, 0, 0) \neq (1, 0, 1)(1, 1, 0)(m_1, m_2, m_3) \in N_1 \times N_2 \times N_3$. By Definition 2.1, we get $(1, 0, 1)(1, 1, 0) \in \sqrt{(N_1 \times N_2 \times N_3 : M_1 \times M_2 \times M_3)}$ or $(1, 0, 1)(m_1, m_2, m_3) \in N$ or $(1, 1, 0)^n(m_1, m_2, m_3) \in N$, for some positive integer n. So $m_2 \in N_2$ or $m_3 \in N_3$, a contradiction. Therefore $N = N_1 \times M_2 \times N_3$ or $N = N_1 \times N_2 \times M_3$. If $N = N_1 \times M_2 \times N_3$, then $(0, 1, 0) \in (N : M_1 \times M_2 \times M_3)$. By Theorem 2.13, $\{0\} \times M_2 \times \{0\} = (0, 1, 0)^2 N \subseteq (N : N_1 \times M_2 \times N_3)^2 N = \{(0, 0, 0)\}$, which is a contradiction. Hence $N = \{(0, 0, 0)\}$. □

Theorem 2.19. *Let N_i be a submodule of an R_i-module M_i, for $i = 1, 2, 3$. If $N \neq \{(0, 0, 0)\}$ and N is a 2-absorbing semi-primary submodule of $M_1 \times M_2 \times M_3$, then N is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2 \times M_3$.*

Proof. Similar to the proof of Theorem 2.18 □

The above theorem shows the relationship between 2-absorbing semi-primary and weakly 2-absorbing semi-primary submodules in $R_1 \times R_2 \times R_3$-modules. From the above theorem, we have the following corollary.

Corollary 2.11. *Let N_i be a submodule of an R_i-module M_i, for $i = 1, 2, 3$ with $N \neq \{(0, 0, 0)\}$. Then N is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2 \times M_3$ if and only if N is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2 \times M_3$.*

Proof. This follows from Theorem 2.18. □

Corollary 2.12. *Let N_i be a submodule of an R_i-module M_i, for $i = 1, 2, \ldots, k \geq 3$ with $N \neq \{(0, 0, \ldots, 0)\}$. Then N is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2 \times \ldots \times M_k$ if and only if N is a weakly 2-absorbing semi-primary submodule of $M_1 \times M_2 \times \ldots \times M_k$.*

Proof. This follows from Theorem 2.19. □

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