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A NEW TYPE OF CONNECTED SETS VIA BIOPERATIONS

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Abstract. The purpose of this paper is to introduce the notion of $\alpha_{(\gamma,\gamma')}$-separated sets and study their properties in topological spaces, then we introduce the notions of $\alpha_{(\gamma,\gamma')}$-connected and $\alpha_{(\gamma,\gamma')}$-disconnected sets. We discuss the characterizations and properties of $\alpha_{(\gamma,\gamma')}$-connected sets and then properties under $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous functions. The $\alpha_{(\gamma,\gamma')}$-components in a space X is also introduced.

1. Introduction

Njastad [5] introduced α-open sets in a topological space and studied some of their properties. Ibrahim [1] introduced and discussed an operation of a topology $\alpha O(X)$ into the power set $P(X)$ of a space X and also in [2] he introduced the notion of $\alpha O(X,\tau)_{(\gamma,\gamma')}$, which is the collection of all $\alpha_{(\gamma,\gamma')}$-open sets in a topological space (X,τ). In addition, Ibrahim [3] introduced the concept of $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-closed and $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous functions and investigated some of their basic properties. Mishra [4] introduced α-τ-disconnectedness and α-τ-connectedness in topological spaces. In this paper, the author introduce and study the characterizations and properties of $\alpha_{(\gamma,\gamma')}$-connected and $\alpha_{(\gamma,\gamma')}$-disconnected spaces and then properties under $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous functions.

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2. Preliminaries

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) denotes a topological spaces on which no separation axioms is assumed unless explicitly stated. For a subset A of a space X, $Cl(A)$ and $Int(A)$ represent the closure of A and the interior of A, respectively.

Definition 2.1. [5] A subset A of a topological space (X, τ) is said to be α-open if $A \subseteq Int(Cl(Int(A)))$. The complement of an α-open set is said to be α-closed. The family of all α-open (resp., α-closed) sets in a topological space (X, τ) is denoted by $\alpha O(X, \tau)$ (resp., $\alpha C(X, \tau)$).
The intersection of all α-closed sets containing A is called the α-closure of A and is denoted by $\alpha Cl(A)$.

Definition 2.2. [4] The subsets A and B of a topological space (X, τ) are called α-τ-separated sets if $(\alpha Cl(A) \cap B) \cup (A \cap \alpha Cl(B)) = \phi$.

Definition 2.3. *[1] An operation $\gamma : \alpha O(X, \tau) \to P(X)$ is a mapping satisfying the condition, $V \subseteq V^\gamma$ for each $V \in \alpha O(X, \tau)$. We call the mapping γ an operation on $\alpha O(X, \tau)$. The operation $id : \alpha O(X, \tau) \to P(X)$ is defined by $V^{id} = V$ for any set $V \in \alpha O(X, \tau)$. This operation is called the identity operation on $\alpha O(X, \tau)$.*

Definition 2.4. [2] A nonempty subset A of (X, τ) is said to be $\alpha_{(\gamma,\gamma')}$-open if for each $x \in A$, there exist α-open sets U and V of X containing x such that $U^\gamma \cup V^{\gamma'} \subseteq A$. A subset F of (X, τ) is said to be $\alpha_{(\gamma,\gamma')}$-closed if its complement $X \setminus F$ is $\alpha_{(\gamma,\gamma')}$-open. The set of all $\alpha_{(\gamma,\gamma')}$-open sets of (X, τ) is denoted by $\alpha O(X, \tau)_{(\gamma,\gamma')}$.

Definition 2.5. [2] Let A be a subset of a topological space (X, τ).

(1) The union of all $\alpha_{(\gamma,\gamma')}$-open sets contained in A is called the $\alpha_{(\gamma,\gamma')}$-interior of A and is denoted by $\alpha_{(\gamma,\gamma')}$-$Int(A)$.
(2) The intersection of all $\alpha_{(\gamma,\gamma')}$-closed sets containing A is called the $\alpha_{(\gamma,\gamma')}$-closure of A and denoted by $\alpha_{(\gamma,\gamma')}$-$Cl(A)$.

Proposition 2.1. [2] Let A and B be subsets of (X, τ). Then the following hold:

(1) $A \subseteq \alpha_{(\gamma,\gamma')}$-$Cl(A)$.
(2) If $A \subseteq B$, then $\alpha_{(\gamma,\gamma')}$-$Cl(A) \subseteq \alpha_{(\gamma,\gamma')}$-$Cl(B)$.
(3) A is $\alpha_{(\gamma,\gamma')}$-closed if and only if $\alpha_{(\gamma,\gamma')}$-$Cl(A) = A$.
(4) $\alpha_{(\gamma,\gamma')}$-$Cl(A)$ is $\alpha_{(\gamma,\gamma')}$-closed.

Proposition 2.2. [2] For a point $x \in X$, $x \in \alpha_{(\gamma,\gamma')}$-$Cl(A)$ if and only if $V \cap A \neq \phi$ for every $\alpha_{(\gamma,\gamma')}$-open set V containing x.

Definition 2.6. [3] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-closed if for $\alpha_{(\gamma,\gamma')}$-closed set A of X, $f(A)$ is $\alpha_{(\beta,\beta')}$-closed in Y.

Proposition 2.3. [3] Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then, f is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-closed if and only if $\alpha_{(\beta,\beta')}$-$Cl(f(A)) \subseteq f(\alpha_{(\gamma,\gamma')}$-$Cl(A))$ for every subset A of X.

Theorem 2.1. [3] Suppose that $f : (X, \tau) \to (Y, \sigma)$ is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous. Then,

(1) $f^{-1}(V)$ is $\alpha_{(\gamma,\gamma')}$-open for every $\alpha_{(\beta,\beta')}$-open set V of (Y, σ).

(2) For each point $x \in X$ and each $\alpha_{(\beta,\beta')}$-open W of (Y, σ) containing $f(x)$, there exist $\alpha_{(\gamma,\gamma')}$-open U of (X, τ) containing x such that $f(U) \subseteq W$.

3. $\alpha_{(\gamma,\gamma')}$-Connected and $\alpha_{(\gamma,\gamma')}$-Disconnected Sets

Throughout this section, let $\gamma, \gamma' : \alpha O(X, \tau) \to P(X)$ be operations on $\alpha O(X, \tau)$ and $\beta, \beta' : \alpha O(Y, \sigma) \to P(Y)$ be operations on $\alpha O(Y, \sigma)$.

Definition 3.1. Two subsets A and B of a topological space (X, τ) are called $\alpha_{(\gamma,\gamma')}$-separated if $(\alpha_{(\gamma,\gamma')}$-$Cl(A) \cap B) \cup (A \cap \alpha_{(\gamma,\gamma')}$-$Cl(B)) = \phi$.

Remark 3.1. *Each two $\alpha_{(\gamma,\gamma')}$-separated sets are always disjoint, since $A \cap B \subseteq A \cap \alpha_{(\gamma,\gamma')}$-$Cl(B) = \phi$. The converse may not be true in general, as it is shown in the following example.*

Example 3.1. Let $X = \{1, 2, 3\}$ and $\tau = \{\phi, X, \{2\}\}$. For each $A \in \alpha O(X)$, we define two operations γ and γ', respectively, by

$$A^{\gamma} = A^{\gamma'} = \begin{cases} A & \text{if } 3 \in A \\ X & \text{if } 3 \notin A. \end{cases}$$

Since $\alpha O(X, \tau)_{(\gamma,\gamma')} = \{\phi, X, \{2, 3\}\}$, then $\{2\}$ and $\{3\}$ are disjoint subsets of X, but not $\alpha_{(\gamma,\gamma')}$-separated.

From the fact that $\alpha Cl(A) \subseteq \alpha_{(\gamma,\gamma')}$-$Cl(A)$, for every subset A of X. Then every $\alpha_{(\gamma,\gamma')}$-separated set is α-τ-separated. But the converse may not be true as shown in the following example.

Example 3.2. Let $X = \{1, 2, 3, 4\}$ and $\tau = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}$. For each $A \in \alpha O(X)$, we define two operations γ and γ', respectively, by

$$A^{\gamma} = A^{\gamma'} = \begin{cases} A & \text{if } 4 \in A \\ X & \text{if } 4 \notin A. \end{cases}$$

Since $\alpha O(X, \tau)_{(\gamma,\gamma')} = \{\phi, X, \{1, 2, 4\}\}$, then the subsets $\{3\}$ and $\{4\}$ are α-τ-separated, but not $\alpha_{(\gamma,\gamma')}$-separated.

Theorem 3.1. If A and B are any two nonempty subsets in a space X, then the following statements are true:

(1) If A and B are $\alpha_{(\gamma,\gamma')}$-separated, $A_1 \subseteq A$ and $B_1 \subseteq B$, then A_1 and B_1 are also $\alpha_{(\gamma,\gamma')}$-separated.

(2) If $A \cap B = \phi$ such that each of A and B are both $\alpha_{(\gamma,\gamma')}$-closed ($\alpha_{(\gamma,\gamma')}$-open), then A and B are $\alpha_{(\gamma,\gamma')}$-separated.

(3) If each of A and B is $\alpha_{(\gamma,\gamma')}$-closed ($\alpha_{(\gamma,\gamma')}$-open) and if $H = A \cap (X \setminus B)$ and $G = B \cap (X \setminus A)$, then H and G are $\alpha_{(\gamma,\gamma')}$-separated.

Proof. (1) Since $A_1 \subseteq A$, then $\alpha_{(\gamma,\gamma')}$-$Cl(A_1) \subseteq \alpha_{(\gamma,\gamma')}$-$Cl(A)$. Then, $B \cap \alpha_{(\gamma,\gamma')}$-$Cl(A) = \phi$ implies $B_1 \cap \alpha_{(\gamma,\gamma')}$-$Cl(A) = \phi$ and $B_1 \cap \alpha_{(\gamma,\gamma')}$-$Cl(A_1) = \phi$. Similarly $A_1 \cap \alpha_{(\gamma,\gamma')}$-$Cl(B_1) = \phi$. Hence, A_1 and B_1 are $\alpha_{(\gamma,\gamma')}$-separated.

(2) Since $A = \alpha_{(\gamma,\gamma')}$-$Cl(A)$, $B = \alpha_{(\gamma,\gamma')}$-$Cl(B)$ and $A \cap B = \phi$, then $\alpha_{(\gamma,\gamma')}$-$Cl(A) \cap B = \phi$ and $\alpha_{(\gamma,\gamma')}$-$Cl(B) \cap A = \phi$. Hence, A and B are $\alpha_{(\gamma,\gamma')}$-separated. If A and B are $\alpha_{(\gamma,\gamma')}$-open, then their complements are $\alpha_{(\gamma,\gamma')}$-closed. Hence, $\alpha_{(\gamma,\gamma')}$-$Cl(A) \subseteq X \setminus B$ and $\alpha_{(\gamma,\gamma')}$-$Cl(B) \subseteq X \setminus A$. Therefore, A and B are $\alpha_{(\gamma,\gamma')}$-separated.

(3) If A and B are $\alpha_{(\gamma,\gamma')}$-open, then $X \setminus A$ and $X \setminus B$ are $\alpha_{(\gamma,\gamma')}$-closed. Since $H \subseteq X \setminus B$, $\alpha_{(\gamma,\gamma')}$-$Cl(H) \subseteq \alpha_{(\gamma,\gamma')}$-$Cl(X \setminus B) = X \setminus B$ and so $\alpha_{(\gamma,\gamma')}$-$Cl(H) \cap B = \phi$. Thus $G \cap \alpha_{(\gamma,\gamma')}$-$Cl(H) = \phi$. Similarly, $H \cap \alpha_{(\gamma,\gamma')}$-$Cl(G) = \phi$. Hence H and G are $\alpha_{(\gamma,\gamma')}$-separated. If A and B are $\alpha_{(\gamma,\gamma')}$-closed, then $\alpha_{(\gamma,\gamma')}$-$Cl(H) \subseteq A$ and $\alpha_{(\gamma,\gamma')}$-$Cl(G) \subseteq B$. Thus, H and G are $\alpha_{(\gamma,\gamma')}$-separated.

□

Theorem 3.2. The sets A and B of a space X are $\alpha_{(\gamma,\gamma')}$-separated if and only if there exist U and V in $\alpha O(X,\tau)_{(\gamma,\gamma')}$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \phi$ and $B \cap U = \phi$.

Proof. Let A and B be $\alpha_{(\gamma,\gamma')}$-separated sets. Set $V = X \setminus \alpha_{(\gamma,\gamma')}$-$Cl(A)$ and $U = X \setminus \alpha_{(\gamma,\gamma')}$-$Cl(B)$. Then $U, V \in \alpha O(X,\tau)_{(\gamma,\gamma')}$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \phi$, $B \cap U = \phi$. On the other hand, let $U, V \in \alpha O(X,\tau)_{(\gamma,\gamma')}$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \phi$, $B \cap U = \phi$. Since $X \setminus V$ and $X \setminus U$ are $\alpha_{(\gamma,\gamma')}$-closed, then $\alpha_{(\gamma,\gamma')}$-$Cl(A) \subseteq X \setminus V \subseteq X \setminus B$ and $\alpha_{(\gamma,\gamma')}$-$Cl(B) \subseteq X \setminus U \subseteq X \setminus A$. Thus $\alpha_{(\gamma,\gamma')}$-$Cl(A) \cap B = \phi$ and $\alpha_{(\gamma,\gamma')}$-$Cl(B) \cap A = \phi$. □

Theorem 3.3. In any topological space (X,τ), the following statements are equivalent:

(1) ϕ and X are the only $\alpha_{(\gamma,\gamma')}$-open and $\alpha_{(\gamma,\gamma')}$-closed sets in X.
(2) X is not the union of two disjoint nonempty $\alpha_{(\gamma,\gamma')}$-open sets.
(3) X is not the union of two disjoint nonempty $\alpha_{(\gamma,\gamma')}$-closed sets.
(4) X is not the union of two nonempty $\alpha_{(\gamma,\gamma')}$-separated sets.

Proof. (1) ⇒ (2): Suppose (2) is false and that $X = A \cup B$, where A, B are disjoint nonempty $\alpha_{(\gamma,\gamma')}$-open sets. Since $X \setminus A = B$ is $\alpha_{(\gamma,\gamma')}$-open and nonempty, we have that A is a nonempty proper $\alpha_{(\gamma,\gamma')}$-open and $\alpha_{(\gamma,\gamma')}$-closed set in X, which shows that (1) is false.

(2) ⇔ (3): This is clear.

(3) $\Rightarrow$ (4): If (4) is false, then $X = A \cup B$, where A, B are nonempty and $\alpha_{(\gamma,\gamma')}$-separated. Since $\alpha_{(\gamma,\gamma')}$-$Cl(B) \cap A = \phi$, we conclude that $\alpha_{(\gamma,\gamma')}$-$Cl(B) \subseteq B$, so B is $\alpha_{(\gamma,\gamma')}$-closed. Similarly, A must be $\alpha_{(\gamma,\gamma')}$-closed. Therefore, (3) is false.

(4) $\Rightarrow$ (1): Suppose (1) is false and that A is a nonempty proper $\alpha_{(\gamma,\gamma')}$-open and $\alpha_{(\gamma,\gamma')}$-closed subset of X. Then, $B = X \setminus A$ is nonempty, $\alpha_{(\gamma,\gamma')}$-open and $\alpha_{(\gamma,\gamma')}$-closed, so A and B are $\alpha_{(\gamma,\gamma')}$-separated and $X = A \cup B$, so (4) is false. □

Definition 3.2. A subset C of a space X is said to be $\alpha_{(\gamma,\gamma')}$-disconnected if there are nonempty $\alpha_{(\gamma,\gamma')}$-separated subsets A and B of X such that $C = A \cup B$, otherwise C is called $\alpha_{(\gamma,\gamma')}$-connected. If C is $\alpha_{(\gamma,\gamma')}$-disconnected, such a pair of sets A, B will be called an $\alpha_{(\gamma,\gamma')}$-disconnection of C.

Example 3.3. Let $X = \{1, 2, 3\}$ and $\tau = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}$. For each $A \in \alpha O(X)$, we define two operations γ and γ', respectively, by

$$A^{\gamma} = A^{\gamma'} = \begin{cases} A & \text{if } 3 \in A \\ Cl(A) & \text{if } 3 \notin A. \end{cases}$$

Then, X is $\alpha_{(\gamma,\gamma')}$-disconnected because there exist a pair $\{1\}, \{2, 3\}$ subsets of X such that $\{1\} \cup \{2, 3\} = X$, and $(\alpha_{(\gamma,\gamma')}\text{-}Cl(\{1\}) \cap \{2, 3\}) \cup (\{1\} \cap \alpha_{(\gamma,\gamma')}\text{-}Cl(\{2, 3\})) = (\{1\} \cap \{2, 3\}) \cup (\{1\} \cap \{2, 3\}) = \phi$.

Example 3.4. Let $X = \{1, 2, 3\}$ and $\tau = \{\phi, X, \{1\}, \{3\}, \{1, 3\}\}$. For each $A \in \alpha O(X)$, we define two operations γ and γ', respectively, by

$$A^{\gamma} = A^{\gamma'} = \begin{cases} A & \text{if } 2 \in A \\ X & \text{if } 2 \notin A. \end{cases}$$

Then, X is $\alpha_{(\gamma,\gamma')}$-connected, since there does not exist a pair A, B of nonempty $\alpha_{(\gamma,\gamma')}$-separated subsets of X such that $X = A \cup B$.

Remark 3.2. Every indiscrete space is $\alpha_{(\gamma,\gamma')}$-connected.

Remark 3.3. Every discrete space contains more than one element is $\alpha_{(id,id')}$-disconnected.

Remark 3.4. A space X is $\alpha_{(\gamma,\gamma')}$-connected if any (therefore all) of the conditions $(1) - (4)$ in Theorem 3.3 hold.

Remark 3.5. According to the Definition 3.2 and Remark 3.4, a space X is $\alpha_{(\gamma,\gamma')}$-disconnected if we can write $X = A \cup B$, where the following (equivalent) statements are true:

(1) A and B are disjoint, nonempty and $\alpha_{(\gamma,\gamma')}$-open.
(2) A and B are disjoint, nonempty and $\alpha_{(\gamma,\gamma')}$-closed.
(3) A and B are nonempty and $\alpha_{(\gamma,\gamma')}$-separated.

Theorem 3.4. A space X is $\alpha_{(\gamma,\gamma')}$-disconnected if and only if there exists a nonempty proper subset A of X which is both $\alpha_{(\gamma,\gamma')}$-open and $\alpha_{(\gamma,\gamma')}$-closed in X.

Proof. Follows from Remark 3.5. □

Definition 3.3. Let A be a subset of a space X, then the $\alpha_{(\gamma,\gamma')}$-boundary of A is defined as $\alpha_{(\gamma,\gamma')}$-$Cl(A) \setminus \alpha_{(\gamma,\gamma')}$-$Int(A)$ and is denoted by $\alpha_{(\gamma,\gamma')}$-$Bd(A)$.

Proposition 3.1. Let A be any subset of a topological space (X, τ). Then, the following statements are hold:

(1) $\alpha_{(\gamma,\gamma')}$-$Cl(A) = \alpha_{(\gamma,\gamma')}$-$Int(A) \cup \alpha_{(\gamma,\gamma')}$-$Bd(A)$.
(2) $\alpha_{(\gamma,\gamma')}$-$Bd(A) = \alpha_{(\gamma,\gamma')}$-$Cl(A) \cap \alpha_{(\gamma,\gamma')}$-$Cl(X \setminus A)$.

Proof. Obvious. □

Theorem 3.5. A space X is $\alpha_{(\gamma,\gamma')}$-connected if and only if every nonempty proper subset of X has a nonempty $\alpha_{(\gamma,\gamma')}$-boundary.

Proof. Suppose that a nonempty proper subset A of an $\alpha_{(\gamma,\gamma')}$-connected space X has empty $\alpha_{(\gamma,\gamma')}$-boundary. Since $\alpha_{(\gamma,\gamma')}$-$Cl(A) = \alpha_{(\gamma,\gamma')}$-$Int(A) \cup \alpha_{(\gamma,\gamma')}$-$Bd(A)$. Thus, A is both $\alpha_{(\gamma,\gamma')}$-closed and $\alpha_{(\gamma,\gamma')}$-open. By Theorem 3.4, X is $\alpha_{(\gamma,\gamma')}$-disconnected. This contradiction, hence proves that A has a nonempty $\alpha_{(\gamma,\gamma')}$-boundary.
Conversely, suppose X is $\alpha_{(\gamma,\gamma')}$-disconnected. Then by Theorem 3.4, X has a nonempty proper subset A which is both $\alpha_{(\gamma,\gamma')}$-closed and $\alpha_{(\gamma,\gamma')}$-open. Then, $\alpha_{(\gamma,\gamma')}$-$Cl(A) = A$, $\alpha_{(\gamma,\gamma')}$-$Cl(X \setminus A) = X \setminus A$ and $\alpha_{(\gamma,\gamma')}$-$Cl(A) \cap \alpha_{(\gamma,\gamma')}$-$Cl(X \setminus A) = \phi$. So A has empty $\alpha_{(\gamma,\gamma')}$-boundary, this is a contradiction. Hence, X is $\alpha_{(\gamma,\gamma')}$-connected. □

Lemma 3.1. Suppose M, N are $\alpha_{(\gamma,\gamma')}$-separated subsets of X. If $C \subseteq M \cup N$ and C is $\alpha_{(\gamma,\gamma')}$-connected, then $C \subseteq M$ or $C \subseteq N$.

Proof. Since $C \cap M \subseteq M$ and $C \cap N \subseteq N$, then $C \cap M$ and $C \cap N$ are $\alpha_{(\gamma,\gamma')}$-separated and $C = C \cap (M \cup N) = (C \cap M) \cup (C \cap N)$. But C is $\alpha_{(\gamma,\gamma')}$-connected so $(C \cap M)$ and $(C \cap N)$ can not form an $\alpha_{(\gamma,\gamma')}$-disconnection of C. Therefore, either $C \cap M = \phi$, so $C \subseteq N$ or $C \cap N = \phi$, so $C \subseteq M$. □

Theorem 3.6. Suppose C and C_i $(i \in I)$ are $\alpha_{(\gamma,\gamma')}$-connected subsets of X and that for each i, C_i and C are not $\alpha_{(\gamma,\gamma')}$-separated. Then, $S = C \cup C_i$ is $\alpha_{(\gamma,\gamma')}$-connected.

Proof. Suppose that $S = M \cup N$, where M and N are $\alpha_{(\gamma,\gamma')}$-separated. By Lemma 3.1, either $C \subseteq M$ or $C \subseteq N$. Without loss of generality, assume $C \subseteq M$. By the same reasoning we conclude that for each i, either $C_i \subseteq M$ or $C_i \subseteq N$. But if some $C_i \subseteq N$, then C and C_i would be $\alpha_{(\gamma,\gamma')}$-separated. Hence every $C_i \subseteq M$. Therefore, $N = \phi$ and the pair M, N is not an $\alpha_{(\gamma,\gamma')}$-disconnection of S. □

Corollary 3.1. Suppose that for each $i \in I$, C_i is an $\alpha_{(\gamma,\gamma')}$-connected subset of X and that for all $i \neq j$, $C_i \cap C_j \neq \phi$. Then, $\cup\{C_i : i \in I\}$ is $\alpha_{(\gamma,\gamma')}$-connected.

Proof. If $I = \phi$, then $\cup\{C_i : i \in I\} = \phi$ is $\alpha_{(\gamma,\gamma')}$-connected. If $I \neq \phi$, pick $i_0 \in I$ and let C_{i_0} be the central set C in Theorem 3.6. For all $i \in I$, $C_i \cap C_{i_0} \neq \phi$, so C_i and C_{i_0} are not $\alpha_{(\gamma,\gamma')}$-separated. By Theorem 3.6, $\cup\{C_i : i \in I\}$ is $\alpha_{(\gamma,\gamma')}$-connected. □

Corollary 3.2. Suppose that for all $x, y \in X$, there exists an $\alpha_{(\gamma,\gamma')}$-connected set $C_{xy} \subseteq X$ with $x, y \in C_{xy}$. Then, X is $\alpha_{(\gamma,\gamma')}$-connected.

Proof. Certainly $X = \phi$ is $\alpha_{(\gamma,\gamma')}$-connected. If $X \neq \phi$, choose $a \in X$. By hypothesis there is, for each $y \in X$, an $\alpha_{(\gamma,\gamma')}$-connected set C_{ay} containing both a and y. By Corollary 3.1, $X = \cup\{C_{ay} : y \in X\}$ is $\alpha_{(\gamma,\gamma')}$-connected. □

Corollary 3.3. Suppose C is an $\alpha_{(\gamma,\gamma')}$-connected subset of X and $A \subseteq X$. If $C \subseteq A \subseteq \alpha_{(\gamma,\gamma')}\text{-}Cl(C)$, then A is $\alpha_{(\gamma,\gamma')}$-connected.

Proof. For each $a \in A$, $\{a\}$ and C are not $\alpha_{(\gamma,\gamma')}$-separated. By Theorem 3.6, $A = C \cup \bigcup\{\{a\} : a \in A\}$ is $\alpha_{(\gamma,\gamma')}$-connected. □

Remark 3.6. In particular, the $\alpha_{(\gamma,\gamma')}$-closure of an $\alpha_{(\gamma,\gamma')}$-connected set is $\alpha_{(\gamma,\gamma')}$-connected.

Theorem 3.7. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Consider the following statements.

(1) f is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous.
(2) $f^{-1}(V) \subseteq \alpha_{(\gamma,\gamma')}\text{-}Int(f^{-1}(V))$ for every $\alpha_{(\beta,\beta')}$-open set V of Y.
(3) $f(\alpha_{(\gamma,\gamma')}\text{-}Cl(A)) \subseteq \alpha_{(\beta,\beta')}\text{-}Cl(f(A))$ for every subset A of X.
(4) $\alpha_{(\gamma,\gamma')}\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(\alpha_{(\beta,\beta')}\text{-}Cl(B))$ for every subset B of Y.

Then, the following implications are true: (1) ⇒ (2) ⇒ (3) ⇒ (4).

Proof. (1) ⇒ (2). Let V be any $\alpha_{(\beta,\beta')}$-open set of Y and $x \in f^{-1}(V)$. Then, $f(x) \in V$. Since f is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous, there exists an $\alpha_{(\gamma,\gamma')}$-open set U of X containing x such that $f(U) \subseteq V$ and hence $U \subseteq f^{-1}(V)$, this implies that $x \in \alpha_{(\gamma,\gamma')}\text{-}Int(f^{-1}(V))$. Thus, it follows that $f^{-1}(V) \subseteq \alpha_{(\gamma,\gamma')}\text{-}Int(f^{-1}(V))$.

(2) ⇒ (3). Let A be any subset of X and $f(x) \notin \alpha_{(\beta,\beta')}\text{-}Cl(f(A))$. Then, by Proposition 2.2, there exists an $\alpha_{(\beta,\beta')}$-open set V of Y containing $f(x)$ such that $V \cap f(A) = \phi$ and hence $f^{-1}(V) \cap A = \phi$. Also $f(x) \in V$ implies $x \in f^{-1}(V)$. Then by (2) we obtain that $x \in \alpha_{(\gamma,\gamma')}\text{-}Int(f^{-1}(V))$. Hence, there exists an

$\alpha_{(\gamma,\gamma')}$-open set U of X containing x such that $U \subseteq f^{-1}(V)$. Then $U \cap A = \phi$ and so $x \notin \alpha_{(\gamma,\gamma')}$-$Cl(A)$. This implies $f(x) \notin f(\alpha_{(\gamma,\gamma')}$-$Cl(A))$. Thus, $f(\alpha_{(\gamma,\gamma')}$-$Cl(A)) \subseteq \alpha_{(\beta,\beta')}$-$Cl(f(A))$.

(3) $\Rightarrow$ (4). Let B be any subset of Y. Since $f(f^{-1}(B)) \subseteq B$, so, we have $\alpha_{(\beta,\beta')}$-$Cl(f(f^{-1}(B))) \subseteq \alpha_{(\beta,\beta')}$-$Cl(B)$. Also, $f^{-1}(B) \subseteq X$, then by (3), we have $f(\alpha_{(\gamma,\gamma')}$-$Cl(f^{-1}(B))) \subseteq \alpha_{(\beta,\beta')}$-$Cl(f(f^{-1}(B))) \subseteq \alpha_{(\beta,\beta')}$-$Cl(B)$. Thus, $\alpha_{(\gamma,\gamma')}$-$Cl(f^{-1}(B)) \subseteq f^{-1}(\alpha_{(\beta,\beta')}$-$Cl(B))$. □

Corollary 3.4. Let $f : X \to Y$ be an $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous and injective function. If K is $\alpha_{(\gamma,\gamma')}$-connected in X, then $f(K)$ is $\alpha_{(\beta,\beta')}$-connected in Y.

Proof. Suppose that $f(K)$ is $\alpha_{(\beta,\beta')}$-disconnected in Y. There exist two $\alpha_{(\beta,\beta')}$-separated sets P and Q of Y such that $f(K) = P \cup Q$. Set $A = K \cap f^{-1}(P)$ and $B = K \cap f^{-1}(Q)$. Since $f(K) \cap P \neq \phi$, then $K \cap f^{-1}(P) \neq \phi$ and so $A \neq \phi$. Similarly $B \neq \phi$. Now, $A \cup B = (K \cap f^{-1}(P)) \cup (K \cap f^{-1}(Q)) = K \cap (f^{-1}(P) \cup f^{-1}(Q)) = K \cap f^{-1}(P \cup Q) = K \cap f^{-1}(f(K)) = K$. Since f is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous, then by Theorem 3.7 , $\alpha_{(\gamma,\gamma')}$-$Cl(f^{-1}(Q)) \subseteq f^{-1}(\alpha_{(\beta,\beta')}$-$Cl(Q))$ and $B \subseteq f^{-1}(Q)$, then $\alpha_{(\gamma,\gamma')}$-$Cl(B) \subseteq f^{-1}(\alpha_{(\beta,\beta')}$-$Cl(Q))$. Since $P \cap \alpha_{(\beta,\beta')}$-$Cl(Q) = \phi$, then $A \cap \alpha_{(\gamma,\gamma')}$-$Cl(B) \subseteq A \cap f^{-1}(\alpha_{(\beta,\beta')}$-$Cl(Q)) \subseteq f^{-1}(P) \cap f^{-1}(\alpha_{(\beta,\beta')}$-$Cl(Q)) = \phi$ and then $A \cap \alpha_{(\gamma,\gamma')}$-$Cl(B) = \phi$. Similarly, $B \cap \alpha_{(\gamma,\gamma')}$-$Cl(A) = \phi$. Thus, A and B are $\alpha_{(\gamma,\gamma')}$-separated. Therefore, K is $\alpha_{(\gamma,\gamma')}$-disconnected, this is contradiction. Hence, $f(K)$ is $\alpha_{(\beta,\beta')}$-connected. □

Theorem 3.8. If $f : (X, \tau) \to (Y, \sigma)$ is an onto $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous function and X is $\alpha_{(\gamma,\gamma')}$-connected, then Y is $\alpha_{(\beta,\beta')}$-connected.

Proof. Suppose that Y is $\alpha_{(\beta,\beta')}$-disconnected and A, B is an $\alpha_{(\beta,\beta')}$-disconnection of Y. By Remark 3.5, A and B are both $\alpha_{(\beta,\beta')}$-open sets. Since f is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous, so by Theorem 2.1, $f^{-1}(A)$ and $f^{-1}(B)$ are both nonempty $\alpha_{(\gamma,\gamma')}$-open sets in X. Now, $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$, and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\phi) = \phi$. Then by Remark 3.5, $f^{-1}(A), f^{-1}(B)$ is a pair of $\alpha_{(\gamma,\gamma')}$-disconnection of X. This contradiction shows that Y is $\alpha_{(\beta,\beta')}$-connected. □

Corollary 3.5. For a bijective $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-closed function $f : X \to Y$, if C is $\alpha_{(\beta,\beta')}$-connected in Y, then $f^{-1}(C)$ is $\alpha_{(\gamma,\gamma')}$-connected in X.

Proof. Suppose that $f^{-1}(C)$ is $\alpha_{(\gamma,\gamma')}$-disconnected in X. There exist two $\alpha_{(\gamma,\gamma')}$-separated sets M and N of X such that $f^{-1}(C) = M \cup N$. Set $K = C \cap f(M)$ and $L = C \cap f(N)$. Since $C = f(M) \cup f(N)$, then $C \cap f(M) \neq \phi$ and so $K \neq \phi$. Similarly $L \neq \phi$. Now, $K \cup L = (C \cap f(M)) \cup (C \cap f(N)) = C \cap (f(M) \cup f(N)) = C \cap f(M \cup N) = C \cap f(f^{-1}(C)) = C$. Since f is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-closed, then by Proposition 2.3, $\alpha_{(\beta,\beta')}$-$Cl(f(N)) \subseteq f(\alpha_{(\gamma,\gamma')}$-$Cl(N))$ and $L \subseteq f(N)$, then $\alpha_{(\beta,\beta')}$-$Cl(L) \subseteq f(\alpha_{(\gamma,\gamma')}$-$Cl(N))$. Since $M \cap \alpha_{(\gamma,\gamma')}$-$Cl(N) = \phi$, then $K \cap \alpha_{(\beta,\beta')}$-$Cl(L) \subseteq K \cap f(\alpha_{(\gamma,\gamma')}$-$Cl(N)) \subseteq f(M) \cap f(\alpha_{(\gamma,\gamma')}$-$Cl(N)) = \phi$ and then $K \cap \alpha_{(\beta,\beta')}$-$Cl(L) = \phi$. Similarly, $L \cap \alpha_{(\beta,\beta')}$-$Cl(K) = \phi$. Thus, K and L are $\alpha_{(\beta,\beta')}$-separated. Therefore, C is $\alpha_{(\beta,\beta')}$-disconnected, this is contradiction. Hence, $f^{-1}(C)$ is $\alpha_{(\gamma,\gamma')}$-connected. □

Definition 3.4. A set C is called a maximal $\alpha_{(\gamma,\gamma')}$-connected set if it is $\alpha_{(\gamma,\gamma')}$-connected and if $C \subseteq D \subseteq X$ where D is $\alpha_{(\gamma,\gamma')}$-connected, then $C = D$. A maximal $\alpha_{(\gamma,\gamma')}$-connected subset C of a space X is called an $\alpha_{(\gamma,\gamma')}$-component of X. If X is itself $\alpha_{(\gamma,\gamma')}$-connected, then X is the only $\alpha_{(\gamma,\gamma')}$-component of X.

Theorem 3.9. For each $x \in X$, there is exactly one $\alpha_{(\gamma,\gamma')}$-component of X containing x.

Proof. For any $x \in X$, let $C_x = \bigcup\{A : x \in A \subseteq X$ and A is $\alpha_{(\gamma,\gamma')}$-connected$\}$. Then, $\{x\} \in C_x$, since C_x is a union of $\alpha_{(\gamma,\gamma')}$-connected sets each containing x, C_x is $\alpha_{(\gamma,\gamma')}$-connected by Corollary 3.1. If $C_x \subseteq D$ and D is $\alpha_{(\gamma,\gamma')}$-connected, then D was one of the sets A in the collection whose union defines C_x, so $D \subseteq C_x$ and therefore $C_x = D$. Therefore, C_x is an $\alpha_{(\gamma,\gamma')}$-component of X that contains x. □

Corollary 3.6. A space X is the union of its $\alpha_{(\gamma,\gamma')}$-components.

Proof. Follows from Theorem 3.9. □

Corollary 3.7. Two $\alpha_{(\gamma,\gamma')}$-components are either disjoint or coincide.

Proof. Let C_x and C_y be $\alpha_{(\gamma,\gamma')}$-components and $C_x \neq C_y$. If $p \in C_x \cap C_y$, then by Corollary 3.1, $C_x \cup C_y$ would be an $\alpha_{(\gamma,\gamma')}$-connected set strictly larger than C_x. Therefore, $C_x \cap C_y = \phi$. □

Theorem 3.10. Each $\alpha_{(\gamma,\gamma')}$-connected subset of X is contained in exactly one $\alpha_{(\gamma,\gamma')}$-component of X.

Proof. Let A be an $\alpha_{(\gamma,\gamma')}$-connected subset of X which is not in exactly one $\alpha_{(\gamma,\gamma')}$-component of X. Suppose that C_1 and C_2 are $\alpha_{(\gamma,\gamma')}$-components of X such that $A \subseteq C_1$ and $A \subseteq C_2$. Since $C_1 \cap C_2 \neq \phi$ and by Corollary 3.1, $C_1 \cup C_2$ is another $\alpha_{(\gamma,\gamma')}$-connected set which contains C_1 as well as C_2, a contradiction to the fact that C_1 and C_2 are $\alpha_{(\gamma,\gamma')}$-components. This proves that A is contained in exactly one $\alpha_{(\gamma,\gamma')}$-component of X. □

Theorem 3.11. A nonempty $\alpha_{(\gamma,\gamma')}$-connected subset of X which is both $\alpha_{(\gamma,\gamma')}$-open and $\alpha_{(\gamma,\gamma')}$-closed is $\alpha_{(\gamma,\gamma')}$-component.

Proof. Suppose that A is $\alpha_{(\gamma,\gamma')}$-connected subset of X which is both $\alpha_{(\gamma,\gamma')}$-open and $\alpha_{(\gamma,\gamma')}$-closed. By Theorem 3.10, A is contained in exactly one $\alpha_{(\gamma,\gamma')}$-component C of X. If A is a proper subset of C, then $C = (C \cap A) \cup (C \cap (X \setminus A))$ and $(C \cap A), (C \cap (X \setminus A))$ is an $\alpha_{(\gamma,\gamma')}$-disconnection of C, which is a contradiction. Thus, $A = C$. □

Theorem 3.12. Every $\alpha_{(\gamma,\gamma')}$-component of X is $\alpha_{(\gamma,\gamma')}$-closed.

Proof. Suppose that C is an $\alpha_{(\gamma,\gamma')}$-component of X. Then, by Remark 3.6, $\alpha_{(\gamma,\gamma')}$-$Cl(C)$ is $\alpha_{(\gamma,\gamma')}$-connected containing $\alpha_{(\gamma,\gamma')}$-component C of X. This implies that $C = \alpha_{(\gamma,\gamma')}$-$Cl(C)$ and hence C is $\alpha_{(\gamma,\gamma')}$-closed. □

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