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ON $\alpha_{(\gamma,\gamma')}$-SEPARATION AXIOMS

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ABSTRACT. The purpose of this paper is to introduce and study new separation axioms by using the notions of α-open and α-bioperations. Also, we analyze the relations with some well known separation axioms.

1. Introduction

The study of α-open sets was initiated and explored by Njastad [8]. Maheshwari and Thakur [7] and Maki, Devi and Balachandran [6] introduced and studied a new separation axiom called α-separation axiom. Ibrahim [2] introduced and discussed an operation of a topology $\alpha O(X)$ into the power set $P(X)$ of a space X and also he introduced the concept of α_γ-open sets. Khalaf, Jafari and Ibrahim [4] introduced the notion of $\alpha O(X,\tau)_{[\gamma,\gamma']}$, which is the collection of all $\alpha_{[\gamma,\gamma']}$-open sets in a topological space (X,τ) and also they defined the $\alpha_{[\gamma,\gamma']}$-$T_i$ [5] $(i = 0, \frac{1}{2}, 1, 2)$ in topological spaces. In this paper, the author introduce and study the $\alpha_{(\gamma,\gamma')}$-$T_i$ spaces $(i = 0, \frac{1}{2}, 1, 2)$ and investigate relations among these spaces.

2. Preliminaries

Throughout this paper, (X,τ) represent nonempty topological space on which no separation axioms are assumed, unless otherwise mentioned. The closure and the interior of a subset A of X are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a topological space (X,τ) is said to be α-open [8] if $A \subseteq Int(Cl(Int(A)))$. The complement of an α-open set is said to be α-closed. The intersection of all

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α-closed sets containing A is called the α-closure of A and is denoted by $\alpha Cl(A)$. The family of all α-open (resp. α-closed) sets in a topological space (X,τ) is denoted by $\alpha O(X,\tau)$ (resp. $\alpha C(X,\tau)$). An operation $\gamma : \alpha O(X,\tau) \to P(X)$ [2] is a mapping satisfying the condition, $V \subseteq V^{\gamma}$ for each $V \in \alpha O(X,\tau)$. We call the mapping γ an operation on $\alpha O(X,\tau)$. A subset A of X is called an α_{γ}-open set [2] if for each point $x \in A$, there exists an α-open set U of X containing x such that $U^{\gamma} \subseteq A$. The complement of an α_{γ}-open set is called α_{γ}-closed. The set of all α_{γ}-open sets of X is denote by $\alpha O(X,\tau)_{\gamma}$. An operation γ on $\alpha O(X,\tau)$ is said to be α-regular [2] if for every α-open sets U and V containing $x \in X$, there exists an α-open set W containing x such that $W^{\gamma} \subseteq U^{\gamma} \cap V^{\gamma}$. An operation γ on $\alpha O(X,\tau)$ is said to be α-open [2] if for every α-open set U containing $x \in X$, there exists an α_{γ}-open set V of X such that $x \in V$ and $V \subseteq U^{\gamma}$. A subset A of X is said to be $\alpha_{[\gamma,\gamma']}$-open [4] if for each $x \in A$, there exist α-open sets U and V of X containing x such that $U^{\gamma} \cap V^{\gamma'} \subseteq A$. A subset F of (X,τ) is said to be $\alpha_{[\gamma,\gamma']}$-closed if its complement $X \setminus F$ is $\alpha_{[\gamma,\gamma']}$-open.

We recall the following definitions and results from [3].

Definition 2.1. A non-empty subset A of (X,τ) is said to be $\alpha_{(\gamma,\gamma')}$-open if for each $x \in A$, there exist α-open sets U and V of X containing x such that $U^{\gamma} \cup V^{\gamma'} \subseteq A$. A subset F of (X,τ) is said to be $\alpha_{(\gamma,\gamma')}$-closed if its complement $X \setminus F$ is $\alpha_{(\gamma,\gamma')}$-open. The set of all $\alpha_{(\gamma,\gamma')}$-open sets of (X,τ) is denoted by $\alpha O(X,\tau)_{(\gamma,\gamma')}$.

Definition 2.2. Let A be a subset of a topological space (X,τ). The intersection of all $\alpha_{(\gamma,\gamma')}$-closed sets containing A is called the $\alpha_{(\gamma,\gamma')}$-closure of A and denoted by $\alpha_{(\gamma,\gamma')}$-$Cl(A)$.

Definition 2.3. For a subset A of (X,τ), we define $\alpha Cl_{(\gamma,\gamma')}(A)$ as follows: $\alpha Cl_{(\gamma,\gamma')}(A) = \{x \in X : (U^{\gamma} \cup W^{\gamma'}) \cap A \neq \phi$ holds for every α-open sets U and W containing $x\}$.

Proposition 2.1. If A is $\alpha_{(\gamma,\gamma')}$-open, then A is α_{γ}-open for any operation γ'.

Proposition 2.2. For a point $x \in X$, $x \in \alpha_{(\gamma,\gamma')}$-$Cl(A)$ if and only if $V \cap A \neq \phi$ for every $\alpha_{(\gamma,\gamma')}$-open set V containing x.

Remark 2.1. If γ and γ' are α-regular operations, then $\alpha O(X,\tau)_{(\gamma,\gamma')}$ forms a topology on X.

Proposition 2.3. If A is $\alpha_{(\gamma,\gamma')}$-open, then A is α-open.

Definition 2.4. *A subset A of (X,τ) is said to be an* $\alpha_{(\gamma,\gamma')}$-generalized closed *(briefly,* $\alpha_{(\gamma,\gamma')}$-g.closed*) set if* $\alpha_{(\gamma,\gamma')}$-$Cl(A) \subseteq U$ *whenever $A \subseteq U$ and U is an* $\alpha_{(\gamma,\gamma')}$*-open set in (X,τ).*

Remark 2.2. Every $\alpha_{(\gamma,\gamma')}$-closed set is $\alpha_{(\gamma,\gamma')}$-g.closed.

Proposition 2.4. For each $x \in X$, $\{x\}$ is $\alpha_{(\gamma,\gamma')}$-closed or $X \setminus \{x\}$ is $\alpha_{(\gamma,\gamma')}$-g.closed in (X,τ).

Proposition 2.5. The following statements $(1), (2)$ and (3) are equivalent for a subset A of (X, τ).

(1) A is $\alpha_{(\gamma,\gamma')}$-g.closed in (X, τ).

(2) $\alpha_{(\gamma,\gamma')}$-$Cl(\{x\}) \cap A \neq \phi$ for every $x \in \alpha_{(\gamma,\gamma')}$-$Cl(A)$.

(3) $\alpha_{(\gamma,\gamma')}$-$Cl(A) \setminus A$ does not contain any non-empty $\alpha_{(\gamma,\gamma')}$-closed set.

Definition 2.5. A topological space (X, τ) is said to be:

(1) α-T_0 [6] if for any two distinct points $x, y \in X$, there exists an α-open set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

(2) α-T_1 [6] if for any two distinct points $x, y \in X$, there exist two α-open sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.

(3) α-T_2 [7]) if for any two distinct points $x, y \in X$, there exist two α-open sets U and V containing x and y, respectively, such that $U \cap V = \phi$.

(4) α-$T_{\frac{1}{2}}$ [1] if every (α,α)-g-closed set(X, τ) is α-closed.

Definition 2.6. [5] A topological space (X, τ) is said to be:

(1) $\alpha_{[\gamma,\gamma']}$-$T_0$ if for each pair of distinct points x, y in X, there exist α-open sets U and V such that $x \in U \cap V$ and $y \notin U^{\gamma} \cap V^{\gamma'}$, or $y \in U \cap V$ and $x \notin U^{\gamma} \cap V^{\gamma'}$.

(2) $\alpha_{[\gamma,\gamma']}$-$T_1$ if for each pair of distinct points x, y in X, there exist α-open sets U and V containing x and α-open sets W and S containing y such that $y \notin U^{\gamma} \cap V^{\gamma'}$ and $x \notin W^{\gamma} \cap S^{\gamma'}$.

(3) $\alpha_{[\gamma,\gamma']}$-$T_2$ if for each pair of distinct points x, y in X, there exist α-open sets U and V containing x and α-open sets W and S containing y such that $(U^{\gamma} \cap V^{\gamma'}) \cap (W^{\gamma} \cap S^{\gamma'}) = \phi$.

Proposition 2.6. [5] A topological space (X, τ) is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$ if and only if for each $x \in X$, $\{x\}$ is either $\alpha_{[\gamma,\gamma']}$-closed or $\alpha_{[\gamma,\gamma']}$-open.

3. $\alpha_{(\gamma,\gamma')}$-Separation Axioms

Throughout this section, let γ and γ' be operations on $\alpha O(X, \tau)$.

Definition 3.1. A topological space (X, τ) is said to be:

(1) $\alpha_{(\gamma,\gamma')}$-$T_{\frac{1}{2}}$ if every $\alpha_{(\gamma,\gamma')}$-g.closed set is $\alpha_{(\gamma,\gamma')}$-closed.

(2) $\alpha_{(\gamma,\gamma')}$-$T_0$ if for each pair of distinct points x, y in X, there exist α-open sets U and V such that $x \in U \cap V$ and $y \notin U^{\gamma} \cup V^{\gamma'}$, or $y \in U \cap V$ and $x \notin U^{\gamma} \cup V^{\gamma'}$.

(3) $\alpha_{(\gamma,\gamma')}$-$T_1$ if for each pair of distinct points x, y in X, there exist α-open sets U and V containing x and α-open sets W and S containing y such that $y \notin U^{\gamma} \cup V^{\gamma'}$ and $x \notin W^{\gamma} \cup S^{\gamma'}$.

(4) $\alpha_{(\gamma,\gamma')}$-$T_2$ if for each pair of distinct points x, y in X, there exist α-open sets U and V containing x and α-open sets W and S containing y such that $(U^{\gamma} \cup V^{\gamma'}) \cap (W^{\gamma} \cup S^{\gamma'}) = \phi$.

Remark 3.1. It follows from Remark 2.2 that (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_{\frac{1}{2}}$ if and only if the $\alpha_{(\gamma,\gamma')}$-g.closedness coincides with the $\alpha_{(\gamma,\gamma')}$-closedness.

Remark 3.2. For any two distinct points x and y of a space, the $\alpha_{(\gamma,\gamma')}$-$T_0$-axiom requires that there exist α-open sets U, V, W and S satisfying one of the following conditions :

(1) $x \in U \cap V$, $y \in W \cap S$, $y \notin U^\gamma \cup V^{\gamma'}$ and $x \notin W^\gamma \cup S^{\gamma'}$.

(2) $x \in U \cap V$, $x \in W \cap S$, $y \notin U^\gamma \cup V^{\gamma'}$ and $y \notin W^\gamma \cup S^{\gamma'}$.

(3) $y \in U \cap V$, $y \in W \cap S$, $x \notin U^\gamma \cup V^{\gamma'}$ and $x \notin W^\gamma \cup S^{\gamma'}$.

(4) $y \in U \cap V$, $x \in W \cap S$, $x \notin U^\gamma \cup V^{\gamma'}$ and $y \notin W^\gamma \cup S^{\gamma'}$.

Proposition 3.1. Let (X, τ) be a topological space. If X is $\alpha_{(\gamma,\gamma')}$-$T_0$, then for each distinct points x, y in X, there exists an α-open set W such that $x \in W$ and $y \notin W^\gamma \cap W^{\gamma'}$, or $y \in W$ and $x \notin W^\gamma \cap W^{\gamma'}$.

Proof. Let x, y be two distinct points. Since X is $\alpha_{(\gamma,\gamma')}$-$T_0$, then there exist two α-open sets U and V such that:

Case 1: $x \in U \cap V$ and $y \notin U^\gamma \cup V^{\gamma'}$, or

Case 2: $y \in U \cap V$ and $x \notin U^\gamma \cup V^{\gamma'}$.

For Case 1 above, we have the following possible case, say Case1-1.

Case 1-1: $x \in U \cap V, y \notin U^\gamma$ and $y \notin V^{\gamma'}$; that is, $x \in U$ and $y \notin U^\gamma$. Then, $x \in U$ and $y \notin U^\gamma \cap U^{\gamma'}$, because $U^\gamma \cap U^{\gamma'} \subseteq U^\gamma$ and $y \notin U^\gamma$ hold.

Thus, for Case 1, we can say that there exists an α-open set W such that $x \in W$ and $y \notin W^\gamma \cap W^{\gamma'}$.

For Case 2 above, we have the following possible case, say Case 2-1.

Case 2-1: $y \in U \cap V, x \notin U^\gamma$ and $x \notin V^{\gamma'}$; that is, $y \in U$ and $x \notin U^\gamma$. Then, $y \in U$ and $x \notin U^\gamma \cap U^{\gamma'}$, because $U^\gamma \cap U^{\gamma'} \subseteq U^\gamma$ and $x \notin U^\gamma$ hold.

Thus, for Case 2, we can say that there exists an α-open set W such that $y \in W$ and $x \notin W^\gamma \cap W^{\gamma'}$. $\square$

Proposition 3.2. A topological space (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_{\frac{1}{2}}$ if and only if for each $x \in X$, $\{x\}$ is either $\alpha_{(\gamma,\gamma')}$-closed or $\alpha_{(\gamma,\gamma')}$-open.

Proof. Necessity: Suppose $\{x\}$ is not $\alpha_{(\gamma,\gamma')}$-closed. Then, by Proposition 2.4, $X \setminus \{x\}$ is $\alpha_{(\gamma,\gamma')}$-g.closed. Since (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_{\frac{1}{2}}$, $X \setminus \{x\}$ is $\alpha_{(\gamma,\gamma')}$-closed, that is $\{x\}$ is $\alpha_{(\gamma,\gamma')}$-open.

Sufficiency: Let A be any $\alpha_{(\gamma,\gamma')}$-g.closed set in (X, τ) and $x \in \alpha_{(\gamma,\gamma')}$-$Cl(A)$. It suffices to prove that $x \in A$ for the following two cases:

Case 1. $\{x\}$ is $\alpha_{(\gamma,\gamma')}$-closed: for this case, by Proposition 2.5, it is shown that $\{x\} \not\subseteq \alpha_{(\gamma,\gamma')}$-$Cl(A) \setminus A$; and so $x \in A$.

Case 2. $\{x\}$ is $\alpha_{(\gamma,\gamma')}$-open: for this case, we have that $\{x\} \cap A \neq \phi$ by Proposition 2.2 and so $x \in A$. Hence, A is $\alpha_{(\gamma,\gamma')}$-closed; and so (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_{\frac{1}{2}}$. $\square$

Proposition 3.3. A topological space (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_1$ if and only if for each $x \in X$, $\{x\}$ is $\alpha_{(\gamma,\gamma')}$-closed.

Proof. Necessity: Let x be a point of X. Suppose $y \in X \setminus \{x\}$. Then, there exist α-open sets W and S containing y and $x \notin W^\gamma \cup S^{\gamma'}$. Consequently $y \in W^\gamma \cup S^{\gamma'} \subseteq X \setminus \{x\}$, that is $X \setminus \{x\}$ is $\alpha_{(\gamma,\gamma')}$-open.
Sufficiency: Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X \setminus \{x\}$ and $x \in X \setminus \{y\}$. Hence $X \setminus \{y\}$ is an $\alpha_{(\gamma,\gamma')}$-open set containing x, so there exist α-open sets U and V containing x such that $U^\gamma \cup V^{\gamma'} \subseteq X \setminus \{y\}$. Similarly $X \setminus \{x\}$ is an $\alpha_{(\gamma,\gamma')}$-open set containing y, so there exist α-open sets W and S containing y such that $W^\gamma \cup S^{\gamma'} \subseteq X \setminus \{x\}$. Accordingly X is an $\alpha_{(\gamma,\gamma')}$-$T_1$ space. □

Proposition 3.4. The following statements are equivalent for a topological space (X, τ).

(1) (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_2$.
(2) Let $x \in X$. For each $y \neq x$, there exist α-open sets U and V containing x such that $y \notin \alpha Cl_{(\gamma,\gamma')}(U^\gamma \cup V^{\gamma'})$.
(3) For each $x \in X$, $\cap\{\alpha Cl_{(\gamma,\gamma')}(U^\gamma \cup V^{\gamma'}) : U, V \in \alpha O(X, \tau) \text{ and } x \in U \cap V\} = \{x\}$.

Proof. (1) ⇒ (2): Let $x \in X$. For each $y \neq x$, it follows from (1) that there exist α-open sets U and V containing x and α-open sets W and S containing y such that $(U^\gamma \cup V^{\gamma'}) \cap (W^\gamma \cup S^{\gamma'}) = \phi$. This implies that $y \notin \alpha Cl_{(\gamma,\gamma')}(U^\gamma \cup V^{\gamma'})$.
(2) ⇒ (3): Set $B(z) = \cap\{\alpha Cl_{(\gamma,\gamma')}(U^\gamma \cup V^{\gamma'}) : U, V \in \alpha O(X, \tau) \text{ and } z \in U \cap V\}$, where $z \in X$. Let $x \in X$. We claim that $B(x) = \{x\}$. Indeed, y be any point of X with $x \neq y$. It follows from (2) that there exist α-open sets U and V such that $x \in U \cap V$ and $y \notin \alpha Cl_{(\gamma,\gamma')}(U^\gamma \cup V^{\gamma'})$. Thus, we have that $y \notin B(x)$ and so $\{x\} = B(x)$, because $\{x\} \subseteq B(x) \subseteq \alpha Cl_{(\gamma,\gamma')}(U^\gamma \cup V^{\gamma'})$ hold.
(3) ⇒ (1): Let $x, y \in X$ with $x \neq y$. By (3), it is assumed that $B(x) = \{x\}$ where $B(x)$ is defined in the proof of (2) ⇒ (3) above. Then, there exist α-open sets U and V such that $y \notin \alpha Cl_{(\gamma,\gamma')}(U^\gamma \cup V^{\gamma'})$; and hence $(U^\gamma \cup V^{\gamma'}) \cap (W^\gamma \cup S^{\gamma'}) = \phi$ for some α-open sets W and S containing y. Therefore, (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_2$. □

Proposition 3.5. Let (X, τ) be a topological space. Then:

(1) If (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_2$, then it is $\alpha_{(\gamma,\gamma')}$-$T_1$.
(2) If (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_1$, then it is $\alpha_{(\gamma,\gamma')}$-$T_{\frac{1}{2}}$.
(3) If (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_{\frac{1}{2}}$, then it is $\alpha_{(\gamma,\gamma')}$-$T_0$.

Proof. 1. The proof is follows from Definition 3.1.

2. The proof is obvious by Propositions 3.2 and 3.3.

3. Let x and y be any two distinct points of (X, τ). By Proposition 3.2, the singleton $\{x\}$ is $\alpha_{(\gamma,\gamma')}$-closed or $\alpha_{(\gamma,\gamma')}$-open.

Case 1. $\{x\}$ is $\alpha_{(\gamma,\gamma')}$-closed: for this case, $X \setminus \{x\}$ is an $\alpha_{(\gamma,\gamma')}$-open set containing y; and so there exist α-open sets W and S containing y such that $W^\gamma \cup S^{\gamma'} \subseteq X \setminus \{x\}$. Thus we have that $y \in W \cap S$ and $x \notin W^\gamma \cup S^{\gamma'}$.

Case 2. $\{x\}$ is $\alpha_{(\gamma,\gamma')}$-open: for this case, there exist α-open sets U and V containing x such that $U^\gamma \cup V^{\gamma'} \subseteq \{x\}$. This implies that $x \in U \cap V$ and $y \notin U^\gamma \cup V^{\gamma'}$.

Therefore, we have X is $\alpha_{(\gamma,\gamma')}$-$T_0$. □

Remark 3.3. The following examples show that all converses of Proposition 3.5 can not be reserved.

Example 3.1. Let $X = \{1, 2, 3\}$ and τ be a discrete topology on X. For each $A \in \alpha O(X)$, we define two operations γ and γ', respectively, by

$$A^\gamma = A^{\gamma'} = \begin{cases} A & \text{if } A = \{1,2\} \text{ or } \{1,3\} \text{ or } \{2,3\}, \\ X & \text{otherwise,} \end{cases}$$

Then, it is shown directly that each singleton is $\alpha_{(\gamma,\gamma')}$-closed in (X, τ). By Proposition 3.3, (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_1$. But, we can show that $(U^\gamma \cup V^{\gamma'}) \cap (W^\gamma \cup S^{\gamma'}) \neq \phi$ holds for any α-open sets U, V, W and S. This implies (X, τ) is not $\alpha_{(\gamma,\gamma')}$-$T_2$.

Example 3.2. Let $X = \{1, 2, 3\}$ and $\tau = \{\phi, X, \{1\}, \{1, 2\}, \{1, 3\}\}$ be a topology on X. For each $A \in \alpha O(X, \tau)$ we define two operations γ and γ', respectively, by $A^\gamma = A^{\gamma'} = A$. Then, it is shown directly that each singleton is $\alpha_{(\gamma,\gamma')}$-closed or $\alpha_{(\gamma,\gamma')}$-open in (X, τ). By Proposition 3.2, (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_{\frac{1}{2}}$. However, by Proposition 3.3, (X, τ) is not $\alpha_{(\gamma,\gamma')}$-$T_1$, in fact, a singleton $\{1\}$ is not $\alpha_{(\gamma,\gamma')}$-closed.

Example 3.3. Let $X = \{1, 2, 3\}$ and $\tau = \{\phi, X, \{1\}, \{1, 2\}\}$ be a topology on X. For each $A \in \alpha O(X)$ we define two operations γ and γ', respectively, by

$$A^\gamma = A^{\gamma'} = \begin{cases} A & \text{if } 2 \notin A, \\ X & \text{if } 2 \in A. \end{cases}$$

Then, (X, τ) is not $\alpha_{(\gamma,\gamma')}$-$T_{\frac{1}{2}}$ because a singleton $\{3\}$ is neither $\alpha_{(\gamma,\gamma')}$-open nor $\alpha_{(\gamma,\gamma')}$-closed. It is shown directly that (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_0$.

Remark 3.4. From Proposition 3.5 and Examples 3.1, 3.2 and 3.3, the following implications hold and none of the implications is reversible:

$$\alpha_{(\gamma,\gamma')}\text{-}T_2 \longrightarrow \alpha_{(\gamma,\gamma')}\text{-}T_1 \longrightarrow \alpha_{(\gamma,\gamma')}\text{-}T_{\frac{1}{2}} \longrightarrow \alpha_{(\gamma,\gamma')}\text{-}T_0,$$

where $A \to B$ represents that A implies B.

Proposition 3.6. If (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_i$, then it is α-T_i, where $i = 0, \frac{1}{2}, 1, 2$.

Proof. The proofs for $i = 0, 2$ follow from their definitions. The proof for $i = 1$ (resp. $i = \frac{1}{2}$) follows from Propositions 2.3 and 3.3 (resp. Proposition 3.2). $\square$

Remark 3.5. The following example shows that all converses of Proposition 3.6 can not be reserved.

Example 3.4. Let $X = \{1, 2, 3\}$ and τ be a discrete topology on X. For each $A \in \alpha O(X, \tau)$ we define two operations γ and γ', respectively, by $A^{\gamma} = A^{\gamma'} = X$. Then, (X, τ) is α-T_i but it is not $\alpha_{(\gamma,\gamma')}$-$T_i$, where $i = 0, \frac{1}{2}, 1, 2$.

Proposition 3.7. If (X, τ) is $\alpha_{(\gamma,\gamma')}$-$T_i$, then it is $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i = 0, \frac{1}{2}, 1, 2$.

Proof. The proofs for $i = 0, 1, 2$ follow from Definitions 3.1 and 2.6. The proof for $i = \frac{1}{2}$ follow from Propositions 3.2 and 2.6. $\square$

Remark 3.6. The following examples show that all converses of Proposition 3.7 can not be reserved.

Example 3.5. Let $X = \{1, 2, 3\}$ and τ be a discrete topology on X.

(1) For each $A \in \alpha O(X)$ we define two operations γ and γ', respectively, by

$$A^{\gamma} = A^{\gamma'} = \begin{cases} A & \text{if } A = \{1, 2\} \text{ or } \{1, 3\} \text{ or } \{2, 3\}, \\ X & \text{otherwise.} \end{cases}$$

Then, (X, τ) is $\alpha_{[\gamma,\gamma']}$-$T_2$ but not $\alpha_{(\gamma,\gamma')}$-$T_2$.

(2) For each $A \in \alpha O(X)$ we define two operations γ and γ', respectively, by

$$A^{\gamma} = \begin{cases} A & \text{if } A = \{1, 2\} \text{ or } \{1, 3\}, \\ X & \text{otherwise,} \end{cases}$$

and

$$A^{\gamma'} = \begin{cases} A & \text{if } A = \{2, 3\}, \\ X & \text{if } A \neq \{2, 3\}. \end{cases}$$

Then, (X, τ) is $\alpha_{[\gamma,\gamma']}$-$T_i$ but not $\alpha_{(\gamma,\gamma')}$-$T_i$, where $i = \frac{1}{2}, 1$.

(3) For each $A \in \alpha O(X)$ we define two operations γ and γ', respectively, by

$$A^{\gamma} = \begin{cases} A & \text{if } A = \{1\}, \\ X & \text{if } A \neq \{1\}, \end{cases}$$

and

$$A^{\gamma'} = \begin{cases} A & \text{if } A = \{2\}, \\ X & \text{if } A \neq \{2\}. \end{cases}$$

Then, (X, τ) is $\alpha_{[\gamma,\gamma']}$-$T_0$ but not $\alpha_{(\gamma,\gamma')}$-$T_0$.

Remark 3.7. From Propositions 3.5, 3.6 and 3.7, for distinct operations γ and $\gamma^{'}$ we have the following diagram. We note that implications in the following diagram are not reversible by Remarks 3.3, 3.5 and 3.6:

$$\begin{tikzcd}
\alpha_{[\gamma,\gamma']}\text{-}T_2 \arrow[r] & \alpha_{[\gamma,\gamma']}\text{-}T_1 \arrow[r] & \alpha_{[\gamma,\gamma']}\text{-}T_{\frac{1}{2}} \arrow[r] & \alpha_{[\gamma,\gamma']}\text{-}T_0 \\
\alpha_{(\gamma,\gamma')}\text{-}T_2 \arrow[r] \arrow[u] \arrow[d] & \alpha_{(\gamma,\gamma')}\text{-}T_1 \arrow[r] \arrow[u] \arrow[d] & \alpha_{(\gamma,\gamma')}\text{-}T_{\frac{1}{2}} \arrow[r] \arrow[u] \arrow[d] & \alpha_{(\gamma,\gamma')}\text{-}T_0 \arrow[u] \arrow[d] \\
\alpha\text{-}T_2 \arrow[r] & \alpha\text{-}T_1 \arrow[r] & \alpha\text{-}T_{\frac{1}{2}} \arrow[r] & \alpha\text{-}T_0
\end{tikzcd}$$

where $A \to B$ represents that A implies B.

Proposition 3.8. Suppose that γ and $\gamma^{'}$ are α-regular operations on $\alpha O(X,\tau)$. A space (X,τ) is $\alpha_{(\gamma,\gamma')}$-$T_i$ if and only if an associated space $(X, \alpha O(X,\tau)_{(\gamma,\gamma')})$ is T_i, where $i = 1, 1/2$.

Proof. It follows from Remark 2.1 that a subset A is $\alpha_{(\gamma,\gamma')}$-open in (X,τ) if and only if A is open in $(X, \alpha O(X,\tau)_{(\gamma,\gamma')})$. Therefore, the proof for $i = \frac{1}{2}$ (resp. $i = 1$) follows from Propositions 3.2 (resp. Proposition 3.3). □

Proposition 3.9. Let γ and $\gamma^{'}$ be α-regular operations on $\alpha O(X,\tau)$. If $(X, \alpha O(X,\tau)_{(\gamma,\gamma')})$ is T_i, then (X,τ) is $\alpha_{(\gamma,\gamma')}$-$T_i$, where $i = 0, 2$.

Proof. The proof for $i = 0$ (resp. $i = 2$) follows from the T_0-separation property (resp. Hausdorffness) of $(X, \alpha O(X,\tau)_{(\gamma,\gamma')})$, the concept of $\alpha_{(\gamma,\gamma')}$-open sets Definitions 2.1 and 3.1 (2) (resp. Definition 3.1 (4)).

□

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