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PARABOLIC EQUATIONS IN MUSIELAK-ORLICZ-SOBOLEV SPACES

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ABSTRACT. We prove in this paper the existence of solutions of nonlinear parabolic problems in Musielak-Orlicz-Sobolev spaces. An approximation and a trace results in inhomogeneous Musielak-Orlicz-Sobolev spaces have also been provided.

1. INTRODUCTION

Let Ω a bounded open subset of $\mathbb{R}^n$ and let Q be the cylinder $\Omega \times (0,T)$ with some given $T > 0$.

This paper is concerned with the existence of solutions for boundary value problems for quasi-linear parabolic equations of the form

$$(1) \qquad \begin{cases} \frac{\partial u}{\partial t} + A(u) = f \text{ in } Q \\ u(x,t) = 0 \text{ on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \text{ in } \Omega \end{cases}$$

where A is a Leray-Lions operator of the form:

$$A(u) = -\text{ div } (a(x,t,u,\nabla u)) + a_0(x,t,u,\nabla u),$$

with the coefficients a and a_0 satisfying the classical Leray-Lions conditions. Consider first the case where a and a_0 have polynomial growth with respect to u and ∇u. Therefore A is a bounded operator from $L^p(0,T,W_0^{1,p}(\Omega)), 1 < p < \infty$,into its dual. In this setting, it is well known that problems of the form (1) were solved by Lions [16], and Brzis and Browder [9] in the case where $p \geq 2$, and by Landes [14] and Landes and Mustonen [15] when $1 < p < 2$. See also [6, 7] for related topics.

In the case where a and a_0 satisfy a more general growth with respect to u and ∇u (for example of exponential or logarithmic type), it is shown in [10] that the adequate space in which (1) can be studied is the inhomogeneous Orlicz-Sobolev space $W^{1,x}L_M(Q)$, where the N-function M is related to the actual growth of a and a_0. The solvability of (1) in this setting was proved by Donaldson [10] and Robert [18] when A is monotone, and by Elmahi [11] and Elmahi-Meskine [12].

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Our purpose in this paper is to prove existence theorems for the problem (1) in the setting of inhomogeneous Musielak-Orlicz-Sobolev spaces $W^{1,x}L_{\varphi}(Q)$ by applying some new approximation result in inhomogeneous Musielak-Orlicz-Sobolev spaces (see Theorem 1), as it is done in the setting of Orlicz-Sobolev spaces (see [12]), which allows us, on the one hand, to regularize a test function by smooth ones with converging time derivatives (and thus enlarge the set of test functions in order to cover the solution u and then get the energy equality), and, on the other hand, to prove a trace result (see Lemma 3) which states that if $u \in W_0^{1,x}L_{\varphi}(Q) \cap L^2(Q)$ such that $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\psi}(Q) + L^2(Q)$, then $u \in C([0,T], L^2(\Omega))$, showing that the assumption $u_0 \in L^2(\Omega)$ cannot be weakened.

Our result generalizes that of the Elmahi-Meskine in [12] to the case of inhomogeneous Musielak-Orlicz-Sobolev spaces.

Let us point out that our result can be applied in the particular case when $\varphi(x,t) = t^p(x)$, in this case we use the notations $L^{p(x)}(\Omega) = L_{\varphi}(\Omega)$, and $W^{m,p(x)}(\Omega) = W^m L_{\varphi}(\Omega)$. These spaces are called Variable exponent Lebesgue and Sobolev spaces.

For some classical and recent results on elliptic and parabolic problems in Orlicz-sobolev spaces and a Musielak-Orlicz-Sobolev spaces, we refer to [1, 2, 5, 10, 11, 12].

2. PRELIMINARIES

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. Standard reference is [17]. We also include the definition of inhomogeneous Musielak-Orlicz-Sobolev spaces and some preliminaries Lemmas to be used later.

Musielak-Orlicz-Sobolev spaces : Let Ω be an open subset of $\mathbb{R}^n$.
A Musielak-Orlicz function φ is a real-valued function defined in $\Omega \times \mathbb{R}_+$ such that :

a): $\varphi(x,t)$ is an N-function i.e. convex, nondecreasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,t) > 0$ for all $t > 0$ and

$$\begin{aligned} \lim_{t \longrightarrow 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} &= 0 \\ \lim_{t \longrightarrow \infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} &= 0. \end{aligned}$$

b): $\varphi(.,t)$ is a Lebesgue measurable function

Now, let $\varphi_x(t) = \varphi(x,t)$ and let φ_x^{-1} be the non-negative reciprocal function with respect to t, i.e the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x, \phi_x^{-1}) = t.$$

For any two Musielak-Orlicz functions φ and γ we introduce the following ordering :

c): if there exists two positives constants c and T such that for almost everywhere $x \in \Omega$:

$$\varphi(x,t) \le \gamma(x,ct) \text{ for } t \ge T$$

we write $\varphi \prec \gamma$ and we say that γ dominates φ globally if $T = 0$ and near infinity if $T > 0$.

d): if for every positive constant c and almost everywhere $x \in \Omega$ we have

$$\lim_{t\to 0}(\sup_{x\in\Omega} \frac{\varphi(x,ct)}{\gamma(x,t)}) = 0 \text{ or } \lim_{t\to\infty}(\sup_{x\in\varphi} \frac{\varphi(x,ct)}{\gamma(x,t)}) = 0$$

we write $\varphi \prec\prec \gamma$ at 0 or near ∞ respectively, and we say that φ increases essentially more slowly than γ at 0 or near infinity respectively.

In the sequel the measurability of a function $u : \Omega \mapsto R$ means the Lebesgue measurability.

We define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx$$

where $u : \Omega \mapsto \mathbb{R}$ is a measurable function.
The set

$$K_\varphi(\Omega) = \{u : \Omega \to R \text{ mesurable } /\varrho_{\varphi,\Omega}(u) < +\infty\}.$$

is called the Musielak-Orlicz class (the generalized Orlicz class).

The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$.
Equivelently:

$$L_\varphi(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ mesurable } /\varrho_{\varphi,\Omega}(\frac{|u(x)|}{\lambda}) < +\infty, \text{ for some } \lambda > 0\right\}$$

Let

$$\psi(x,s) = \sup_{t\ge 0}\{st - \varphi(x,t)\},$$

ψ is the Musielak-Orlicz function complementary to (or conjugate of) $\varphi(x,t)$ in the sense of Young with respect to the variable s.

On the space $L_\varphi(\Omega)$ we define the Luxemburg norm:

$$||u||_{\varphi,\Omega} = \inf\{\lambda > 0/ \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx, \le 1\}.$$

and the so-called Orlicz norm :

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_\psi \le 1} \int_\Omega |u(x)v(x)|dx.$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [17].

The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $E_\psi(\Omega)^* = L_\varphi(\Omega)$ [17].

The following conditions are equivalent:

e): $E_\varphi(\Omega) = K_\varphi(\Omega)$

f): $K_\varphi(\Omega) = L_\varphi(\Omega)$
g): φ has the Δ_2 property.

We recall that φ has the Δ_2 property if there exists $k > 0$ independent of $x \in \Omega$ and a nonnegative function h , integrable in Ω such that $\varphi(x, 2t) \leq k\varphi(x, t) + h(x)$ for large values of t, or for all values of t, according to whether Ω has finite measure or not.

Let us define the modular convergence: we say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \varrho_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall|\alpha| \leq m \quad D^\alpha u \in L_\varphi(\Omega)\}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ with nonnegative integers α_i; $|\alpha| = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives.
The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space.

Now, the functional

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \varrho_{\varphi,\Omega}(D^\alpha u),$$

for $u \in W^m L_\varphi(\Omega)$ is a convex modular. and

$$||u||^m_{\varphi,\Omega} = \inf\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1\}$$

is a norm on $W^m L_\varphi(\Omega)$.
The pair $\langle W^m L_\varphi(\Omega), ||u||^m_{\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition :

$$\text{there exist a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \geq c,$$

as in [17].

The space $W^m L_\varphi(\Omega)$ will always be identified to a $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \prod L_\varphi$.
Let $W^m_0 L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W^m E_\varphi(\Omega)$ be the space of functions u such that u and its distribution derivatives up to order m lie in $E_\varphi(\Omega)$, and let $W^m_0 E_\varphi(\Omega)$ be the (norm) closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m}L_{\psi}(\Omega) = \{f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^{\alpha} f_{\alpha} \text{ with } f_{\alpha} \in L_{\psi}(\Omega)\}$$

$$W^{-m}E_{\psi}(\Omega) = \{f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^{\alpha} f_{\alpha} \text{ with } f_{\alpha} \in E_{\psi}(\Omega)\}$$

As we did for $L_{\varphi}(\Omega)$, we say that a sequence of functions $u_n \in W^m L_{\varphi}(\Omega)$ is modular convergent to $u \in W^m L_{\varphi}(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

From [17], for two complementary Musielak-Orlicz functions φ and ψ the following inequalities hold :

h) : the young inequality :

$$t.s \leq \varphi(x,t) + \psi(x,s) \text{ for } t,s \geq 0,\ x \in \Omega$$

i) : the Hölder inequality :

$$\left|\int_{\Omega} u(x)v(x)\ dx\right| \leq ||u||_{\varphi,\Omega}|||v|||_{\psi,\Omega}.$$

for all $u \in L_{\varphi}(\Omega)$ and $v \in L_{\psi}(\Omega)$.

Inhomogeneous Musielak-Orlicz-Sobolev spaces :

Let Ω an bounded open subset of $\mathbb{R}^n$ and let $Q = \Omega\times]0,T[$ with some given T ¿ 0. Let φ be a Musielak function. For each $\alpha \in \mathbb{N}^n$, denote by D_x^{α} the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^n$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows.

$$W^{1,x}L_{\varphi}(Q) = \{u \in L_{\varphi}(Q) : \forall|\alpha| \leq 1\ D_x^{\alpha}u \in L_{\varphi}(Q)\}$$

and

$$W^{1,x}E_{\varphi}(Q) = \{u \in E_{\varphi}(Q) : \forall|\alpha| \leq 1\ D_x^{\alpha}u \in E_{\varphi}(Q)\}$$

The last space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\leq m} \|D_x^{\alpha}u\|_{\varphi,Q}.$$

We can easily show that they form a complementary system when Ω is a Lipschitz domain [4]. These spaces are considered as subspaces of the product space $\Pi L_{\varphi}(Q)$ which has $(N+1)$ copies. We shall also consider the weak topologies $\sigma(\Pi L_{\varphi}, \Pi E_{\psi})$ and $\sigma(\Pi L_{\varphi}, \Pi L_{\psi})$. If $u \in W^{1,x}L_{\varphi}(Q)$ then the function : $t \longmapsto u(t) = u(t,.)$ is defined on $(0,T)$ with values in $W^1L_{\varphi}(\Omega)$. If, further, $u \in W^{1,x}E_{\varphi}(Q)$ then this function is a $W^1E_{\varphi}(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds : $W^{1,x}E_{\varphi}(Q) \subset L^1(0,T;W^1E_{\varphi}(\Omega))$. The space $W^{1,x}L_{\varphi}(Q)$ is not in general separable, if $u \in W^{1,x}L_{\varphi}(Q)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto \|u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_\varphi(Q)$ is defined as the (norm) closure in $W^{1,x}E_\varphi(Q)$ of $\mathcal{D}(Q)$. We can easily show as in [4] that when Ω a Lipschitz domain then each element u of the closure of $\mathcal{D}(Q)$ with respect of the weak * topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is limit, in $W^{1,x}L_\varphi(Q)$, of some subsequence $(u_i) \subset \mathcal{D}(Q)$ for the modular convergence; i.e., there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$,

$$\int_Q \varphi(x, (\frac{D_x^\alpha u_i - D_x^\alpha u}{\lambda}))\,dx\,dt \to 0 \text{ as } i \to \infty,$$

this implies that (u_i) converges to u in $W^{1,x}L_\varphi(Q)$ for the weak topology $\sigma(\Pi L_M, \Pi L_\psi)$. Consequently

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi E_\psi)} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi L_\psi)},$$

this space will be denoted by $W_0^{1,x}L_\psi(Q)$. Furthermore, $W_0^{1,x}E_\varphi(Q) = W_0^{1,x}L_\varphi(Q) \cap \Pi E_\varphi$.

Poincaré's inequality also holds in $W_0^{1,x}L_\varphi(Q)$ i.e. there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_\varphi(Q)$ one has

$$\sum_{|\alpha|\leq 1} \|D_x^\alpha u\|_{\varphi,Q} \leq C \sum_{|\alpha|=1} \|D_x^\alpha u\|_{\varphi,Q}.$$

Thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_\varphi(Q)$. We have then the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_\varphi(Q) & F \\ W_0^{1,x}E_\varphi(Q) & F_0 \end{pmatrix},$$

F being the dual space of $W_0^{1,x}E_\varphi(Q)$. It is also, except for an isomorphism, the quotient of ΠL_ψ by the polar set $W_0^{1,x}E_\varphi(Q)^\perp$, and will be denoted by $F = W^{-1,x}L_\psi(Q)$ and it is shown that

$$W^{-1,x}L_\psi(Q) = \Big\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_\psi(Q) \Big\}.$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\psi,Q}$$

where the inf is taken on all possible decompositions

$$f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha, \quad f_\alpha \in L_\psi(Q).$$

The space F_0 is then given by

$$F_0 = \Big\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_\psi(Q) \Big\}$$

and is denoted by $F_0 = W^{-1,x}E_\psi(Q)$.

The following technical lemmas are important for the proof of our main result.

Lemma 1. If $u \in W_0^{1,1}(\Omega)$, then $||u_\sigma - u||_{1,\Omega} \leq \sigma ||\nabla u||_{1,\Omega}$, where $u_\sigma = u * \rho_\sigma$ and where (ρ_σ) is a mollifier sequence in $\mathbb{R}^N$.

Lemma 2. Let φ be an Musielak-Orlicz function. Let (u_n) be a bounded sequence in $W_0^{1,x}L_\varphi(Q) \cap L^\infty(0,T;L^1(\Omega))$. If $u_n(t) \rightharpoonup u(t)$ weakly in $L^1(\Omega)$ for almost every $t \in [0,T]$, then $u_n \to u$ strongly in $L^1(Q)$.

Proof.For each $v \in W_0^{1,x}L_\varphi(Q)$, denote $v_\sigma(x,t) = \int_{\mathbb{R}^N} v(y,t)\rho_\sigma(x-y)dy$, where $v(y,t) = 0$ if $y \notin \Omega$ and where (ρ_σ) is a mollifier sequence in $\mathbb{R}^N$.
Since $u_n(t) \rightharpoonup u(t)$ weakly in $L^1(\Omega)$, we have $u_{n\sigma}(x,t) \to u_\sigma(x,t)$ almost everywhere in Q
and $u_{n\sigma}(t) \to u_\sigma(t)$ strongly in $L^1(\Omega)$ for almost every $t \in [0,T]$,we have

$$\int_\Omega |u_n(t) - u_k(t)|dx \leq \int_\Omega |u_n(t) - u_{n\sigma}(t)|dx + \int_\Omega |u_{n\sigma}(t) - u_{k\sigma}(t)|dx + \int_\Omega |u_{k\sigma}(t) - u_k(t)|dx$$
$$\leq \sigma(\int_\Omega |\nabla u_n(t)|dx + \int_\Omega |\nabla u_k(t)|dx) + ||u_{n\sigma}(t) - u_{k\sigma}(t)||_{1,\Omega}$$

Integrating this over $[0,T]$ yields

$$\int_Q |u_n(t) - u_k(t)|dxdt \leq \sigma(\int_Q |\nabla u_n(t)|dxdt + \int_Q |\nabla u_k(t)|dxdt) + \int_0^T ||u_{n\sigma}(t) - u_{k\sigma}(t)||_{1,\Omega}dt$$

which gives, since $L_\varphi(Q) \subset L^1(Q)$ with continuous imbedding,

$$\int_Q |u_n(t) - u_k(t)|dxdt \leq \sigma C_1(||\nabla u_n||_{\varphi,Q} + ||\nabla u_k||_{\varphi,Q}) + \int_0^T ||u_{n\sigma}(t) - u_{k\sigma}(t)||_{1,\Omega}dt$$

where C_1 and C_2 are constants which do not depend on n and k such that $||v||_{1,Q} \leq C_1||v||_{\varphi,Q}$ for all $v \in L_\varphi(Q)$ and $||\nabla u_n||_{\varphi,Q} \leq C_2$ for all n. Consequently, we obtain :

$$\int_Q |u_n(t) - u_k(t)|dx \leq 2C_1C_2\sigma + \int_0^T ||u_{n\sigma}(t) - u_{k\sigma}(t)||_{1,\Omega}dt.$$

Since $||u_{n\sigma}(t) - u_{k\sigma}(t)||_{1,\Omega} \to 0$ almost everywhere in $[0,T]$ when $n,k \to \infty$ and $||u_{n\sigma}(t)||_{L^1(\Omega)} \leq ||u_n(t)||_{L^1(\Omega)} \leq C$ uniformly with respect to n and $t \in [0,T]$,we deduce by using Lebesgue's theorem that

$$\int_0^T ||u_{n\sigma}(t) - u_{k\sigma}(t)||_{1,\Omega}dt \to 0$$

as $n,k \to \infty$ implying, since σ is arbitrary, that $\int_Q |u_n(t) - u_k(t)|dxdt \to 0$ when n and $k \to \infty$.
Hence (u_n) is a Cauchy sequence in $L^1(Q)$ and thus $u_n \to u$ strongly in $L^1(Q)$.

3. APPROXIMATION AND TRACE RESULTS

In this section, Ω is a bounded Lipschitz domain in $\mathbb{R}^N$ and I is a subinterval of $\mathbb{R}$ (possibly unbounded) and $Q = \Omega \times I$. It is easy to see that Q also satisfies Lipschitz domain.

Definition. We say that $u_n \to u$ in $W^{-1,x}L_\psi(Q) + L^2(Q)$ for the modular convergence if we can write

$$u_n = \sum_{|\alpha|\leq 1} D_x^\alpha u_n^\alpha + u_n^0 \text{ and } u = \sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$$

with $u_n^\alpha \to u^\alpha$ in $L_\psi(Q)$ for modular convergence for all $|\alpha| \leq 1$
and $u_n^\alpha \to u^\alpha$ strongly in $L^2(Q)$.

We shall prove the following approximation theorem, which plays a fundamental role in the prove of our main results.
Theorem 1. If $u \in W^{1,x}L_\varphi(Q) \cap L^2(Q)$ (respectively $W_0^{1,x}L_\varphi(Q) \cap L^2(Q)$) and $\frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(Q) + L^2(Q)$, then there exists a sequence (v_j) in $\mathcal{D}(\overline{Q})$ (respectively $\mathcal{D}(\overline{I}, \mathcal{D}(\Omega))$) such that $v_j \to u$ in $W^{1,x}L_\varphi(Q) \cap L^2(Q)$ and $\frac{\partial v_j}{\partial t} \to \frac{\partial u}{\partial t}$ in $W^{-1,x}L_\psi(Q) + L^2(Q)$ for the modular convergence.

Proof. Let $u \in W^{1,x}L_\varphi(Q) \cap L^2(Q)$ such that $\frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(Q) + L^2(Q)$ and let $\varepsilon > 0$ be given. Writing $\frac{\partial u}{\partial t} = \sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$, where $u^\alpha \in L_\psi(Q)$ for all $|\alpha| \leq 1$ and $u^0 \in L^2(Q)$, we will show that there exists $\lambda > 0$(depending only on u and N)
and there exists $v \in \mathcal{D}(\overline{Q})$ for which we can write $\frac{\partial v}{\partial t} = \sum_{|\alpha|\leq 1} D_x^\alpha v^\alpha + v^0$ with $v^\alpha, v^0 \in \mathcal{D}(\overline{Q})$ such that

$$\int_Q \varphi(x, \frac{D_x^\alpha v - D_x^\alpha u}{\lambda})dxdt \leq \varepsilon, \forall |\alpha| \leq 1, \tag{2}$$

$$||v - u||_{L^2(Q)} \leq \varepsilon, \tag{3}$$

$$||v^0 - u^0||_{L^2(Q)} \leq \varepsilon, \tag{4}$$

$$\int_Q \psi(x, \frac{v^\alpha - u^\alpha}{\lambda})dxdt \leq \varepsilon, \forall |\alpha| \leq 1, \tag{5}$$

The equation (2) flows from a slight adaptation of the arguments of [4],
(3) and (4) flow also from classical approximation results.
Regrading the equation (5) it is enough to prove that $\mathcal{D}(\overline{Q})$ is dense in $L_\psi(Q)$, for this end we use the fact that the log-Hölder continuity(commutes with the complementarity) i.e :if φ is log-HÖlder the its complementary ψ also it is, and proceed as in [4] (with φ and ψ interchanged) and using of course $\mathbb{R}^{N+1}$ instead of $\mathbb{R}^N$ and $Q = \Omega \times (0, T)$ instead of Ω.
These facts lead us to prove that

$$||K_\varepsilon f||_{\psi,Q} \leq C||f||_{\psi,Q}, \forall f \in L_\psi(Q)$$

(with $K_\varepsilon f(x,t) = k_\varepsilon^{-1} \int_Q K_\varepsilon(x-y)f(k_\varepsilon y, t)dy$,$K_\varepsilon(x) = \frac{1}{\varepsilon^N}K(\frac{x}{\varepsilon})$ and $K(x)$ is a measurable function with support in the ball $B_R = B(0,R)$ see [4]).
And then we deduce that $\mathcal{D}(\overline{Q})$ is dense in $L_\psi(Q)$ for the modular convergence which gives the desired conclusion.
The case of $W_0^{1,x}L_\varphi(Q) \cap L^2(Q)$ is similar to the above arguments as in [4].

Remark 1. If, in the statement of Theorem 1, one consider $\Omega \times \mathbb{R}$ instead of Q, we have $\mathcal{D}(\Omega \times \mathbb{R})$ is dense in $u \in W_0^{1,x}L_\varphi(\Omega \times \mathbb{R}) \cap L^2(\Omega \times \mathbb{R}) : \frac{\partial u}{\partial t} \in W_0^{1,x}L_\psi(\Omega \times \mathbb{R}) + L^2(\Omega \times \mathbb{R})$ for the modular convergence. This follows trivially from the fact that $\mathcal{D}(\mathbb{R}, \mathcal{D}(\Omega)) \equiv \mathcal{D}(\Omega \times \mathbb{R})$.
A first application of Theorem 1 is the following trace result generalizing a classical result which states that if u belong to $L^2(a, b; H_0^1(\Omega))$ and $\frac{\partial u}{\partial t}$ belongs to $L^2(a, b; H^{-1}(\Omega))$, then u is in $C([a, b], L^2(\Omega))$.

Lemma 3. Let $a < b \in \mathbb{R}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then
$\{u \in W_0^{1,x}L_\varphi(\Omega\times(a,b))\cap L^2(\Omega\times(a,b)) : \frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(\Omega\times(a,b))+L^2(\Omega\times(a,b))\}$
is a subset of $C([a,b], L^2(\Omega))$.

Proof. Let $u \in W_0^{1,x}L_\varphi(\Omega \times (a,b)) \cap L^2(\Omega \times (a,b))$ such that $W^{-1,x}L_\psi(\Omega \times (a,b)) + L^2(\Omega \times (a,b))$. After two consecutive reflection first with respect to $t = b$ and then with respect to $t = a$,
$\hat{u}(x,t) = u(x,t)\chi_{(a,b)} + u(x, 2b-t)\chi_{(b,2b-a)}$ on $\Omega \times (a, 2b-a)$
$\tilde{u}(x,t) = \hat{u}(x,t)\chi_{(a,2b-a)} + \hat{u}(x, 2a-t)\chi_{(3a-2b,a)}$ on $\Omega \times (3a-2b, 2b-a)$,
we get a function $\tilde{u} \in W_0^{1,x}L_\varphi(\Omega \times (3a-2b, 2b-a)) \cap L^2(\Omega \times (3a-2b, 2b-a))$ such that $\frac{\partial \tilde{u}}{\partial t} \in W^{-1,x}L_\psi(\Omega \times (3a-2b, 2b-a)) + L^2(\Omega \times (3a-2b, 2b-a))$.
Now, by letting a function $\eta \in \mathcal{D}(\mathbb{R})$ with $\eta = 1$ on $[a,b]$ and supp $\eta \subset (3a-2b, 2b-a)$, setting $\overline{u} = \eta\tilde{u}$, and using standard arguments (see [[8],Lemme IV,Remarque 10,p.158]), we have $\overline{u} = u$ on $\Omega \times (a,b)$ $\tilde{u} \in W_0^{1,x}L_\varphi(\Omega \times \mathbb{R}) \cap L^2(\Omega \times \mathbb{R})$ $\frac{\partial \tilde{u}}{\partial t} \in W^{-1,x}L_\psi(\Omega \times \mathbb{R}) + L^2(\Omega \times \mathbb{R})$.
Now let $v_j \in \mathcal{D}(\Omega \times \mathbb{R})$ be the sequence given by Theorem 1 corresponding to $\overline{u}$, that is,

$$v_j \to \overline{u} \in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R})\cap L^2(\Omega\times\mathbb{R}) \text{ and } \frac{\partial v_j}{\partial t} \to \frac{\partial \overline{u}}{\partial t} \in W^{-1,x}L_\psi(\Omega\times\mathbb{R})+L^2(\Omega\times\mathbb{R})$$

for the modular convergence.
We have

$$\int_\Omega (v_i(\tau) - v_j(\tau))^2 dx = 2\int_\Omega \int_{-\infty}^{\tau} (v_i - v_j)(\frac{\partial v_i}{\partial t} - \frac{\partial v_j}{\partial t})dxdt \to 0, \text{ as } i,j \to \infty$$

from which one deduces that v_j is a Cauchy sequence in $C(\mathbb{R}, L^2(\Omega))$, and since the limit
of v_j in $L^2(\Omega \times \mathbb{R})$ is $\overline{u}$, we have $v_j \to \overline{u}$ in$C(\mathbb{R}, L^2(\Omega))$. Consequently, $u \in C([a,b], L^2(\Omega))$.

4. EXISTENCE RESULT

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N(N \geq 2)$, $T > 0$ and set $Q = \Omega \times (0,T)$.
Throughout this section, we denote $Q_\tau = \Omega \times (0,\tau)$ for every $\tau \in [0,T]$.
Let φ and γ two Musielak-Orlicz functions such that $\gamma \ll \varphi$.
Consider a second-order operator $A : D(A) \subset W^{1,x}L_\varphi(Q) \to W^{-1,x}L\psi(Q)$ of the form

$$A(u) = -div(a(x,t,u,\nabla u)) + a_0(x,t,u,\nabla u),$$

where $a : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N a_0 : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathodory functions, for almost every$(x,t) \in \Omega \times [0,T]$ and all $s \in \mathbb{R}, \xi \neq \xi^* \in \mathbb{R}^N$,

$$|a(x,t,s,\xi)| \le \beta(c(x,t) + \psi_x^{-1}\gamma(x,\vartheta|s|) + \psi_x^{-1}\varphi(x,\vartheta|\xi|)) \tag{6}$$

$$|a_0(x,t,s,\xi)| \le \beta(c(x,t) + \psi_x^{-1}\gamma(x,\vartheta|s|) + \psi_x^{-1}\varphi(x,\vartheta|\xi|)) \tag{7}$$

$$(a(x,t,s,\xi) - a(x,t,s,\xi^*))(\xi - \xi^*) > 0 \tag{8}$$

$$a(x,t,s,\xi)\xi + a_0(x,t,s,\xi)s \ge \alpha\varphi(x, \frac{|\xi|}{\lambda}) - d(x,t) \tag{9}$$

with $c(x,t) \in E_\psi(Q), c \ge 0, d(x,t) \in L^1(Q), \alpha, \beta, \vartheta > 0$. Furthermore, let

$$f \in W^{-1,x}E_\psi(Q) \tag{10}$$

Consider then the following parabolic initial-boundary value problem.

$$\begin{cases} \frac{\partial u}{\partial t} + A(u) = f \text{ in } Q \\ u(x,t) = 0 \text{ on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \text{ in } \Omega \end{cases} \tag{11}$$

where u_0 is a given function in $L^2(\Omega)$.
We shall prove the following existence theorem.

Theorem 2. Assume that (6)-(10) hold true. Then there exists at least one distributional solution $u \in D(A) \cap W_0^{1,x}L_\varphi(Q) \cap \mathcal{C}(([0,T], L^2(\Omega))$ of (11) satisfying $u(x,0) = u_0(x)$ for almost every $x \in \Omega$. Furthermore, for all $\tau \in [0,T]$, we have

$$\langle \frac{\partial \phi}{\partial t}, u \rangle_{Q_\tau} + [\int_\Omega u(t)\phi(t)dx]_0^\tau + \int_{Q_\tau} [a(x,t,u,\nabla u)\nabla\phi + a_0(x,t,u,\nabla u)\phi]dxdt = \langle f, \phi \rangle_{Q_\tau} \tag{12}$$

for every $\phi \in W_0^{1,x}L_\varphi(Q) \cap L^2(Q)$ with $\frac{\partial \phi}{\partial t} \in W^{-1,x}L_\psi(Q) + L^2(Q)$
and for $\phi = u$, which gives the energy equality

$$\frac{1}{2}\int_\Omega u^2(\tau)dx - \frac{1}{2}\int_\Omega u_0^2 dx + \int_{Q_\tau} [a(x,t,u,\nabla u)\nabla u + a_0(x,t,u,\nabla u)u]dxdt = \langle f, u \rangle_{Q_\tau}$$

Remark 2. Note that all the terms in (12) make sense. Indeed, it easy to see that the first, third, and fourth terms are well defined.
For the second one, we have by the trace result in Lemma 3 that $\phi \in C([0,T], L^2(\Omega))$, from which we can easily show that the second term of (12) makes sense.
Note also that taking $\phi \in \mathcal{D}(Q)$ in (12) shows that u is a distributional solution of (11).

Remark 3. If $a_0 \equiv 0$ and $a(x,t,s,\xi) \equiv a(x,t,\xi)$ does not depend on s, then the solution u is unique. Ineed, let $v \in W_0^{1,x}L_\phi(Q) \cap L^2(Q)$ be another solution of (11). Using $u - v$ as a test function in both equations corresponding to u and v with $\tau = T$, we get

$$\frac{1}{2}[\int_\Omega (u(t) - v(t))^2 dx]_0^\tau + \int_{Q_\tau} [a(x,t,u,\nabla u) - a(x,t,u,\nabla v)][\nabla u - \nabla v]dxdt = 0, \tag{13}$$

which implies that, by (8) and the fact that $u(0) = v(0), \nabla u = \nabla v$. This gives, again by(13),
$u(t) = v(t)$ for almost every $t \in [0,T]$ and hence $u = v$.

Remark 4. Note that the trace result in Lemma3 shows that the assumption $u_0 \in L^2(\Omega)$ cannot be weakened in order to get a distributional solution for the Cauchy-Dirichlet problem (11).

Remark 5. As in the elliptic case (see, [5]), γ is introduced instead of φ in (6) and (7) only to guarantee the boundedness in $L_\psi(Q)$ of $\psi_x^{-1}\gamma(x,\vartheta|u_n|)$ and $\psi_x^{-1}\gamma(x,\vartheta|\nabla u_n|)$ whenever u_n is bounded in $W^{1,x}L_\varphi(Q)$.

In the elliptic case, one usually takes $\gamma=\varphi$ in the term $\psi_x^{-1}\gamma(x,\vartheta|u_n|)$ since u_n is bounded in a smaller space $L_\theta(\Omega)$ with $\varphi\ll\theta$; see [5].
However, in the parabolic case, we cannot conclude that there is the boundedness. Nevertheless, we can take $\gamma=\varphi$ if one of the following assertions holds true.
(1) φ satisfies a $\triangle_2$ condition near infinity.
(2) A is monotone, that is $\langle A(u)-A(v),u-v\rangle\geq 0$ for all $u,v\in D(A)\cap W_0^{1,x}L_\varphi(Q)$.
Indeed, suppose first that φ satisfies a $\triangle_2$ condition.Therefore (6) and (7),now with $\gamma=\varphi$, imply that, for all $\varepsilon>0$,

$$|a(x,t,s,\xi)|\leq\beta_\varepsilon(c_\varepsilon(x,t)+\psi_x^{-1}\varphi(x,\varepsilon|s|)+\psi_x^{-1}\varphi(x,\varepsilon|\xi|)),$$
$$|a_0(x,t,s,\xi)|\leq\beta_\varepsilon(c_\varepsilon(x,t)+\psi_x^{-1}\varphi(x,\varepsilon|s|)+\psi_x^{-1}\varphi(x,\varepsilon|\xi|))$$

which allows us to deduce the boundedness in $L_\psi(Q)$ of $a(x,t,u_n,\nabla u_n)$ and $a(x,t,u_n,\nabla u_n)$.
Assume now that A is monotone. We have, for all $\phi\in W_0^{1,x}E_\varphi(Q),\langle A(u_n)-A(\phi),u_n-\phi\rangle\geq 0$. This gives $\langle A(u_n),\phi\rangle\leq\langle A(u_n),u_n\rangle-\langle A(\phi),u_n-\phi\rangle$, which implies that, since u_n is bounded in $W_0^{1,x}L_\varphi(Q)$ and $\langle A(u_n),u_n\rangle$ is bounded from above, thanks to the a priori estimates,

$$\langle A(u_n),\phi\rangle\leq C_\phi \text{ for all } \phi\in W_0^{1,x}E_\varphi(Q),$$

where C_ϕ is a constant depending on ϕ but not n. Therefore, the Banach-Steinhauss theorem applies so that we can obtain the boundedness of $A(u_n)$ in $W^{-1,x}L_\psi(Q)$.
Proof of Theorem 2. We will use a Galerkin method due to Landes and Musten [15]. For the Galerkin method, we choose a sequence $\{w_1,w_2,.....\}$ in $\mathcal{D}(\Omega)$ such that $\bigcup_{n=1}^\infty V_n$ with

$$V_n=\ \text{span}\{w_1,w_2,....,w_n\}$$

is dense in $H_0^m(\Omega)$ with m sufficiently large such that $H_0^m(\Omega)$ is continuously embedded in $C^1(\overline{\Omega})$. For any $v\in H_0^m(\Omega)$, there exists a sequence $(v_k)\subset\bigcup_{n=1}^\infty V_n$ such that $v_k\to v$ in $H_0^m(\Omega)$ and $C^1(\overline{\Omega})$ too.
We denote further $\mathcal{V}_n=C([0,T],V_n)$. It is easy to see that the closure of $\bigcup_{n=1}^\infty\mathcal{V}_n$ with respect to the norm

$$||v||_{C^{1,0}(Q)}=sup_{|\alpha|\leq 1}\{|D_x^\alpha v(x,t)|:(x,t)\in Q\}$$

contains $\mathcal{D}(\Omega)$. This implies that, for any $f\in W^{-1,x}E_\psi(Q)$, there exists a sequence $(f_k)\subset\bigcup_{n=1}^\infty\mathcal{V}_n$ such that $f_k\to f$ strongly in $W^{-1,x}E_\psi(Q)$. Indeed,let $\varepsilon>0$ be given . Writing $f=\sum_{|\alpha|\leq 1}D_x^\alpha f^\alpha$ for all $|\alpha|\leq 1$, there exists $g^\alpha\in\mathcal{D}(Q)$ such that $||f^\alpha-g^\alpha||_{\psi,Q}\leq\frac{\varepsilon}{(2N+2)}$. Moreover, by setting $g=\sum_{|\alpha|\leq 1}D_x^\alpha g^\alpha$, we see that $g\in\mathcal{D}(Q)$, and so there exists $\phi\in\bigcup_{n=1}^\infty\mathcal{V}_n$ such that $||g-\phi||_{\infty,Q}\leq\frac{\varepsilon}{(2meas(Q))}$. We deduce then that

$$||f-\phi||_{W^{-1,x}L_\psi(Q)}\leq\sum_{|\alpha|\leq 1}||f^\alpha-g^\alpha||_{\psi,Q}+||g-\phi||_{\psi,Q}$$

For any $u_0\in L^2(\Omega)$, there is a sequence $u_{0k}\subset\bigcup_{n=1}^\infty V_n$ such that $u_{0k}\to u_0$ in $L^2(\Omega)$.

We divide the proof into three steps.

Step 1(a priori estimates): As in [15], by using [[14],Lemma1], we find that there exists a Gelerkin solution u_n of (13) in the following sense.

$$u_n \in \mathcal{V}_n, \frac{\partial u_n}{\partial t} \in L^1(0,T;V_n), u_n(0) = u_{0n}, \tag{14}$$

and for all $\phi \in \mathcal{V}_n$ and all $\tau \in [0,T]$

$$\int_{Q_\tau} \frac{\partial u_n}{\partial t}\phi dxdt + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla\phi dxdt + \int_{Q_\tau} a_0(x,t,u_n,\nabla u_n)\phi dxdt = \int_{Q_\tau} f\phi dxdt.$$

Letting $\phi = u_n$ in (13) with $\tau = T$ and using (9) yields

$$||u_n||_{W_0^{1,x}L_\varphi(Q)} \leq C, ||u_n||_{L^\infty(0,T;L^2(\Omega))} \leq C,$$
$$\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n dxdt + \int_Q a_0(x,t,u_n,\nabla u_n)u_n dxdt \leq C,$$

where here and below C denotes a constant not depending on n. Using (7) and the fact that $\gamma \ll \varphi$, it is easy to see that $a_0(x,t,u_n,\nabla u_n)$ is bounded in $L_\psi(Q)$. This implies that

$$\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \leq C.$$

To prove that $a(x,t,u_n,\nabla u_n)$ is bounded in $(L_\psi(Q))^N$, let $\phi \in (E_\varphi(Q))^N$, with $||\phi||_{\varphi,Q} = 1$. In view of (8), we have

$$\int_Q [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\phi)][\nabla u_n - \phi]dxdt \geq 0,$$

which gives

$$\int_Q [a(x,t,u_n,\nabla u_n)\phi dxdt \leq \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n dxdt - \int_Q a(x,t,u_n,\phi)[\nabla u_n - \phi]dxdt,$$

and since, thinks to (6), $a(x,t,u_n,\phi)$ is uniformly bounded in $(L_\psi(Q))^N$, we deduce that

$$\int_Q a(x,t,u_n,\nabla u_n)\phi dxdt \leq C \text{ for all } \phi \in (E_\varphi(Q))^N, ||\phi||_{\varphi,Q} = 1,$$

which implies that, by the use of the dual norm of $(L_\psi(Q))^N, a(x,t,u_n,\nabla u_n)$ is bounded in $L_\psi(Q))^N$. Hence, for a subsequence and some $h_0 \in L_\psi(Q), h \in (L_\psi(Q))^N$,
$u_n \rightharpoonup u$ in $W_0^{1,x}L_\psi(Q)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and weakly in $L^2(Q)$,
$a_0(x,t,u_n,\nabla u_n) \rightharpoonup h_0, a(x,t,u_n,\nabla u_n) \rightharpoonup h$ in $L_\psi(Q)$ for $\sigma(\Pi L_\psi, \Pi E_\varphi)$.
As in [15], we get $u_n(t) \rightharpoonup u(t)$ in $L^1(\Omega)$ for almost every $t \in [0,T]$, and then, by using Lemma2, we deduce that $u_n \to u$ strongly in $L^1(Q)$ and that, for some subsequence still denoted by u_n, $u_n \to u$ almost everywhere in Q.

Step 2(almost everywhere convergence of the gradients):For every $\tau \in (0,T]$ and for all $\phi \in C^1([0,T],\mathcal{D}(\Omega))$, we get from (10)

$$(15)\int_{Q_\tau} u\frac{\partial\phi}{\partial t}dxdt + [\int_\Omega u(t)\phi(t)dx]_0^\tau + \int_{Q_\tau} h\nabla\phi + \int_{Q_\tau} h_0\phi dxdt = \langle f,\phi\rangle_{Q_\tau},$$

and then, by choosing $\tau = T$ and taking ϕ to be arbitrary in $\mathcal{D}(Q)$, we have $\frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(Q)$.
Consider now the prolongation of u to $\Omega\times\mathbb{R}$ as int the proof of Lemma3. We see that there exists a sequence v_k in $\mathcal{D}(\Omega\times\mathbb{R})$ such that
$v_k \to u$ in $W_0^{1,x}L_\varphi(Q)\cap L^2(Q)$ and $\frac{\partial v_k}{\partial t} \to \frac{\partial u}{\partial t}$ in $W^{-1,x}L_\psi(Q)+L^2(Q)$ for the modular convergence and so (see the proof of Lemma 3), $v_k \to u$ in $C([0,T],L^2(\Omega))$, which implies that, in particular,
$u\in C([0,T],L^2(\Omega))$. Consequently,

$$\lim_{k\to\infty}\int_Q \frac{\partial v_k}{\partial t}(v_k-u)dxdt = 0,$$

which gives, by the use of the fact that $\frac{\partial v_k}{\partial t}\in E_\psi(Q)$,

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_Q \frac{\partial v_k}{\partial t}(v_k-u_n)dxdt = 0.$$

This implies that

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\int_Q \frac{\partial u_n}{\partial t}(v_k-u_n)dxdt \le 0.$$

Since

$$\begin{aligned}\textstyle\int_Q \frac{\partial u_n}{\partial t}(v_k-u_n)dxdt &= \textstyle -\frac{1}{2}[\int_\Omega(u_n(t)-v_k(t))^2dx]_0^T + \int_Q \frac{\partial v_k}{\partial t}(v_k-u_n)dxdt\\ &\le \textstyle\frac{1}{2}||u_{0n}-v_k(0)||^2_{L^2(\Omega)} + \int_Q \frac{\partial v_k}{\partial t}(v_k-u_n)dxdt\end{aligned}$$

and $u_{0n}\to u_0$ in $L^2(\Omega)$ and $v_k(0)\to u(0)$ in $L^2(\Omega)$(note that $u(0)=u_0$ since $u_n(0)\rightharpoonup u(0)$ in $L^1(\Omega)$).
From (14) and (15), we have

$$\begin{aligned}\limsup_{n\to\infty}(\int_\Omega [a(x,t,u_n\nabla u_n)\nabla u_n - h\nabla v_k + a_0(x,t,u_n,\nabla u_n)u_n - h_0 v_k]dxdt)\\ \le \limsup_{n\to\infty}\langle f_n,u_n\rangle - \langle f,v_k\rangle +\\ \limsup_{n\to\infty}(-\int_Q \frac{\partial u_n}{\partial t}u_n dxdt) - \int_Q\frac{\partial v_k}{\partial t}udxdt + [\int_\Omega u(t)v_k(t)dx]_0^T\\ = \langle f,u-v_k\rangle + \limsup_{n\to\infty}\int_Q \frac{\partial u_n}{\partial t}(v_k-u_n)dxdt\end{aligned}$$

Where we have used the fact that

$$\begin{aligned}-\int_Q\frac{\partial v_k}{\partial t}(u)dxdt + [\int_\Omega u(t)v_k(t)dx]_0^T\\ = \lim_{n\to\infty}(-\int_Q\frac{\partial v_k}{\partial t}(u_n)dxdt + [\int_\Omega u_n(t)v_k(t)dx]_0^T)\\ = \lim_{n\to\infty}\int_Q\frac{\partial u_n}{\partial t}(v_k)dxdt.\end{aligned}$$

We deduce that

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\Big(\int_\Omega [a(x,t,u_n,\nabla u_n)\nabla u_n - h\nabla v_k + a_0(x,t,u_n\nabla u_n)(u_n-v_k)]dxdt\Big) \le 0 \tag{16}$$

Since, as can be easily seen,
$\lim_{n\to\infty}\int_\Omega (a(x,t,u_n,\nabla u_n)\nabla v_k - \nabla v_k + a_0(x,t,u_n\nabla u_n)v_k)dxdt) = \int_Q (h\nabla v_k + h_0 v_k)dxdt.$

In the sequel, and for any $r>0$ and any $k\in\mathbb{N}$, we denote by χ_k^r χ^r the characteristic functions of $\{(x,t)\in Q : |\nabla v_k|\le r\}$ and $\{(x,t)\in Q : |\nabla u|\le r\}$, respectivly. We also denote by $\varepsilon(n,k,s)$ all quantities (possibly different) depending on l such that

$$\lim_{s\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\varepsilon(n,k,s)=0,$$

and this will be the order in which the parametres we use will tend to infinity, that is, first n, then k, and finally s. Similarly, we will write only$\varepsilon(n)$, or $\varepsilon(n,k)$,...to mean that the limits are only on the specified parametrers. We have, for any $l>0$,

$$\begin{aligned}
&\int_{\{|u_n|\le l\}}[a(x,t,u_n,\nabla u_n)-a(x,t,u_n,\nabla u.\chi^s)][\nabla u_n-\nabla u.\chi^s]dxdt\\
&-\int_{\{|u_n|\le l\}}[a(x,t,u_n,\nabla u_n)-a(x,t,u_n,\nabla v_k.\chi_k^s)][\nabla u_n-\nabla v_k.\chi_k^s]dxdt\\
&\qquad=\int_{\{|u_n|\le l\}}a(x,t,u_n,\nabla v_k.\chi_k^s)[\nabla u_n-\nabla v_k.\chi_k^s]dxdt\\
&\qquad+\int_{\{|u_n|\le l\}}a(x,t,u_n,\nabla u_n)[\nabla v_k.\chi_k^s-\nabla u.\chi^s]dxdt\\
&\qquad\int_{\{|u_n|\le l\}}a(x,t,u_n,\nabla u.\chi^s)[\nabla u.\chi^s-\nabla u_n]dxdt\\
&\qquad:= I_1+I_2+I_3.
\end{aligned}$$

We shall go to the limit in all integrals I_i (for $i=1,2,3$) as first n, then k and finally s tend to infinity. Starting with I_1 and letting $n\to\infty$ yields

$$I_1=\int_{\{|u|\le l\}}a(x,t,u,\nabla v_k.\chi_k^s)[\nabla u-\nabla v_k.\chi_k^s]dxdt+\varepsilon(n)$$

Since $\chi_{\{|u_n|\le l\}}a(x,t,u_n,\nabla v_k.\chi_k^s)\to\chi_{\{|u|\le l\}}a(x,t,u,\nabla v_k.\chi_k^s)$ strongly in $(E_\psi(Q))^N$ by (6) and the Lebesgue theorem while $\nabla u_n\to\nabla u$ in $(L_\varphi(Q))^N$. This implies, by letting $k\to\infty$ in the integral of last side, that

$$I_1=\int_{\{|u|\le l\}\cap\{|\nabla u|>s\}}a(x,t,u,0)\nabla u dxdt+\varepsilon(n,k),$$

from which we get $I_1=\varepsilon(n,k,s)$, since the first term of the last side goes to 0 as $s\to\infty$.
For I_2, we have, by letting $n\to\infty$,

$$I_2=\int_{\{|u|\le l\}\cap\{|\nabla u|>s\}}h[\nabla v_k.\chi_k^s-\nabla u.\chi^s]dxdt+\varepsilon(n),$$

and so, by letting $k\to\infty$ in the integral of last side and using the fact that $\nabla v_k\chi_k^s\to\nabla u\chi^s$ strongly in $(E_\varphi(Q))^N$, we deduce that $I_2=\varepsilon(n,k)$.

For the third term I_3, we have, by letting $n \to \infty$,

$$I_3 = -\int_{\{|u|\leq l\}\cap\{|\nabla u|>s\}} a(x,t,u,0)\nabla u dxdt + \varepsilon(n,k),$$

and since the first term of the last side tends to zero as $s \to \infty$, we obtain $I_3 = \varepsilon(n,k,s)$.
We have then proved that

$$\int_{\{|u_n|\leq l\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u.\chi^s)][\nabla u_n - \nabla u.\chi^s]dxdt$$

$$= \int_{\{|u_n|\leq l\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla v_k.\chi_k^s)][\nabla u_n - \nabla v_k.\chi_k^s]dxdt + \varepsilon(n,k,s).$$

For all $s \geq r > 0$ and all $l \geq \delta > 0$, we have

$$(17)\ 0 \leq \int_{\{|u_n|\leq\delta\}\cap\{|\nabla u|\leq r\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]dxdt$$

$$\leq \int_{\{|u_n|\leq l\}\cap\{|\nabla u|\leq s\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]dxdt$$

$$\leq \int_{\{|u_n|\leq l\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u.\chi^s)][\nabla u_n - \nabla u.\chi^s]dxdt$$

$$= \int_{\{|u_n|\leq l\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla v_k.\chi_k^s)][\nabla u_n - \nabla v_k.\chi_k^s]dxdt + \varepsilon(n,k,s)$$

$$= -\int_{\{|u_n|\leq l\}} a(x,t,u_n,\nabla v_k.\chi_k^s)[\nabla u_n - \nabla v_k.\chi_k^s]dxdt$$

$$+ \int_Q [a(x,t,u_n,\nabla u_n)(\nabla u_n - \nabla v_k) + a_0(x,t,u_n,\nabla u_n)(u_n - v_k)]dxdt$$

$$-(\int_{\{|u_n|\leq l\}} a(x,t,u_n,\nabla u_n)[\nabla u_n - \nabla v_k]dxdt + \int_Q a_0(x,t,u_n,\nabla u_n)(u_n - v_k)]dxdt)$$

$$+ \int_{\{|u_n|\leq l\}\cap\{|v_k|>s\}} a(x,t,u_n,\nabla u_n)\nabla v_k dxdt + \varepsilon(n,k,s)$$

$$:= J_1 + J_2 + J_3 + J_4 + \varepsilon(n,k,s).$$

We shall go to the limit sup first over n and next over k and finally over s in all integrals of the last side.
First of all, note that $J_1 = -I_1 = \varepsilon(n,k,s)$ and that, thanks to (16),

$$\limsup_{k\to\infty} \limsup_{k\to\infty} J_2 \leq 0.$$

The third integral reads
$J_3 = -\int_{\{|u_n|>l\}} [a(x,t,u_n,\nabla u_n)(\nabla u_n - \nabla v_k) + a_0(x,t,u_n,\nabla u_n)(u_n - v_k)]dxdt$
$-\int_{\{|u_n|\leq l\}} a_0(x,t,u_n,\nabla u_n)(u_n - v_k)dxdt,$
and, by using (9),

$$J_3 \leq \int_{\{|u_n|>l\}} [a(x,t,u_n,\nabla u_n)\nabla v_k + a_0(x,t,u_n,\nabla u_n)v_k]dxdt$$

$$-\int_{\{|u_n|>l\}} d(x,t)dxdt - \int_{\{|u_n|\leq l\}} a_0(x,t,u_n,\nabla u_n)(u_n - v_k)dxdt,$$

which gives
$\limsup_{n\to\infty} J_3 \leq \int_{\{|u|\geq l\}} (h\nabla v_k + h_0 v_k)dxdt - \int_{\{|u|\geq l\}} d(x,t)dxdt - \int_{\{|u|\leq l\}} h_0(u - v_k)dxdt$, where we have used the strong convergence of $\chi_{\{|u_n|>l\}}|\nabla v_k|$ and $\chi_{\{|u_n|>l\}}|v_k|$ and $\chi_{\{|u_n|\leq l\}} u_n$ in $E_\varphi(Q)$
as $n \to \infty$. This implies that

$$\limsup_{k\to\infty} \limsup_{n\to\infty} J_3 \leq \int_{\{|u|\geq l\}} (h\nabla u + h_0 u)dxdt - \int_{\{|u|\geq l\}} d(x,t)dxdt$$

since $v_k \to u$ in $W_0^{1,x}L_\phi(Q)$ for the modular convergence. For J_4, we have

$$\lim_{n\to\infty} J_4 = \int_{\{|u|\leq l\}\cap\{|\nabla v_k|>s\}} h\nabla v_k dxdt$$

since $\chi_{\{|u_n|\leq l\}\cap\{|\nabla v_k|>s\}}\nabla v_k \to \chi_{\{|u|\leq l\}\cap\{|\nabla v_k|>s\}}\nabla v_k$ strongly in $(E_\varphi(Q))^N$ as $n \to \infty$. This implies that

$$\lim_{k\to\infty} \lim_{n\to\infty} J_4 = \int_{\{|u|\leq l\}\cap\{|u|\geq s\}} h\nabla u dxdt \leq \int_{\{|u|\geq s\}} |h\nabla u| dxdt$$

and thus

$$\limsup_{s\to\infty} \lim_{k\to\infty} \lim n \to \infty J_4 \leq 0.$$

Combining these estimates with (17) and passing to the limit sup first over n, then over k, and finally over s, we deduce that

$$0 \leq \limsup_{n\to\infty} \int_{\{|u_n|\leq\delta, |\nabla u|\leq r\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]dxdt$$
$$\leq \int_{\{|u_n|\geq l\}} (h\nabla u + h_0 u - d(x,t))dxdt,$$

in which we can let $l \to \infty$ to get
$\lim_{n\to\infty} \int_{\{|u_n|\leq\delta,|\nabla u|\leq r\}} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]dxdt = 0$, and thus, as the elliptic case (see [1]), we deduce that, for a subsequence still denoted by u_n,

$$\nabla u_n \to \nabla u \text{ a.e in } Q,$$

and so $h = a(x,t,u,\nabla u)$ and $h_0 = a_0(x,t,u,\nabla u)$. Therefore, we get for every $\tau \in (0,T]$ and for all $\phi \in C^1([O,\tau], \mathcal{D}(\Omega))$,

$$(18) - \int_{Q_\tau} u\frac{\partial\phi}{\partial t}dxdt + [\int_\Omega u(t)\phi(t)dx]_0^\tau + \int_{Q_\tau} [a(x,t,u,\nabla u)\nabla\phi + a_0(x,t,u,\nabla u)\phi]dxdt = \langle f, \phi\rangle_{Q_\tau}.$$

Step 3 (passage to the limit): Let $v \in W_0^{1,x}L_\varphi(Q) \cap L^2(Q)$ such that $\frac{\partial v}{\partial t}$ in $W^{-1,x}L_\psi(Q) + L^2(Q)$. There exists a prolongation $\overline{v}$ of v such that (see proof of Lemma 3)

$$(19) \overline{v} = v \text{ on } Q, \overline{v} \in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R}) \cap L^2(\Omega\times\mathbb{R}), \frac{\partial\overline{v}}{\partial t} \in W^{-1,x}L_\psi(\Omega\times\mathbb{R}) + L^2(\Omega\times\mathbb{R}).$$

By Theorem 1 (see also Remark 1),there exists a sequence $w_j \subset \mathcal{D}(\Omega\times\mathbb{R})$ such that

$$(20) w_j \to \overline{v} \in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R}) \cap L^2(\Omega\times\mathbb{R}) \text{ and } \frac{\partial w_j}{\partial t} \to \frac{\partial\overline{v}}{\partial t} \in W^{-1,x}L_\psi(\Omega\times\mathbb{R}) + L^2(\Omega\times\mathbb{R})$$

for the modular convergence.
Letting $\phi = w_j \chi_{(0,\tau)}$ (which belongs to $C^1([0,\tau], \mathcal{D}(\Omega))$) as a test function in (18), we get, for every $\tau \in (0,T]$,

$$-\int_{Q_\tau} u \frac{\partial w_j}{\partial t} dxdt + [\int_\Omega u(t) w_j(t) dx]_0^\tau + \int_{Q_\tau} [a(x,t,u,\nabla u)\nabla w_j + a_0(x,t,u,\nabla u) w_j] dxdt = \langle f, w_j \rangle_{Q_\tau}. \tag{21}$$

We shall now go to the limit as $j \to \infty$ in all terms of (21). In view of (20), there is no problem with passing to the limit in the first and last three terms. For the second one, observe that, as in the proof of Lemma 3, we have $w_j \to v$ in $C([0,T], L^2(\Omega))$, and since, for all $t \in [0,T]$, $u(t)$ is in $L^2(\Omega)$, we have, for every $t \in [0,T]$,

$$\int_\Omega u(t) w_j(t) dx \to \int_\Omega u(t) v(t) dx.$$

Letting $j \to \infty$ in both sides of (21)

$$-\langle \frac{\partial v}{\partial t}, u \rangle_{Q_\tau} + [\int_\Omega u(t) v(t) dx]_0^\tau + \int_{Q_\tau} [a(x,t,u,\nabla u)\nabla v + a_0(x,t,u,\nabla u) v] dxdt = \langle f, v \rangle_{Q_\tau}.$$

To prove the energy equality, it suffices to take $v = u$ in the above equality (note that this is possible since $u \in W_0^{1,x} L_\varphi(Q) \cap L^2(Q)$ and $\frac{\partial u}{\partial t} \in W^{-1,x} L_\psi(Q)$).
This gives

$$-\langle \frac{\partial u}{\partial t}, u \rangle_{Q_\tau} + [\int_\Omega u^2(t) dx]_0^\tau + \int_{Q_\tau} [a(x,t,u,\nabla u)\nabla u + a_0(x,t,u,\nabla u) u] dxdt = \langle f, u \rangle_{Q_\tau},$$

and since, as can by easily seen,

$$\langle \frac{\partial u}{\partial t}, u \rangle_{Q_\tau} = \frac{1}{2}(\int_\Omega u^2(\tau) dx - \int_\Omega u_0^2 dx),$$

we get the desired equality. This completes the proof of Theorem 2.

References

[1] M.L.Ahmed Oubeid,A. Benkirane and M. Ould Mohamedhen val; *Nonlinear elliptic equations involiving measur data in Musielak-Orlicz-Sobolev spaces*, J.A. Diff. Eq and App Volume 4, Number 1, pp. 43-57 (2013)

[2] A. Benkirane and M. Ould Mohamedhen val; *An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces*, Bull. Belg. Math. Soc. Simon Stevin Volume 20, Number 1 (2013), 57-75.

[3] A. Benkirane, J. Douieb, M. Ould Mohamedhen val; *An approximation theorem in Musielak-Orlicz-Sobolev spaces*, Comment. Math. (Prace Matem.) Vol. 51, No. 1 (2011), 109-120.

[4] A. Benkirane and M. Ould Mohamedhen val; *Some approximation properties in Musielak-Orlicz-Sobolev spaces*, Thai.J. Math. Vol. 10, No. 2 (2012), 371-381.

[5] A. Benkirane and M. Ould Mohamedhen val; *Variational inequalities in Musielak-Orlicz-Sobolev spaces*, to appear in Bull. Belg. Math. Soc. Simon Stevin.

[6] L. Boccardo and F. Murat; *Strongly nonlinear Cauchy problems with gradient dependent lower order nonlinearity*, Pitman Research Notes in Mathematics 208 (Pitman, 1989) 247-254.

[7] L. Boccardo and F. Murat; *Almost everywhere convergence of the gradients*, Nonlinear Anal. 19 (1992) 581-597.

[8] H.Brzis; *Analyse fonctionnelle,thorie et applications,3rd end*, (Masson, Paris,1992).

[9] H.Brzis and F.E.Browder; *Strongly nonlinear parabolic initial boundary value problems*, Proc.Nat.Acad.Sci.U.S.A.76(1979)38-40.

[10] T.Donaldson; *Inhomogeneous Orlicz-Sobolev spaces and nonlinrar parabolic initial boundary value problems*, J.Differential Equations 16(1974)201-256.

[11] A.Elmahi; *Strongly nonlinear parabolic initial-boundary value problems in Orlicz spaces*,Electron. J. Differential Equations.09 (2002) 203-220.

[12] A.Elmahi, D.Meskine; *Parabolic equations in orlicz spaces*,J.London Math.Soc.72 (2)(2005) 410-428
[13] J.P. Gossez, *Somme approximation properties in Orlicz Sobolev spaces*, Studia Math. 74 (1982), 17-24.
[14] R.Landes and V.Mustonen, *On the existence of weak solutions for quasilinear parabolic initial-boundary value problems* , Proc.Roy.Soc. Edinburgh Sect. A 89 (1981), 217-237.
[15] R.Landes and V.Mustonen, *A strongly nonlinear parabolic initial-boundary value problem*, Ark. Mat. 25 (1987), 29-40.
[16] J.L.Lions; *Quelques mthodes de rsolution des problmes aux limites non linaires* ;(Gauthiers-Villars,1969).
[17] J. Musielak; *Modular spaces and Orlicz spaces* ;Lecture Notes in Math. 1034 (1983).
[18] J. Robert; *Inequations variationnelles paraboliques fortement non lineaires* ;(Gauthiers-Villars,1969).

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