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A STUDY OF NON-ASSOCIATIVE ORDERED SEMIGROUPS IN TERMS OF SEMILATTICES VIA SMALLEST (DOUBLE-FRAMED SOFT) IDEALS

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Abstract. Soft set theory, introduced by Molodtsov has been considered as a successful mathematical tool for modeling uncertainties. A double-framed soft set is a generalization of a soft set, consisting of union soft sets and intersectional soft sets. An ordered $\mathcal{AG}$-groupoid can be referred to as a non-associative ordered semigroup, as the main difference between an ordered semigroup and an ordered $\mathcal{AG}$-groupoid is the switching of an associative law. In this paper, we define the smallest left (right) ideals in an ordered $\mathcal{AG}$-groupoid and use them to characterize a strongly regular class of a unitary ordered $\mathcal{AG}$-groupoid along with its semilattices and double-framed soft (briefly DFS) l-ideals (r-ideals). We also give the concept of an ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid and investigate its structural properties by using the generated ideals and DFS l-ideals (r-ideals). These concepts will verify the existing characterizations and will help in achieving more generalized results in future works.

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1. Introduction

The concept of soft set theory was introduced by Molodtsov in [17]. This theory can be used as a generic mathematical tool for dealing with uncertainties. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applied to many different fields [1,2,5–9]. At present, the research work on soft set theory in algebraic fields is progressing rapidly [20, 22–24]. A soft set is a parameterized family of subsets of the universe set. In the real world, the parameters of this family arise from the view point of fuzzy set theory. Most of the researchers of algebraic structures have worked on the fuzzy aspect of soft sets. Soft set theory is applied in the field of optimization by Kovkov in [13]. Several similarity measures have been discussed in [16], decision making problems have been studied in [22], reduction of fuzzy soft sets and its applications in decision making problems have been analyzed in [14]. The notions of soft numbers, soft derivatives, soft integrals and many more have been formulated in [15]. This concept have been used for forecasting the export and import volumes in international trade [26].

Recently, Jun et al. further extended the notion of softs set into double-framed soft sets and defined double-framed soft subalgebra of BCK/BCI algebra and studied the related properties in [8]. Jun et al. also defined the concept of a double-framed soft ideal (briefly, DFS ideal) of a BCK/BCI-algebra and gave many valuable results for this theory. In [12], Khan et al. have applied the idea of double-framed soft set to ordered semigroups and defined prime and irreducible DFS ideals of an ordered semigroup over a universe set U. Khan et al. have also characterized different classes of an ordered semigroup by using different DFS ideals.

In the present paper, we apply the idea given by Jun et al. in [8], to ordered $\mathcal{AG}$-groupoids. We introduce and investigate the notions of DFS l-ideals and DFS r-ideals, and study the relationship between these DFS ideals in detail. As an application of our results we get characterizations of a strongly regular class of a unitary ordered $\mathcal{AG}$-groupoid (an ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid) in terms of its semilattices, one-sided (two-sided) ideals based on DFS-sets and generated commutative monoids.

2. Preliminaries

An $\mathcal{AG}$-groupoid is a non-associative and a non-commutative algebraic structure lying in a grey area between a groupoid and a commutative semigroup. Commutative law is given by $abc = cba$ in ternary operations. By putting brackets on the left of this equation, i.e. $(ab)c = (cb)a$, in 1972, M. A. Kazim and M. Naseeruddin introduced a new algebraic structure called a left almost semigroup abbreviated as an $\mathcal{LA}$-semigroup [10]. This identity is called the left invertive law. P. V. Protic and N. Stevanovic called the same structure an Abel-Grassmann's groupoid abbreviated as an AG-groupoid [21].

This structure is closely related to a commutative semigroup because a commutative $\mathcal{AG}$-groupoid is a semigroup [18]. It was proved in [10] that an $\mathcal{AG}$-groupoid S is medial, that is, $ab \cdot cd = ac \cdot bd$ holds for all

a,b,c,$d \in S$. An $\mathcal{AG}$-groupoid may or may not contain a left identity. The left identity of an $\mathcal{AG}$-groupoid permits the inverses of elements in the structure. If an $\mathcal{AG}$-groupoid contains a left identity, then this left identity is unique [18]. In an $\mathcal{AG}$-groupoid S with left identity, the paramedial law $ab \cdot cd = dc \cdot ba$ holds for all a,b,c,$d \in S$. By using medial law with left identity, we get $a \cdot bc = b \cdot ac$ for all a,b,$c \in S$. We should genuinely acknowledge that much of the ground work has been done by M. A. Kazim, M. Naseeruddin, Q. Mushtaq, M. S. Kamran, P. V. Protic, N. Stevanovic, M. Khan, W. A. Dudek and R. S. Gigon. One can be referred to [3, 4, 11, 18, 19, 21, 25] in this regard.

An $\mathcal{AG}$-groupoid $(S, \cdot)$ together with a partial order $\leq$ on S that is compatible with an $\mathcal{AG}$-groupoid operation, meaning that for $x, y, z \in S$, $x \leq y \Rightarrow zx \leq zy$ and $xz \leq yz$, is called an ordered $\mathcal{AG}$-groupoid [28].

Let us define a binary operation "$\circ_e$" (e-sandwich operation) on an ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ with left identity e as follows:

$$a \circ_e b = ae \cdot b, \ \forall \ a, b \in S.$$

Then $(\mathcal{S}, \circ_e, \leq)$ becomes an ordered semigroup [28].

Note that an ordered $\mathcal{AG}$-groupoid is the generalization of an ordered semigroup because if an ordered $\mathcal{AG}$-groupoid has a right identity then it becomes an ordered semigroup.

Let $\emptyset \neq A \subseteq S$, we denote $(A]$ by $(A] := \{x \in S/x \leq a$ for some $a \in A\}$. If $A = \{a\}$, then we write $(\{a\}]$. For $\emptyset \neq A, B \subseteq S$, we denote $AB =: \{ab/a \in A, b \in B\}$.

• Let S be an ordered $\mathcal{AG}$-groupoid. By an ordered $\mathcal{AG}$-subgroupoid of S, we means a nonempty subset A of S such that $(A^2] \subseteq A$.

• A nonempty subset A of an ordered $\mathcal{AG}$-groupoid S is called a left (right) ideal of S if:

(i) $SA \subseteq A$ $(AS \subseteq A)$;

(ii) if $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Equivalently: A nonempty subset A of an ordered $\mathcal{AG}$-groupoid S is called a left (right) ideal of S if $(SA] \subseteq A$ $((AS] \subseteq A)$.

• By two-sided ideal or simply ideal, we mean a nonempty subset of an ordered $\mathcal{AG}$-groupoid S which is both left and right ideal of S.

Lemma 2.1. *[28] Let S be an ordered $\mathcal{AG}$-groupoid and $\emptyset \neq A, B \subseteq S$. Then the following hold:*

(i) $A \subseteq (A]$;

(ii) If $A \subseteq B$, then $(A] \subseteq (B]$;

(iii) $(A](B] \subseteq (AB]$;

(iv) $(A] = ((A]]$;

(vi) $((A](B]] = (AB]$;

(vii) $(T] = T$, for every ideal T of S;

(viii) $(SS] = S = SS$, *if* S *has a left identity.*

3. Soft Sets

In [24], Sezgin and Atagun introduced some new operations on soft set theory and defined soft sets in the following way.

Let U be an initial universe set, E a set of parameters, $P(U)$ the power set of U and $A \subseteq E$. Then a *soft set* f_A over U is a function defined by:

$$f_A : E \to P(U) \text{ such that } f_A(x) = \emptyset, \text{ if } x \notin A.$$

Here f_A is called an *approximate function*. A soft set over U can be represented by the set of ordered pairs

$$f_A = \{(x,\ f_A(x)) :\ x \in E,\ f_A(x) \in P(U)\}.$$

It is clear that a soft set is a parameterized family of subsets of U. The set of all soft sets is denoted by $S(U)$.

• Let $f_A,\, f_B \in S(U)$. Then f_A is a soft subset of f_B, denoted by $f_A \mathrel{\widetilde{\subseteq}} f_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in S$. Two soft sets $f_A,\, f_B$ are said to be equal soft sets if $f_A \mathrel{\widetilde{\subseteq}} f_B$ and $f_B \mathrel{\widetilde{\subseteq}} f_A$ and is denoted by $f_A \mathrel{\widetilde{=}} f_B$. The union of f_A and f_B, denoted by $f_A \mathbin{\widetilde{\cup}} f_B$, is defined by $f_A \mathbin{\widetilde{\cup}} f_B = f_{A\cup B}$, where $f_{A\cup B}(x) = f_A(x) \cup f_B(x)$, $\forall$ $x \in E$. In a similar way, we can define the intersection of f_A and f_B.

• Let S be an ordered $\mathcal{AG}$-groupoid, let $f_A,\, f_B \in S(U)$. Then the *soft product* [24] of f_A and f_B, denoted by $f_A \mathbin{\widetilde{\circ}} f_B$, is defined as follows:

$$(f_A \mathbin{\widetilde{\circ}} f_B)(x) = \begin{cases} \bigcup\limits_{(y,z)\in A_x} \{f_A(y) \cap g_B(z)\} & \text{if } A_x \neq \emptyset \\ \emptyset & \text{if } A_x = \emptyset \end{cases}$$

where $A_x = \{(y, z) \in S \times S / x \leq yz\}$.

• A double-framed soft pair $\left\langle (f_A^+,\ f_A^-; A\right\rangle$ is called a *double-framed soft set* (briefly, DFS-set of A) [8] of A over U, where f_A^+ and f_A^- are mappings from A to $P(U)$. The set of all DFS-sets of A over U will be denoted by $DFS(U)$.

• Let $f_A = \left\langle (f_A^+,\ f_A^-); A\right\rangle$ and $g_A = \left\langle (g_A^+,\ g_A^-); A\right\rangle$ be two double-framed soft sets of an ordered $\mathcal{AG}$-groupoid S over U. Then the *uni-int soft product* [12], denoted by $f_A \diamond g_A = \left\langle (f_A^+ \mathbin{\widetilde{\circ}} g_A^+,\ f_A^- \mathbin{\widetilde{\star}} g_A^-); A\right\rangle$ is defined to be a double-framed soft set of S over U, in which $f_A^+ \mathbin{\widetilde{\circ}} g_A^+$ and $f_A^- \mathbin{\widetilde{\star}} g_A^-$ are mapping from S to $P(U)$, given as follows:

$$f_A^+ \,\tilde{\circ}\, g_A^+ : S \longrightarrow P(U), x \longmapsto \begin{cases} \bigcup\limits_{(y,z)\in A_x} \{f_A^+(y) \cap g_A^+(z)\} & \text{if } A_x \neq \emptyset \\ \emptyset & \text{if } A_x = \emptyset, \end{cases}$$

$$f_A^- \,\tilde{\star}\, g_A^- : S \longrightarrow P(U), x \longmapsto \begin{cases} \bigcap\limits_{(y,z)\in A_x} \{f_A^-(y) \cup g_A^-(z)\} & \text{if } A_x \neq \emptyset \\ U & \text{if } A_x = \emptyset. \end{cases}$$

• Let $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ and $g_A = \langle (g_A^+,\ g_A^-); A\rangle$ be two double-framed soft sets over a common universe set U. Then $\langle (f_A^+,\ f_A^-); A\rangle$ is called a *double-framed soft subset* (*briefly, DFS-subset*) [12] of $\langle (g_A^+,\ g_A^-); A\rangle$, denote by $\langle (f_A^+,\ f_A^-); A\rangle \sqsubseteq \langle (g_A^+,\ g_A^-); A\rangle$ if:

(i) $A \subseteq B$;

(ii) $(\forall e \in A) \begin{pmatrix} f_A^+ \text{ and } g_A^+ \text{ are identical approximations } (f_A^+(e) \subseteq g_A^+(e)) \\ f_A^- \text{ and } g_A^- \text{ are identical approximations } (f_A^-(e) \supseteq g_A^-(e)) \end{pmatrix}$.

• For two DFS-sets $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ and $g_A = \langle (g_A^+,\ g_A^-); A\rangle$ over U are said to be *equal*, denoted by $\langle (f_A^+,\ f_A^-); A\rangle = \langle (g_A^+,\ g_A^-); A\rangle$, if $\langle (f_A^+,\ f_A^-); A\rangle \sqsubseteq \langle (g_A^+,\ g_A^-); A\rangle$ and $\langle (g_A^+,\ g_A^-); A\rangle \sqsubseteq \langle (f_A^+,\ f_A^-); A\rangle$.

• For two DFS-sets $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ and $g_A = \langle (g_A^+,\ g_A^-); A\rangle$ over U, the DFS *int-uni* set [12] of $\langle (f_A^+,\ f_A^-); A\rangle$ and $\langle (g_A^+,\ g_A^-); A\rangle$, is defined to be a DFS-set $\langle (f_A^+ \cap g_A^+,\ f_A^- \cup g_A^-); A\rangle$, where $f_A^+ \cap g_A^+$ and $f_A^- \cup g_A^-$ are mapping given as follows:

$$f_A^+ \cap g_A^+ \quad : \quad A \longrightarrow P(U), x \longmapsto f_A^+(x) \cap g_A^+(x);$$

$$f_A^- \cup g_A^- \quad : \quad A \longrightarrow P(U), x \longmapsto f_A^-(x) \cup g_A^-(x).$$

It is denoted by $\langle (f_A^+,\ f_A^-); A\rangle \sqcap \langle (g_A^+,\ g_A^-); A\rangle = \langle (f_A^+ \cap g_A^+,\ f_A^- \cup g_A^-); A\rangle$.

• A double-framed soft set $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ of S over U is called a *double-framed soft $\mathcal{AG}$-subgroupoid* (briefly, DFS $\mathcal{AG}$-subgroupoid) of S over U if it satisfies $f_A^+(xy) \supseteq f_A^+(x) \cap f_A^+(y)$, $f_A^-(xy) \subseteq f_A^-(x) \cup f_A^-(y)$, $\forall\ x,\ y \in S$.

• A double-framed soft set $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ of S over U is called

(i) a *double-framed soft left ideal* (briefly, DFS l-ideal) of S over U if it satisfies:

(a) $f_A^+(xy) \supseteq f_A^+(y)$ and $f_A^-(xy) \subseteq f_A^-(y)$;

(b) $x \leq y \Longrightarrow f_A^+(x) \supseteq f_A^+(y)$ and $f_A^-(x) \subseteq f_A^-(y)$, $\forall\ x,\ y \in S$.

(ii) a *double-framed soft right ideal* (briefly, DFS r-ideal) of S over U if it satisfies:

(a) $f_A^+(xy) \supseteq f_A^+(x)$ and $f_A^-(xy) \subseteq f_A^-(x)$;

(b) $x \leq y \Longrightarrow f_A^+(x) \supseteq f_A^+(y)$ and $f_A^-(x) \subseteq f_A^-(y)$, $\forall\ x,\ y \in S$.

(iii) a *double-framed soft ideal* (briefly, DFS ideal) of S over U, if it is both DFS l-ideal and DFS r-ideal of S over U.

• Let A be a nonempty subset of S. Then the *characteristic double-framed soft mapping* of A, denoted by $\langle (\mathcal{X}_A^+,\ \mathcal{X}_A^-); A\rangle = \mathcal{X}_A$ is defined to be a double-framed soft set, in which $\mathcal{X}_A^+$ and $\mathcal{X}_A^-$ are soft mappings over U, given as follows:

$$\mathcal{X}_A^+ \quad : \quad S \longrightarrow P(U), x \longmapsto \begin{cases} U & \text{if } x \in A \\ \emptyset & \text{if } x \notin A, \end{cases}$$

$$\mathcal{X}_A^- \quad : \quad S \longrightarrow P(U), x \longmapsto \begin{cases} \emptyset & \text{if } x \in A \\ U & \text{if } x \notin A. \end{cases}$$

Note that the characteristic mapping of the whole set S, denoted by $\mathcal{X}_S = \langle (\mathcal{X}_S^+,\ \mathcal{X}_S^-); S \rangle$, is called the *identity double-framed soft mapping*, where $\mathcal{X}_S^+(x) = U$ and $\mathcal{X}_S^-(x) = \emptyset$, $\forall\ x \in S$.

The following result holds for an ordered semigroup [6] just because of the closure property which makes very clear for an ordered $\mathcal{AG}$-groupoid to hold the same Lemma.

Lemma 3.1. *For a nonempty subset A of an ordered $\mathcal{AG}$-groupoid S, the following conditions are equivalent*:

(i) *A is a left ideal (right ideal) of S;*

(ii) *The DFS set $\mathcal{X}_A$ of S over U is a DFS l-ideal (DFS r-ideal) of S over U.*

The following result holds for an ordered semigroup [12] just because of the closure property which makes very clear for an ordered $\mathcal{AG}$-groupoid to hold the same Lemma.

Lemma 3.2. *Let $f_A = \langle (f_A^+, f_A^-); A \rangle$ be any DFS-set of an ordered $\mathcal{AG}$-groupoid S over U. Then f_A is a DFS r-ideal (l-ideal) of S over U if and only if $f_A \diamond \mathcal{X}_S \sqsubseteq f_A$ ($\mathcal{X}_S \diamond f_A \sqsubseteq f_A$).*

- A double-framed soft set $f_A = \langle (f_A^+,\ f_A^-); A \rangle$ of S over U is called *DFS semiprime* if $f_A(x) \sqsupseteq f_A(x^2)$, $\forall\ x \in A$.

Lemma 3.3. *Let A be any right (left) ideal of an ordered $\mathcal{AG}$-groupoid S. Then A is semiprime if and only if $\mathcal{X}_A$ is DFS semiprime.*

Proof. Let A be a right (left) ideal of S, then by Lemma 3.1, $\mathcal{X}_A$ is a DFS r-ideal (DFS l-ideal) of S over U. Let $a^2 \in A$, then $\mathcal{X}_A^+(a) \supseteq \mathcal{X}_A^+(a^2)$, therefore $\mathcal{X}_A^+(a^2) = U \subseteq \mathcal{X}_A^+(a)$, this implies $\mathcal{X}_A^+(a) = U$ and similarly $\mathcal{X}_A^-(a) = \emptyset$. Thus $a \in A$ and therefore A is semiprime. Converse is simple. □

Remark 3.1. *The set $(DFS(U), \diamond, \sqsubseteq)$ forms an ordered $\mathcal{AG}$-groupoid and satisfies all the basic laws.*

Remark 3.2. *If S is an ordered $\mathcal{AG}$-groupoid, then $\mathcal{X}_S \diamond \mathcal{X}_S = \mathcal{X}_S$.*

The following result also holds for an ordered semigroup [12] just because of the closure property which is very trivial for an ordered $\mathcal{AG}$-groupoid to hold the same Lemma.

Lemma 3.4. *Let S be an ordered $\mathcal{AG}$-groupoid. For $\emptyset \neq A, B \subseteq S$, the following assertions hold*:

(i) $A \subseteq B \Leftrightarrow \mathcal{X}_A \sqsubseteq \mathcal{X}_B$;

(ii) $\mathcal{X}_A \sqcap \mathcal{X}_B = \mathcal{X}_{A\cap B}$;

(iii) $\mathcal{X}_A \sqcup \mathcal{X}_B = \mathcal{X}_{A\cup B}$;

(iv) $\mathcal{X}_A \diamond \mathcal{X}_B = \mathcal{X}_{(AB]}$.

4. On DFS strongly regular ordered $\mathcal{AG}$-groupoids

Throughout this paper, let $E = S$, where S is a unitary ordered $\mathcal{AG}$-groupoid, unless otherwise stated. By a unitary ordered $\mathcal{AG}$-groupoid, we shall mean an ordered $\mathcal{AG}$-groupoid with left identity.

4.1. **Basic Results.** This section contains some examples and basic results which will be essential for up coming section.

Theorem 4.1. *Let S be an ordered $\mathcal{AG}$-groupoid. A nonempty subset A of S is a left (resp. right) ideal of S if and only if the DFS-set $\left\langle (g_B^+,\ g_B^-); B\right\rangle$, defined by*

$$g_B^+(x) = \left\{ \begin{array}{c} \gamma_1 \text{ if } x \in A \\ \gamma_2 \text{ if } x \in S\backslash A \end{array} \right\} \text{ and } g_B^-(x) = \left\{ \begin{array}{c} \delta_1 \text{ if } x \in A \\ \delta_2 \text{ if } x \in S\backslash A \end{array} \right\},$$

is a DFS l-ideal (resp. DFS r-ideal) of S over U, where $\gamma_1, \gamma_2, \delta_1, \delta_2 \subseteq U$ such that $\gamma_2 \subseteq \gamma_1$ and $\delta_1 \subseteq \delta_2$.

Proof. Necessity. Let $x, y \in S$ be such that $x \leq y$. If $y \notin A$, then $g_B^+(y) = \gamma_2 \subseteq g_B^+(x)$ and $g_B^-(y) = \delta_2 \supseteq g_B^-(x)$. If $y \in A$, then $\gamma_1 = g_B^+(y)$ and $\delta_2 = g_B^-(y)$. Since $x \leq y \in A$, and A is a left ideal of S, we have $x \in A$. Then $g_B^+(x) = \gamma_1 = g_B^+(y)$ and $g_B^-(x) = \delta_1 = g_B^-(y)$. For $x, y \in S$, we discuss the following two cases.

Case 1. If $x \in S$ and $y \in A$, then $xy \in A$ and we have $g_B^+(x) = \gamma_1 = g_B^+(y)$ and $g_B^-(x) = \delta_1 = g_B^-(y)$.

Case 2. If $x \in S$ and $y \notin A$, then $g_B^+(y) = \gamma_2 \subseteq g_B^+(xy)$ and $g_B^-(y) = \delta_2 \supseteq g_B^-(xy)$. Therefore $\left\langle (g_B^+,\ g_B^-); B\right\rangle$ is a DFS l-ideal of S over U. Similarly we can prove the result for a DFS r-ideal of S over U.

Sufficiency. Assume that $\left\langle (g_B^+,\ g_B^-); B\right\rangle$ is a DFS l-ideal of S over U. Let $x, y \in S$ be such that $x \leq y$. If $y \in A$, then $g_B^+(y) \supseteq \gamma_1$ and $g_B^-(y) \subseteq \delta_1$. Since $g_B^+(x) \supseteq g_B^+(y) \supseteq \gamma_1$ and $g_B^-(x) \subseteq g_B^+(y) \subseteq \delta_1$, we have $x \in A$. Let $x \in S$ and $y \in A$, then $g_B^+(y) = \gamma_1$ and $g_B^-(y) = \delta_1$. By hypothesis, $g_B^+(xy) \supseteq g_B^+(y) = \gamma_1$ and $g_B^-(xy) \subseteq g_B^+(y) = \delta_1$. Hence $xy \in A$. Thus A is a left ideal of S. Similarly, we can show that A is a right ideal of S. □

Example 4.1. *There are six different chemicals which have been used in an experiment. Take a collection of chemicals as the initial universe set U given by*

$$U = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}.$$

Let a set of parameters $E = \{1, 2, 3, 4, 5\}$ be a set of particular properties of each chemical in U with the following type of natures:

1 stands for the parameter "*density*",

2 stands for the parameter "*melting point*",

3 stands for the parameter "*combustion*",

4 stands for the parameter "*enthalpy*",

5 stands for the parameter "*toxicity*".

Let us define the following binary operation and order on a set of parameters E as follows.

$*$	1	2	3	4	5
1	2	2	4	4	5
2	2	2	2	2	5
3	1	2	3	4	5
4	1	2	1	2	5
5	1	5	5	5	5

$$\leq = \{(1,1),(1,2),(3,3),(1,3),(4,4),(1,5),(5,5),(2,2)\}.$$

It is easy to observe that $(E,*,\leq)$ is a unitary ordered $\mathcal{AG}$-groupoid.

Let $A=\{1,2,5\}$ and define a DFS-set $\left\langle (f_A^+,\ f_A^-);A\right\rangle$ of S over U as follows:

$$f_A^+(x)=\left\{\begin{array}{c}\{\gamma_1,\gamma_2\} \text{ if } x=1\\ \{\gamma_1,\gamma_2,\gamma_3\} \text{ if } x=2\\ \{\gamma_5\} \text{ if } x=3\\ \{\gamma_5\} \text{ if } x=4\\ \{\gamma_1,\gamma_2,\gamma_3,\gamma_4\} \text{ if } x=5\end{array}\right\} \text{ and } f_A^-(x)=\left\{\begin{array}{c}\{\gamma_1,\gamma_2,\gamma_3\} \text{ if } x=1\\ \{\gamma_1,\gamma_2\} \text{ if } x=2\\ \{\ \} \text{ if } x=3\\ \{\ \} \text{ if } x=4\\ \{\gamma_2\} \text{ if } x=5\end{array}\right\}.$$

Then it is easy to verify that $\left\langle (f_A^+,\ f_A^-);A\right\rangle$ is a DFS l-ideal of S over U.

Let $B=\{1,2,4\}$ and define a DFS-set $\left\langle (g_B^+,\ g_B^-);B\right\rangle$ of S over U as follows:

$$g_B^+(x)=\left\{\begin{array}{c}\{\gamma_4\} \text{ if } x=1\\ \{\gamma_2,\gamma_3,\gamma_4\} \text{ if } x=2\\ \{\gamma_1,\gamma_2,\gamma_5\} \text{ if } x=3\\ \{\gamma_1,\gamma_2,\gamma_5\} \text{ if } x=4\\ \{\gamma_2,\gamma_4\} \text{ if } x=5\end{array}\right\} \text{ and } g_B^-(x)=\left\{\begin{array}{c}\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\} \text{ if } x=1\\ \{\gamma_2,\gamma_3\} \text{ if } x=2\\ \{\gamma_1,\gamma_2,...,\gamma_5\} \text{ if } x=3\\ \{\gamma_1,\gamma_2,\gamma_3,\gamma_4\} \text{ if } x=4\\ \{\gamma_2,\gamma_3\} \text{ if } x=5\end{array}\right\}.$$

Then it is easy to verify that $\left\langle (g_B^+,\ g_B^-);B\right\rangle$ is a DFS r-ideal of S over U.

Remark 4.1. *Every DFS r-ideal of a unitary ordered $\mathcal{AG}$-groupoid S over U is a DFS l-ideal of S over U but the converse inclusion is not true in general which can be followed from above example.*

Lemma 4.1. *Let R be a right ideal and L be a left ideal of a unitary ordered $\mathcal{AG}$-groupoid S. Then $(RL]$ is a left ideal of S.*

Proof. Let R be a right ideal and L be a left ideal of S. Then by using Lemma 2.1, we get $S(RL] = (SS](RL] \subseteq (SS \cdot RL] = (SR \cdot SL] \subseteq (SR \cdot (SL]] = (SR \cdot L] = ((SS]R \cdot L] \subseteq ((SS)R \cdot L] = ((RS)S \cdot L] \subseteq ((RS]S \cdot L] \subseteq (RL]$,which shows that $(RL]$ is a left ideal of S. □

Lemma 4.2. *Let S be a unitary ordered $\mathcal{AG}$-groupoid. If $a = a^2$ for all $a \in S$, then $R_a = (Sa \cup Sa^2]$ is the smallest right ideal of S containing a.*

Proof. Assume that $a = a^2$ for all $a \in S$. Then by using Lemma 2.1, we have

$$\begin{aligned} (Sa \cup Sa^2]S &= (Sa \cup Sa^2](S] \subseteq ((Sa \cup Sa^2)S] = (Sa \cdot S \cup Sa^2 \cdot S] \\ &= (Sa \cdot SS \cup Sa^2 \cdot SS] = (S \cdot aS \cup S \cdot a^2S] = (a \cdot SS \cup a^2 \cdot SS] \\ &= (a^2 \cdot SS \cup a^2 \cdot SS] = (SS \cdot a^2 \cup SS \cdot a^2] = (Sa \cup Sa^2], \end{aligned}$$

which shows that $(Sa \cup Sa^2]$ is a right ideal of S. It is easy to see that $a \in (Sa \cup Sa^2]$. Let R be another right ideal of S containing a. Since

$$(Sa \cup Sa^2] = (SS \cdot a \cup a \cdot Sa] = (aS \cdot S \cup a \cdot Sa] \subseteq (RS \cdot S \cup RS] \subseteq R,$$

Hence $(Sa \cup Sa^2]$ is the smallest right ideal of S containing a. □

Lemma 4.3. *Let S be a unitary ordered $\mathcal{AG}$-groupoid and $a = a^2$ for all $a \in S$. Then S becomes a commutative monoid.*

Proof. It is simple. □

Corollary 4.1. $R_a = (Sa \cup Sa^2]$ *is the smallest right ideal of an ordered commutative monoid S containing a.*

Lemma 4.4. *Let S be a unitary ordered $\mathcal{AG}$-groupoid and $a \in S$. Then $L_a = (Sa]$ is the smallest left ideal of S containing a.*

Proof. It is simple. □

Theorem 4.2. *Let S be a unitary ordered $\mathcal{AG}$-groupoid and $\emptyset \neq E \subseteq S$. Then the following assertions hold:*

(*i*) *E forms a semilattice, where $E = \{x \in S : x = x^2\}$;*

(*ii*) *E is a singleton set, if $a = ax \cdot a$, $\forall\ a, x \in S$.*

Proof. (i). It is simple.

(ii). Let $y, z \in E$. Then by using (i), we get

$$y = yz \cdot y = zy \cdot y = yy \cdot z = yz = zy = zz \cdot y = yz \cdot z = zy \cdot z = z.$$

□

• Recall that an ordered $\mathcal{AG}^{**}$-groupoid is an ordered $\mathcal{AG}$-groupoid in which $a \cdot bc = b \cdot ac$, $\forall\ a, b, c \in S$. Note that an ordered $\mathcal{AG}^{**}$-groupoid also satisfies the paramedial law as well.

Now let us introduce the concept of an ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid as follows:

• An ordered $\mathcal{AG}^{**}$-groupoid S is called an ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid if $S = (S^2]$.

Corollary 4.2. *Let S be an ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid and $\emptyset \neq E \subseteq S$. Then the following assertions hold*:

(i) *E forms a semilattice, where $E = \{x \in S : x = x^2\}$;*

(ii) *E is a singleton set if $a = ax \cdot a$, $\forall\ a, x \in S$.*

Lemma 4.5. *Let S be an ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid. Then $\langle R\rangle_{a^2} = (Sa^2 \cup a^2]$ $(\langle L\rangle_a = (Sa \cup a])$ is the right (resp. left) ideal of S.*

Proof. Let $a \in S$, then by using Lemma 2.1, we get

$$\begin{aligned}
(Sa^2 \cup a^2]S &= (Sa^2 \cup a^2](S] = ((Sa^2 \cup a^2)S] = (Sa^2 \cdot S \cup a^2S] \\
&= (SS \cdot a^2S \cup SS \cdot aa] = (S \cdot a^2S \cup Sa^2] \\
&= (a^2 \cdot SS \cup Sa^2] = (Sa^2] \subseteq (Sa^2 \cup a^2],
\end{aligned}$$

which is what we set out to prove. Similarly we can prove that $S(Sa \cup a] \subseteq (Sa \cup a]$. □

• An element a of an ordered $\mathcal{AG}$-groupoid S is called a strongly regular element of S, if there exists some x in S such that $a \leq ax \cdot a$ and $ax = xa$, where x is called a pseudo-inverse of a. S is called strongly regular ordered $\mathcal{AG}$-groupoid if all elements of S are strongly regular.

Theorem 4.3. *Let S be an ordered $\mathcal{AG}$-groupoid (an ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid) with left identity. An element a of S is strongly regular if and only if $a \leq ax \cdot ay$ for some $x, y \in S$.*

Proof. Necessity. Let $a \in S$ is strongly regular, then $a \leq ax \cdot a \leq (ax) \cdot (xa)(ax \cdot a) = (ax) \cdot (a \cdot ax)(ax) = (ax) \cdot a((a \cdot ax)x) = ax \cdot ay$, where $(a \cdot ax)x = y \in S$. Thus $a \leq ax \cdot ay$ for some $x, y \in S$.

Sufficiency. Let $a \in S$ such that $a \leq ax \cdot ay$ for some $x, y \in S$, then $a \leq ax \cdot ay = (ay \cdot x)a = (xy \cdot a)a = ua \cdot a$, where $xy = u \in S$. Thus $au \leq (ua \cdot a)u = ua \cdot ua = u(ua \cdot a) \leq ua$, and $a \leq ua \cdot a = au \cdot a$. Thus S is strongly regular. □

Lemma 4.6. *Let $f_A = \left\langle (f_A^+,\ f_A^-); A \right\rangle$ be any DFS r-ideal (DFS l-ideal) of a strongly regular ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid S over U. Then the following assertions hold*:

(*i*) $f_A = f_A \diamond \mathcal{X}_S$ ($f_A = \mathcal{X}_S \diamond f_A$);

(*ii*) f_A *is DFS semiprime.*

Proof. It is simple. □

4.2. **Characterization Problems.** In this section, we generalize the results of an ordered semigroup and get some interesting characterizations which we usually do not find in an ordered semigroup.

From now onward, R (*resp.* L) will denote any right (*resp.* left) ideal of an ordered $\mathcal{AG}$-groupoid S; R_a (*resp.* L_a) will denote any smallest right (*resp.* smallest left) ideal of S containing a. Any DFS r-ideal of an ordered $\mathcal{AG}$-groupoid S (*resp.* DFS l-ideal of S) over U will be denoted by f_A (*resp.* g_B) unless otherwise specified.

Theorem 4.4. *Let f_A, g_B be any DFS l-ideals of a unitary ordered $\mathcal{AG}$-groupoid S. Then the following conditions are equivalent*:

(*i*) *S is strongly regular*;

(*ii*) *S is strongly regular commutative monoid*;

(*iii*) $(R_aL_a] \cap L_a = ((R_a \cdot R_aL_a)L_a \cdot L_a]$, ($a = a^2$, $\forall\ a \in S$);

(*iv*) $(RL] \cap L = ((R \cdot RL)L \cdot L]$;

(*v*) $f_A \sqcap g_B = (f_A \diamond g_B) \diamond f_A$;

(*vi*) *S is strongly regular and $|E| = 1$*, ($a = ax \cdot a$, $\forall\ a, x \in E$);

(*vii*) *S is strongly regular and $\emptyset \neq E \subseteq S$ is semilattice.*

Proof. $(i) \Longrightarrow (vii)$: It can be followed from Theorem 4.2 (i).

$(vii) \Longrightarrow (vi)$: It can be followed from Theorem 4.2 (ii).

$(vi) \Longrightarrow (v)$: Let f_A and g_B be any DFS l-ideals of a strongly regular S over U. Now for $a \in S$, there exist some $x, y \in S$ such that $a \leq ax \cdot ay = ya \cdot xa \leq y(ax \cdot ay) \cdot xa = (ax)(y \cdot ay) \cdot xa = (ay \cdot y)(xa) \cdot xa = (y^2a \cdot xa)(xa)$. Thus $(y^2a \cdot xa, xa) \in A_a$. Therefore

$$\begin{aligned}
((f_A^+ \tilde{\circ}\, g_B^+) \tilde{\circ}\, f_A^+)(a) &= \bigcup_{(y^2a\cdot xa, xa)\in A_a} \left\{ (f_A^+ \tilde{\circ}\, g_B^+)(y^2a \cdot xa) \cap f_A^+(xa) \right\} \\
&\supseteq \bigcup_{y^2a\cdot xa \leq y^2a\cdot xa} \{ f_A^+(y^2a) \cap g_B^+(xa) \} \cap f_A^+(xa) \\
&\supseteq f_A^+(y^2a) \cap g_B^+(xa) \cap f_A^+(xa) \supseteq f_A^+(a) \cap g_B^+(a),
\end{aligned}$$

and similarly, we get

$$\begin{aligned}((f_A^- \tilde{\star} g_B^-) \tilde{\star} f_A^-)(a) &= \bigcap_{(y^2a\cdot xa, xa)\in A_a} \left\{(f_A^- \tilde{\star} g_B^-)(y^2a\cdot xa) \cup f_A^-(xa)\right\} \\ &\subseteq \bigcap_{y^2a\cdot xa \leq y^2a\cdot xa} \{f_A^-(y^2a) \cup g_B^-(xa)\} \cup f_A^-(xa) \\ &\subseteq f_A^-(y^2a) \cup g_B^-(xa) \cup f_A^-(xa) \subseteq f_A^-(a) \cup g_B^-(a),\end{aligned}$$

which shows that $(f_A \diamond g_B) \diamond f_A \sqsupseteq f_A \sqcap g_B$. By using Lemmas 3.2 and 4.6, it is easy to show that $(f_A \diamond g_B) \diamond f_A \sqsubseteq f_A \sqcap g_B$. Thus $f_A \sqcap g_B = (f_A \diamond g_B) \diamond f_A$.

$(v) \implies (iv)$: Let R and L be any right and left ideals of S respectively. Then by using Lemmas 3.1 and 4.1, $\mathcal{X}_{(RL]}$ and $\mathcal{X}_L$ are the DFS l-ideals of S over U. Now by using Lemma 3.4, we get $\mathcal{X}_{(RL]\cap L} = \mathcal{X}_{(RL]} \sqcap \mathcal{X}_L = (\mathcal{X}_{(RL]} \diamond \mathcal{X}_L) \diamond \mathcal{X}_{(RL]} = \mathcal{X}_{((RL]L\cdot(RL]]}$, which give us $(RL] \cap L = ((RL]L \cdot (RL]]$. Now by using Lemma 2.1, we get

$$\begin{aligned}((RL]L \cdot (RL]] &= ((RL)L \cdot RL] = (L^2R \cdot RL] = (LR \cdot RL^2] \\ &= (R(LR \cdot L^2)] = (R(L^2 \cdot RL)] = (R(R \cdot L^2L)] \\ &= (R \cdot RL^3] = (R(R \cdot L^2L)] = (R(L^2 \cdot RL)] \\ &= ((R \cdot RL)L \cdot L].\end{aligned}$$

$(iv) \implies (iii)$: It is simple.

$(iii) \implies (ii)$: Since $(Sa \cup Sa^2]$ is the smallest right ideal of S containing a and $(Sa]$ is the smallest left ideal of S containing a, where $a = a^2$, $\forall\ a \in S$. Thus by using given assumption and Lemma 2.1, we get

$$\begin{aligned}a &\in ((Sa \cup Sa^2](Sa]] \cap (Sa] = (((Sa \cup Sa^2] \cdot (Sa \cup Sa^2](Sa])(Sa] \cdot (Sa]] \\ &= (((Sa \cup Sa^2) \cdot (Sa \cup Sa^2)(Sa))(Sa) \cdot (Sa)] \subseteq (S(Sa) \cdot (Sa)] \\ &= (S^2a \cdot Sa] = (Sa \cdot Sa] = (aS \cdot aS].\end{aligned}$$

Hence by using Lemma 4.2, S is strongly regular commutative monoid.

$(ii) \implies (i)$: It is obvious. □

Theorem 4.5. *Let S be an ordered $\mathcal{AG}$-groupoid. Then the following conditions are equivalent:*

(i) *S is strongly regular;*

(ii) *S is strongly regular commutative monoid;*

(iii) $R_a \cap L_a = (R_a(L_aR_a \cdot R_a)]$, $(a = a^2, \forall\ a \in S)$;

(iv) $R \cap L = (R(LR \cdot R)]$;

(v) $f_A \sqcap g_B = f_A^3 \diamond g_B$;

(vi) *S is strongly regular and* $|E| = 1$, ($a = ax \cdot a$, $\forall\ a, x \in E$);

(vii) *S is strongly regular and* $\emptyset \neq E \subseteq S$ *is semilattice.*

Proof. $(i) \Longrightarrow (vii)$: It can be followed from Theorem 4.2 (i).

$(vii) \Longrightarrow (vi)$: It can be followed from Theorem 4.2 (ii).

$(vi) \Longrightarrow (v)$: Let f_A and g_B be any DFS r-ideal and DFS l-ideal of a strongly regular S over U respectively. From Lemma 3.2, it is easy to show that $f_A^{+3} \diamond g_B^+ \sqsubseteq f_A^+ \sqcap g_B^+$. Now for $a \in S$, there exist some x,$y \in S$ such that

$$\begin{aligned} a &\leq ax \cdot ay \leq (ax \cdot ay)x \cdot (ax \cdot ay)y = y(ax \cdot ay) \cdot x(ax \cdot ay) \\ &= (ax)(y \cdot ay) \cdot (ax)(x \cdot ay) = (ax)(ay^2) \cdot (ax)(a \cdot xy) \\ &= (y^2a)(xa) \cdot (ax)(a \cdot xy) = ((ax)(a \cdot xy))(xa) \cdot y^2a \\ &= ((ax)(a \cdot xy))(ex \cdot a) \cdot y^2a = ((ax)(a \cdot xy))(ax \cdot e) \cdot y^2a \\ &= bc \cdot y^2a = d \cdot y^2a, \text{ where } d = bc = ((ax)(a \cdot xy))(ax \cdot e). \end{aligned}$$

Thus

$$\begin{aligned} ((f_A^+ \tilde{\circ} f_A^+) \tilde{\circ} f_A^+)(d) &= \bigcup_{d \leq bc} \{(f_A^+ \tilde{\circ} f_A^+)(b) \cap f_A^+(c)\} \supseteq (f_A^+ \tilde{\circ} f_A^+)(b) \cap f_A^+(c) \\ &= \bigcup_{b \leq (ax)(a \cdot xy)} \{f_A^+(ax) \cap f_A^+(a \cdot xy)\} \cap f_A^+(ax \cdot e) \\ &\supseteq f_A^+(ax) \cap f_A^+(a \cdot xy) \cap f_A^+(ax \cdot e) \supseteq f_A^+(a). \end{aligned}$$

Therefore

$$(f_A^{+3} \tilde{\circ} g_B^+)(a) = \bigcup_{a \leq d \cdot y^2a} \{((f_A^+ \tilde{\circ} f_A^+) \tilde{\circ} f_A^+)(d) \cap g_B^+(y^2a)\} \supseteq f_A^+(a) \cap g_B^+(a),$$

which shows that $f_A^+ \cap g_B^+ \subseteq f_A^{+3} \tilde{\circ} g_B^+$, and similarly $f_A^- \cup g_B^- \supseteq f_A^{-3} \tilde{\star} g_B^-$. Thus $f_A \sqcap g_B = f_A^3 \diamond g_B$.

$(v) \Longrightarrow (iv)$: Let R and L be any right and left ideals of S respectively. Then by using Lemma 3.1, $\mathcal{X}_R$ and $\mathcal{X}_L$ are the DFS r-ideal and DFS l-ideal of S over U respectively. Now by using Lemma 3.4, we get

$$\mathcal{X}_{R \cap L} = \mathcal{X}_R \sqcap \mathcal{X}_L = ((\mathcal{X}_R \diamond \mathcal{X}_R) \diamond \mathcal{X}_R) \diamond \mathcal{X}_L = \mathcal{X}_{(R^3]} \diamond \mathcal{X}_L = \mathcal{X}_{((R^3]L]},$$

which implies that $R \cap L = ((R^3]L]$. Now by using Lemma 2.1, we get $R \cap L = ((R^3]L] = (R^3L] = (R^2R \cdot L] = (LR \cdot R^2] = (R^2 \cdot RL] = (R \cdot R^2L] = (R(LR \cdot R)]$.

$(iv) \Longrightarrow (iii)$: It is simple.

$(iii) \Longrightarrow (ii)$: Since $(Sa \cup Sa^2]$ is the smallest right ideal of S containing a and $(Sa]$ is the smallest left ideal of S containing a. Thus by using given assumption and Lemma 2.1, we get

$$\begin{aligned} a &\in (Sa \cup Sa^2] \cap (Sa] = ((Sa \cup Sa^2]((Sa](Sa \cup Sa^2] \cdot (Sa \cup Sa^2])] \\ &= ((Sa \cup Sa^2)((Sa)(Sa \cup Sa^2) \cdot (Sa \cup Sa^2))] \subseteq (S(S(Sa \cup Sa^2) \cdot (Sa \cup Sa^2))] \\ &= (S((S^2a \cup S^2a^2)(Sa \cup Sa^2))] = ((S^2a \cup S^2a^2)(S(Sa \cup Sa^2))] \\ &= ((S^2a \cup S^2a^2)(S^2a \cup S^2a^2)] = ((Sa \cup a^2S^2)(Sa \cup a^2S^2)] \\ &= ((Sa \cup S^2a \cdot a)(Sa \cup S^2a \cdot a)] \subseteq ((Sa \cup Sa)(Sa \cup Sa)] \\ &= (Sa \cdot Sa] = (aS \cdot aS]. \end{aligned}$$

Hence by using Lemma 4.2, S is strongly regular commutative monoid.

$(ii) \Longrightarrow (i)$: It is obvious. □

Let S be an ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid. From now onward, R (*resp.* L) will denote any right (*resp.* left) ideal of S; $\langle R \rangle_{a^2}$ will denote a right ideal $(Sa^2 \cup a^2]$ of S containing a^2 and $\langle L \rangle_a$ will denote a left ideal $(Sa \cup a]$ of S containing a; f_A (*resp.* g_B) will denote any DFS r-ideal over U (*resp.* DFS l-ideal over U) of S unless otherwise specified.

Theorem 4.6. *Let S be an ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid. Then S is strongly regular if and only if $\langle R \rangle_{a^2} \cap \langle L \rangle_a = (\langle R \rangle_{a^2}^2 \langle L \rangle_a^2]$ and $\langle R \rangle_{a^2}$ is semiprime.*

Proof. Necessity: Let S be strongly regular. It is easy to see that $(\langle R \rangle_{a^2}^2 \langle L \rangle_a^2] \subseteq \langle R \rangle_{a^2} \cap \langle L \rangle_a$. Let $a \in \langle R \rangle_{a^2} \cap \langle L \rangle_a$. Then there exist some $x, y \in S$ such that

$$\begin{aligned} a &\leq ax \cdot ay \leq (ax \cdot ay)x \cdot (ax \cdot ay)y = (x \cdot ay)(ax) \cdot (y \cdot ay)(ax) \\ &= (a \cdot xy)(ax) \cdot (ay^2)(ax) = (a \cdot xy)(ax) \cdot (xa)(y^2a) \\ &\in (\langle R \rangle_{a^2} S \cdot \langle R \rangle_{a^2} S)(S \langle L \rangle_a \cdot S \langle L \rangle_a) \subseteq \langle R \rangle_{a^2}^2 \langle L \rangle_a^2, \end{aligned}$$

which shows that $\langle R \rangle_{a^2} \cap \langle L \rangle_a = (\langle R \rangle_{a^2}^2 \langle L \rangle_a^2]$. It is easy to see that $\langle R \rangle_{a^2}$ is semiprime.

Sufficiency: Since $(Sa^2\cup a^2]$ and $(Sa\cup a]$ are the right and left ideals of S containing a^2 and a respectively. Thus by using given assumption and Lemma 2.1, we get

$$\begin{aligned} a &\in (Sa^2\cup a^2]\cap(Sa\cup a]=((Sa^2\cup a^2]^2(Sa\cup a]^2] \\ &= ((Sa^2\cup a^2)(Sa^2\cup a)\cdot(Sa\cup a)(Sa\cup a)]\subseteq(S(Sa^2\cup a)\cdot S(Sa\cup a)] \\ &= ((S\cdot Sa^2\cup Sa)(S\cdot Sa\cup Sa)]=((a^2S\cdot S\cup Sa)(aS\cdot S\cup Sa)] \\ &= ((a^2S\cdot S\cup Sa)(aS\cdot S\cup Sa)]=((Sa^2\cup Sa)(Sa\cup Sa)] \\ &= ((a^2S\cup Sa)(Sa\cup Sa)]=((Sa\cdot a\cup Sa)(Sa\cup Sa)]\subseteq((Sa\cup Sa)(Sa\cup Sa)] \\ &= (Sa\cdot Sa]=(aS\cdot aS]. \end{aligned}$$

This implies that S is strongly regular. $\square$

Corollary 4.3. *Let S be an ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid. Then S is strongly regular if and only if $\langle R\rangle_{a^2}\cap\langle L\rangle_a=(\langle L\rangle_a^2\langle R\rangle_{a^2}^2]$ and $\langle R\rangle_{a^2}$ is semiprime.*

Theorem 4.7. *Let S be an ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid. Then the following conditions are equivalent:*

(i) S is strongly regular;
(ii) $\langle R\rangle_{a^2}\cap\langle L\rangle_a=(\langle L\rangle_a^2\langle R\rangle_{a^2}^2]$ and $\langle R\rangle_{a^2}$ is semiprime;
(iii) $R\cap L=(L^2R^2]$ and R semiprime;
(iv) $f_A\sqcap g_B=(f_A\diamond g_B)\diamond(f_A\diamond g_B)$ and f_A is DFS semiprime;
(v) S is strongly regular and $|E|=1$, $(a=ax\cdot a,\ \forall\ a,x\in E)$;
(vi) S is strongly regular and $\emptyset\neq E\subseteq S$ is semilattice.

Proof. $(i)\Longrightarrow(vi)$: It can be followed from Corollary 4.2 (i).

$(vi)\Longrightarrow(v)$: It can be followed from Corollary 4.2 (ii).

$(v)\Longrightarrow(iv)$: Let f_A and g_B be any DFS r-ideal and DFS l-ideal of a strongly regular S over U respectively. From Lemma 3.2, it is easy to show that $(f_A\diamond g_B)\diamond(f_A\diamond g_B)\sqsubseteq f_A\sqcap g_B$. Now for $a\in S$, there exist some $x,y\in S$ such that

$$\begin{aligned} a &\leq ax\cdot ay\leq(ax\cdot ay)x\cdot(ax\cdot ay)y=(ax\cdot ay)\cdot((ax\cdot ay)x)y \\ &= (ax\cdot ay)\cdot(yx)(ax\cdot ay)=(ax\cdot ay)\cdot(ax)(yx\cdot ay) \\ &= (ax\cdot ay)\cdot(ay\cdot yx)(xa)=(ax\cdot ay)\cdot((yx\cdot y)a)(xa) \\ &= (ax)((yx\cdot y)a)\cdot(ay)(xa)=(ax)(ba)\cdot(ay)(xa),\ \text{where } yx\cdot y=b. \end{aligned}$$

Thus $(ax\cdot ba, ay\cdot xa)\in A_a$. Therefore

$$\begin{aligned}
&((f_A^+\,\widetilde{\circ}\,g_B^+)\,\widetilde{\circ}\,(f_A^+\,\widetilde{\circ}\,g_B^+))(a)\\
=&\bigcup_{(ax\cdot ba,ay\cdot xa)\in A_a}\{(f_A^+\,\widetilde{\circ}\,g_B^+)(ax\cdot ba)\cap(f_A^+\,\widetilde{\circ}\,g_B^+)(ay\cdot xa)\}\\
\supseteq&\bigcup_{ax\cdot ba\le ax\cdot ba}\{f_A^+(ax)\cap g_B^+(ba)\}\cap\bigcup_{ay\cdot xa=ay\cdot xa}\{f_A^+(ay)\cap g_B^+(xa)\}\\
\supseteq&\, f_A^+(ax)\cap g_B^+(ba)\cap f_A^+(ay)\cap g_B^+(xa)\supseteq f_A^+(a)\cap g_B^+(a),
\end{aligned}$$

which shows that $(f_A^+\widetilde{\circ}g_B^+)\widetilde{\circ}(f_A^+\widetilde{\circ}g_B^+)\supseteq f_A^+\cap g_B^+$. Similarly we can show that $(f_A^-\widetilde{\star}g_B^-)\widetilde{\star}(f_A^-\widetilde{\star}g_B^-)\subseteq f_A^-\cup g_B^-$. Thus $f_A\sqcap g_B\sqsubseteq(f_A\diamond g_B)\diamond(f_A\diamond g_B)$. Hence $f_A\sqcap g_B=(f_A\diamond g_B)\diamond(f_A\diamond g_B)$. Also by using Lemma 4.6, f_A is DFS semiprime.

$(iv)\implies(iii)$: Let R and L be any left and right ideals of S. Then by using Lemma 3.1, $\mathcal{X}_R$ and $\mathcal{X}_L$ are the DFS r-ideal and DFS l-ideal of S over U respectively. Now by using Lemma 3.4, we get $\mathcal{X}_{R\cap L}=\mathcal{X}_R\sqcap\mathcal{X}_L=(\mathcal{X}_R\diamond\mathcal{X}_L)\diamond(\mathcal{X}_R\diamond\mathcal{X}_L)=(\mathcal{X}_R\diamond\mathcal{X}_R)\diamond(\mathcal{X}_L\diamond\mathcal{X}_L)=\mathcal{X}_{(R^2]}\diamond\mathcal{X}_{(L^2]}=\mathcal{X}_{(R^2L^2]}=\mathcal{X}_{(L^2R^2]}$, which implies that $R\cap L=(L^2R^2]$.

$(iii)\implies(ii)$: It is simple.

$(ii)\implies(i)$: It can be followed from Corollary 4.3. □

Theorem 4.8. *Let S be an ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid. Then the following conditions are equivalent*:

(i) *S is strongly regular*;

(ii) $\langle R\rangle_{a^2}\cap\langle L\rangle_a=(\langle R\rangle_{a^2}\langle L\rangle_a\cdot\langle R\rangle_{a^2}]$ *and* $\langle R\rangle_{a^2}$ *is semiprime*;

(iii) $R\cap L=(RL\cdot R]$ *and R is semiprime*;

(iv) $f_A\sqcap g_B=(f_A\diamond g_B)\diamond f_A$ *and f_A is DFS semiprime*;

(v) *S is strongly regular and* $|E|=1$, $(a=ax\cdot a,\ \forall\ a,x\in E)$;

(vi) *S is strongly regular and $\emptyset\neq E\subseteq S$ is semilattice.*

Proof. $(i)\implies(vi)$: It can be followed from Corollary 4.2 (i).

$(vi)\implies(v)$: It can be followed from Corollary 4.2 (ii).

$(v)\implies(iv)$: Let f_A and g_B be any DFS l-ideals of a strongly regular S over U. Now for $a\in S$, there exist some $x,y\in S$ such that $a\le ax\cdot ay\le ax\cdot(ax\cdot ay)y=((ax\cdot ay)y\cdot x)a=(xy\cdot(ax\cdot ay))a=(ax\cdot(xy\cdot ay))a=(ax\cdot(a\cdot(xy)y))a$.

Thus $(ax\cdot(a\cdot(xy)y),a)\in A_a$. Therefore

$$\begin{aligned}
&((f_A^+\,\tilde{\circ}\,g_B^+)\,\tilde{\circ}\,f_A^+)(a)\\
=&\bigcup_{(ax\cdot(a\cdot(xy)y),a)\in A_a}\{(f_A^+\,\tilde{\circ}\,g_B^+)(ax\cdot(a\cdot(xy)y))\cap g_B^+(a)\}\\
\supseteq&\bigcup_{ax\cdot(a\cdot(xy)y\leq ax\cdot(a\cdot(xy)y}\{f_A^+(ax)\cap g_B^+(a\cdot(xy)y)\}\cap g_B^+(a)\\
&\supseteq f_A^+(ax)\cap g_B^+(a\cdot(xy)y)\cap g_B^+(a)\supseteq f_A^+(a)\cap g_B^+(a),
\end{aligned}$$

which shows that $(f_A^+\,\tilde{\circ}\,g_B^+)\,\tilde{\circ}\,f_A^+\supseteq f_A^+\cap g_B^+$. Similarly we can show that $(f_A^-\,\tilde{\star}\,g_B^-)\,\tilde{\star}\,f_A^-\subseteq f_A^-\cup g_B^-$. Thus $(f_A\diamond g_B)\diamond f_A\sqsupseteq f_A\sqcap g_B$. By using Lemmas 3.2 and 4.6, it is easy to show that $(f_A\diamond g_B)\diamond f_A\sqsubseteq f_A\sqcap g_B$. Thus $f_A\sqcap g_B=(f_A\diamond g_B)\diamond f_A$. Also by using Lemma 4.6, f_A is DFS semiprime.

$(iv)\Longrightarrow(iii)$: Let R and L be any left and right ideals of S. Then by Lemma 3.1, $\mathcal{X}_R$ and $\mathcal{X}_L$ are the DFS r-ideal and DFS l-ideal of S over U respectively. Now by using Lemmas 3.4, 4.1 and 2.1, we get

$$\mathcal{X}_{R\cap L}=\mathcal{X}_R\sqcap\mathcal{X}_L=(\mathcal{X}_R\diamond\mathcal{X}_L)\diamond\mathcal{X}_L=\mathcal{X}_{((RL]\cdot R]}=\mathcal{X}_{(RL\cdot R]},$$

which shows that $R\cap L=(RL\cdot R]$. Also by using Lemma 3.3, R is semiprime.

$(iii)\Longrightarrow(ii)$: It is simple.

$(ii)\Longrightarrow(i)$: Since $(Sa^2\cup a^2]$ and $(Sa\cup a]$ are the right and left ideals of S containing a^2 and a respectively. Thus by using given assumption and Lemma 2.1, we get

$$\begin{aligned}
a &\in (Sa^2\cup a^2]\cap(Sa\cup a]=((Sa^2\cup a^2](Sa\cup a]\cdot(Sa^2\cup a^2]]\\
&= ((Sa^2\cup a^2)(Sa\cup a)\cdot(Sa^2\cup a^2)]\subseteq(S(Sa\cup a)\cdot(Sa^2\cup a^2)]\\
&= ((S^2a\cup Sa)(Sa^2\cup a^2)]=((S^2a\cdot Sa^2)\cup(S^2a\cdot a^2)\cup(Sa\cdot Sa^2)\cup(S^2a\cdot a^2)]\\
&\subseteq ((Sa\cdot a^2S)\cup(Sa\cdot Sa)\cup(Sa\cdot a^2S)\cup(Sa\cdot Sa)]\\
&\subseteq ((Sa\cdot Sa)\cup(Sa\cdot Sa)\cup(Sa\cdot Sa)\cup(Sa\cdot Sa)]=(Sa\cdot Sa]=(aS\cdot aS].
\end{aligned}$$

Hence S is strongly regular. □

5. Conclusions

We have got some interesting and new characterizations which we usually do not find in other algebraic structures. We have considered the following problems in detail:

i) Define and compare DFS left/right ideals of an ordered $\mathcal{AG}$-groupoid and respective examples are provided.

ii) Introduce the concept of an ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid and characterize it by using DFS left/right ideals.

iii) Study the structural properties of a unitary ordered $\mathcal{AG}$-groupoid and ordered $\mathcal{A}^*\mathcal{G}^{**}$-groupoid in terms of its semilattices, strongly regular classes and generated commutative monoids.

This paper generalized the theory of an $\mathcal{AG}$-groupoid in the following ways:

i) In an $\mathcal{AG}$-groupoid (without order) by using the DFS-sets.

ii) In an $\mathcal{AG}$-groupoid (with and without order) by using fuzzy sets instead of DFS-sets.

Some important issues for future work are:

i) To develop strategies for obtaining more valuable results in related areas.

ii) To apply these notions and results for studying DFS expert sets and applications in decision making problems.

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