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SOME PROPERTIES OF GENERALIZED STRONGLY HARMONIC CONVEX FUNCTIONS

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Abstract. In this paper, we introduce a new class of harmonic convex functions with respect to an arbitrary trifunction $F(\cdot,\cdot,\cdot) : K \times K \times [0,1] \to \mathbb{R}$, which is called generalized strongly harmonic convex functions. We study some basic properties of strongly harmonic convex functions. We also discuss the sufficient conditions of optimality for unconstrained and inequality constrained programming under the generalized harmonic convexity. Several special cases are discussed as applications of our results. Ideas and techniques of this paper may motivate further research in different fields.

1. Introduction

The concept of convexity and generalized convexity in the study of optimality to solve mathematical programming, have been extended using innovative ideas and techniques. For example, in earlier papers, Bector and Singh [3] introduced a class of b-vex functions. Chao et al. [4] considered new generalized sub-b-convex functions and sub-b-convex sets. They proved the sufficient conditions of optimality for both unconstrained and inequality constrained sub-b-convex programming. Anderson et. al. [1] and Iscan [5] have investigated various properties of harmonic convex functions. Noor and Noor [9] have shown that the minimum of the differentiable harmonic convex functions on the harmonic convex set can be characterized by a class of variational inequalities, which is called harmonic variational inequality. To the best of our

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knowledge, this field is new one and has not developed as yet. A significant class of convex functions is that of strongly harmonic convex functions introduced by Noor et. al. [11]. For recent applications, generalizations and other aspects of convex and harmonic convex functions, see [2–4,6,10,12–19] and references therein.

Inspired by the research works [2–4, 6], we introduce a new class of harmonic convex functions with respect to an arbitrary function $F(.,.,.)$, which is called strongly generalized harmonic convex function. One can show that the generalized strongly harmonic convex functions is quite general and unified one. Several new and old classes of convex and harmonic convex functions can be obtained from these general harmonic convex functions. We consider the sufficient conditions of optimality for both unconstrained and inequality constrained programming. Some properties of generalized strongly harmonic convex functions and generalized strongly harmonic convex sets are discussed. Results obtained in this paper may be considered as significant improvement of the known results.

2. Preliminaries

In this Section, we recall some basic concepts and results. We also introduce some new concepts and discuss some special cases.

Definition 2.1. [1]. *A set $K = [a, b] \subset \mathbb{R}^n \setminus \{0\}$ is said to be a harmonic convex set, if*

$$\frac{xy}{tx + (1-t)y} \in K, \qquad \forall x, y \in K,\ t \in [0, 1].$$

We now consider the concept of generalized strongly harmonic convex functions with respect to an arbitrary trifunction $F(\cdot, \cdot, \cdot) : K \times K \times [0, 1] \to \mathbb{R}$.

Definition 2.2. *Let K is a nonempty harmonic convex set in $\mathbb{R}^n \setminus \{0\}$. A function $f : K \to \mathbb{R}$ is said to be generalized strongly harmonic convex function on K with respect to map $F(\cdot, \cdot, \cdot) : K \times K \times [0, 1] \to \mathbb{R}$, if and only if,*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \le (1-t)f(x) + tf(y) + F(x, y, t), \qquad \forall x, y \in K, t \in [0, 1]. \tag{2.1}$$

(I). If $F(x, y, t) = 0$ in Definition 2.2, then it reduces to harmonic convex function.

Definition 2.3. [5]. *A function $f : K \to \mathbb{R}$, where K is a nonempty harmonic convex set $\mathbb{R}^n \setminus \{0\}$, is said to be a harmonic convex function on K, if and only if,*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \le (1-t)f(x) + tf(y), \qquad \forall x, y \in K,\ t \in [0, 1].$$

This shows that every harmonic convex function f is harmonic sub-F-convex function with respect to the map $F(x, y, t) = 0$, but the converse may not be true. See also [8–10, 17].

(II). If $F(x,y,t)=t(1-t)\mathfrak{F}(\frac{xy}{x-y})$ in Definition 2.2, then it reduces to $\mathfrak{F}$-strongly harmonic convex function

Definition 2.4. [18]. *A function $f : K \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be $\mathfrak{F}$-strongly harmonic convex function, if there exists a non-negative function $\mathfrak{F} : X\setminus\{0\} \to [0,\infty)$, such that*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le (1-t)f(x)+tf(y)-t(1-t)\mathfrak{F}\left(\frac{xy}{x-y}\right), \quad \forall x,y\in K, t\in(0,1). \tag{2.2}$$

If $t=\frac{1}{2}$, then (2.2) reduces to

$$f\left(\frac{2xy}{x+y}\right) \le \frac{f(x)+f(y)}{2}-\frac{1}{4}\mathfrak{F}\left(\frac{xy}{x-y}\right)$$

and the function$f$ is called F-strongly harmonic mid-convex(harmonic Jensen-convex) function.

(III). If $F(x,y,t) = -ct(1-t) \parallel \frac{x-y}{xy} \parallel$ in Definition 2.2, then it reduces to strongly harmonic convex function with modulus $c>0$.

Definition 2.5. [11]. *A function $f : K \to \mathbb{R}$, where K is a nonempty harmonic convex set in $\mathbb{R}^n\setminus\{0\}$, is said to be strongly harmonic convex function on K with modulus $c>0$, if and only if,*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le (1-t)f(x)+tf(y)-ct(1-t)\left\|\frac{x-y}{xy}\right\|^2, \quad \forall x,y\in I, t\in(0,1).$$

Theorem 2.1. [17]. *Let K be a nonempty harmonic convex set in $\mathbb{R}^n\setminus\{0\}$ and let $f : K \to \mathbb{R}$ be differentiable on K. Then f is harmonic quasi convex, if and only if,*

$$f(x)\le f(y) \Rightarrow \langle f'(y), \frac{xy}{y-x}\rangle \le 0 \quad \forall\ x,y\in K.$$

Definition 2.6. *Let K is a nonempty harmonic convex set in $\mathbb{R}^n\setminus\{0\}$. A function $f : K \to \mathbb{R}$ is said to be generalized strongly harmonic quasi convex function on K with respect to map $F(\cdot,\cdot,\cdot) : K\times K\times[0,1] \to \mathbb{R}$, if and only if,*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le \max\{f(x),f(y)\}+F(x,y,t), \qquad \forall x,y\in K, t\in[0,1]. \tag{2.3}$$

(I). If $F(x,y,t)=0$ in Definition 2.6, then it reduces to harmonic quasi convex function.

Definition 2.7. [20]. *A function $f : K \to \mathbb{R}$, where K is a nonempty harmonic convex set $\mathbb{R}^n\setminus\{0\}$, is said to be a harmonic quasi convex function on K, if and only if,*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le \max\{f(x),f(y)\}, \qquad \forall x,y\in K.$$

We now define a new class of generalized strongly harmonic log-convex functions.

Definition 2.8. *Let K is a nonempty harmonic convex set in $\mathbb{R}^n\setminus\{0\}$. A function $f : K \to \mathbb{R}$ is said to be generalized strongly harmonic* log*-convex function on K with respect to map $F(\cdot,\cdot,\cdot) : K\times K\times[0,1] \to \mathbb{R}$, if and only if,*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le [f(x)]^{1-t}[f(y)]^t+F(x,y,t), \qquad \forall\ x,y\in K,\ t\in[0,1]. \tag{2.4}$$

From (2.4), it follows that

$$\begin{aligned} f\left(\frac{xy}{tx+(1-t)y}\right) &\leq [f(x)]^{1-t}[f(y)]^t + F(x,y,t) \\ &\leq (1-t)f(x) + tf(y) + F(x,y,t) \\ &\leq \max\{f(x), f(y)\} + F(x,y,t), \qquad \forall\, x,y \in K,\ t \in [0,1], \end{aligned}$$

This shows that every generalized strongly harmonic log-convex function is generalized strongly harmonic convex and every generalized strongly harmonic convex function is generalized strongly harmonic quasi convex, but the converse is not true.

3. Main Results

In this section, we introduce the concept of generalized harmonic convex set and harmonic convex functions with respect to an arbitrary map $F(\cdot,\cdot,\cdot)$ and discuss its properties.

Theorem 3.1. *If $f_i : K \to \mathbb{R}$, $(i = 1, 2, 3, ..., m)$ are generalized strongly harmonic convex functions with respect to map $F_i : K \times K \times [0,1] \to R$, respectively. Then the function*

$$f = \sum_{i=1}^{m} a_i f_i, \quad a_1 \geq 0, \quad i = 1, 2, 3, ..., m,$$

is generalized strongly harmonic convex with respect to $F = \sum_{i=1}^{m} a_i b_i$.

Proof. For all $x, y \in K$ and $t \in [0,1]$, we have

$$\begin{aligned} f\left(\frac{xy}{tx+(1-t)y}\right) &= \sum_{i=1}^{m} a_i f_i\left(\frac{xy}{tx+(1-t)y}\right) \\ &\leq \sum_{i=1}^{m} a_i[(1-t)f_i(x) + tf_i(y) + F_i(x,y,t)] \\ &= (1-t)\sum_{i=1}^{m} a_i f_i(x) + t\sum_{i=1}^{m} a_i f_i(y) + \sum_{i=1}^{m} a_i F_i(x,y,t) \\ &= (1-t)f(x) + tf(y) + F(x,y,t). \end{aligned}$$

□

Theorem 3.2. *If the function $f_i : K \to \mathbb{R}$ are generalized strongly harmonic convex functions with respect to $F_i(x,y,t)$ respectively. Then $f = \max\{f_i, i = 1, 2, 3, ..., m\}$ is also generalized strongly harmonic convex with respect to $F = \max\{F_i\}$.*

Proof. Consider,

$$\begin{aligned} f\left(\frac{xy}{tx+(1-t)y}\right) &= \max\left\{f_i\left(\frac{xy}{tx+(1-t)y}\right), i=1,2,3,...,m\right\} \\ &\leq (1-t)\max\{f_i(x)\} + t\max\{f_i(y)\} + \max\{F_i(x,y,t) \\ &= (1-t)f(x) + tf(y) + F(x,y,t). \end{aligned}$$

So, $f(x)$ is harmonic sub-F convex function with respect to
$F(x,y,t) = \max\{f_i, i=1,2,3,...,m\}$. □

Theorem 3.3. *If the function $f : K \to \mathbb{R}$ is a generalized strongly harmonic convex function with respect to $F(x,y,t)$ and $g : \mathbb{R} \to \mathbb{R}$ is a linear function, then $f \circ g$ is generalized strongly harmonic convex with respect to $F' = g \circ F$.*

Proof. Since f is generalized strongly harmonic convex function with respect to $F(x,y,t)$ and g is a an increasing function, it follows that

$$\begin{aligned} (g\circ f)\left(\frac{xy}{tx+(1-t)y}\right) &= g\left(f\left(\frac{xy}{tx+(1-t)y}\right)\right) \\ &\leq g((1-t)f(x) + tf(y) + F(x,y,t) \\ &= (1-t)g(f(x)) + tg(f(y)) + g(F(x,y,t)) \\ &= (1-t)(g\circ f) + t(g\circ f) + (g\circ F)(x,y,t). \end{aligned}$$

That is, $g \circ f$ is generalized strongly harmonic convex function with respect to $F' = g \circ F$ and this completes the proof. □

The following theorem gives a necessary and sufficient characterization of a differentiable generalized strongly harmonic convex function with respect to a map $F(\cdot,\cdot,\cdot)$.

Theorem 3.4. *Let K be a harmonic convex set. If $f : K \to \mathbb{R}$ is a differentiable generalized strongly harmonic convex function on the harmonic convex set K with respect to the map $F(x,y,t)$, then*

(1) $\langle f'(x), \frac{xy}{x-y}\rangle \leq f(y) - f(x) + \lim_{t\to 0^+} \frac{F(x,y,t)}{t}, \quad \forall x,y \in K.$
(2) $\langle f'(x) - f'(y), \frac{xy}{x-y}\rangle \leq \lim_{t\to 0^+} \frac{F(x,y,t)}{t} + \lim_{t\to 0^+} \frac{F(y,x,t)}{t}, \quad \forall x,y \in K.$

Proof. (1). Let f be a generalized strongly harmonic convex function. Then

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)f(x) + tf(y) + F(x,y,t),$$

which can be written as

$$f(y) - f(x) + \frac{F(x,y,t)}{t} \geq \frac{f\left(\frac{xy}{y+t(x-y)}\right) - f(x)}{t}.$$

Since f is differentiable function, so taking the limit in the above inequality, as $t \to 0$, we have

$$f(y) - f(x) + \lim_{t\to 0^+} \frac{F(x,y,t)}{t} \geq \langle f'(x), \frac{xy}{x-y}\rangle, \tag{3.1}$$

which is (1).

Changing the role of x and y, in (3.1), we obtain

$$f(x) - f(y) + \lim_{t\to 0^+} \frac{F(y,x,t)}{t} \geq \langle f'(y), \frac{xy}{y-x}\rangle. \tag{3.2}$$

Adding (3.1) and (3.2), we have

$$\langle f'(x) - f'(y), \frac{xy}{x-y}\rangle \leq \lim_{t\to 0^+} \frac{F(x,y,t)}{t} + \lim_{t\to 0^+} \frac{F(y,x,t)}{t},$$

which is the required (2). This completes the proof. □

Definition 3.1. [17]. *A function $f : K \to \mathbb{R}$, where K is a nonempty harmonic convex set $\mathbb{R}^n \setminus \{0\}$ is said to be a harmonic quasi convex function, if, for each $x, y \in K$ with $f(x) \leq f(y)$, we have $\langle f'(y), \frac{xy}{y-x}\rangle \leq 0$; or equivalently, if $\langle f'(y), \frac{xy}{y-x}\rangle > 0$, then $f(x) > f(y)$.*

Definition 3.2. [17]. *A function $f : K \to \mathbb{R}$, where K is a nonempty harmonic convex set $\mathbb{R}^n \setminus \{0\}$ is said to be a harmonic pseudo-convex function, if, for each $x, y \in K$ with $\langle f'(y), \frac{xy}{y-x}\rangle \geq 0$, we have $f(x) \geq f(y)$; or equivalently, if $f(x) < f(y)$, then $\langle f'(y), \frac{xy}{y-x}\rangle < 0$.*

Theorem 3.5. *Let K be a harmonic convex set and $f : K \to \mathbb{R}$ be a differentiable generalized strongly harmonic convex function on the harmonic convex set K with respect to the map $F(x, y, t)$. If $\lim_{t\to 0} \frac{F(x,y,t)}{t} \leq |f(x) - f(y)|$, $\forall\ x, y \in K$, then f is harmonic quasi-convex. Furthermore, if $\lim_{t\to 0} \frac{F(x,y,t)}{t} < |f(x) - f(y)|$, $\forall\ x, y \in K$, then f is harmonic pseudo-convex.*

Proof. Suppose that $f(x) \leq f(y)$, for any $x, y \in K$ and $t \in (0, 1)$. Then from Theorem 3.4, we have

$$\langle f'(y), \frac{xy}{y-x}\rangle \leq f(x) - f(y) + \lim_{t\to 0^+} \frac{F(y,x,t)}{t}.$$

If $\lim_{t\to 0} \frac{F(y,x,t)}{t} \leq |f(x) - f(y)|$, then $f(x) - f(y) + \lim_{t\to 0} \frac{F(x,y,t)}{t} \leq 0$. So, $\langle f'(y), \frac{xy}{y-x}\rangle \leq 0$. Therefore, from Definition 3.1, we have f is harmonic quasi-convex function.

Similarly, if $f(x) < f(y)$, we also have $\langle f'(x), \frac{xy}{x-y}\rangle < 0$. So, from the Definition 3.2, we have f is harmonic pseudo-convex function. □

Now, we are going to introduce a new concept of generalized harmonic convex set.

Definition 3.3. *Let $K_s \subset \mathbb{R}^{n+1} \setminus \{0\}$ be a nonempty set. A set K_s is said to be generalized harmonic convex with respect to $F(x, y, t) : \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \to \mathbb{R}$, if*

$$\left(\frac{xy}{tx + (1-t)y}, (1-t)\alpha + t\beta + F(x,y,t)\right) \in K_s,\ \ \forall (x,\alpha),(y,\beta) \in K_s,$$

for all $x, y \in \mathbb{R}^n$, *and* $t \in [0,1]$.

We now investigate some characterizations of generalized strongly harmonic convex function $f : K \to \mathbb{R}$ in term of their epigraph $E(f)$.

Definition 3.4. [2]. *Let* K *be a nonempty in* $\mathbb{R}^n$ *and let* $f : K \to \mathbb{R}$ *be a function. Then epigraph of* f*, denoted by* $E(f)$ *is defined by*

$$E(f) = \{(x, \alpha) : x \in K, \alpha \in \mathbb{R}, f(x) \leq \alpha\}.$$

Theorem 3.6. *A function* $f : K \to \mathbb{R}$ *is generalized strongly harmonic convex with respect to* $F(x, y, t) : \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}$, *if and only if, epigraph of* f *is generalized harmonic convex set with respect to* $F(\cdot, \cdot, \cdot)$.

Proof. Suppose that f is harmonic sub-F-convex function with respect to $F(x, y, t)$. Let (x, α) and $(y, \beta) \in E(f)$. Then $x, y \in K$, $f(x) \leq \alpha$ and $f(y) \leq \beta$, we have

$$\begin{aligned} f\left(\frac{xy}{tx + (1-t)y}\right) &\leq (1-t)f(x) + tf(y) + F(x, y, t) \\ &\leq (1-t)\alpha + t\beta + F(x, y, t). \end{aligned}$$

Hence from Definition, one has

$$\left(\frac{xy}{tx + (1-t)y}, (1-t)\alpha + t\beta + F(x, y, t)\right) \in E(f)$$

Thus $E(f)$ is generalized harmonic convex set with respect to F.

Conversely, assume that $E(f)$ is generalized harmonic convex set with respect to F. Let $x, y \in K$, then $(x, f(x))$ and $(y, f(y))$ belong to $E(f)$. Thus for $t \in [0,1]$,

$$\left(\frac{xy}{tx + (1-t)y}, (1-t)f(x) + tf(y) + F(x, y, t)\right) \in E(f).$$

This further follows that,

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)f(x) + tf(y) + F(x, y, t)$$

That is f is a generalized strongly harmonic convex function with respect to $F(., ., .)$. □

Theorem 3.7. *If* K_{s_i} *is a family of generalized harmonic convex set with respect to a map* $F(x, y, t)$, *then* $\cap_{i \in I} K_{s_i}$ *is a generalized harmonic convex set with respect to* $F(x, y, t)$.

Proof. Let $(x, \alpha), (y, \beta) \in \cap_{i \in I} K_{s_i}$, $t \in [0,1]$. Then for each $i \in I$, $(x, \alpha), (y, \beta) \in I_i$. Since K_{s_i} is a generalized harmonic convex set with respect to $F(., ., .)$, it follows that

$$\left(\frac{xy}{tx + (1-t)y}, (1-t)f(x) + tf(y) + F(x, y, t)\right) \in K_{s_i}, \forall i \in I.$$

Thus

$$\left(\frac{xy}{tx+(1-t)y},(1-t)f(x)+tf(y)+F(x,y,t)\right)\in\cap_{i\in I}K_{s_i}.$$

Hence, $\cap_{i\in I}K_{s_i}$ is generalized harmonic convex set with respect to $F(x,y,t)$. □

We apply the above results to the nonlinear programming problem. First, we consider the unconstraint problem.

Theorem 3.8. *Let $f:K\to\mathbb{R}$ be differentiable and generalized strongly harmonic convex function with respect to map $F(\cdot,\cdot,\cdot)$. Consider the optimal problem $\min\{f(x)|x\in K\}$. If $\bar{x}\in K$ and the relation*

$$\langle f'(x),\frac{\bar{x}x}{\bar{x}-x}\rangle-\lim_{t\to 0^+}\frac{F(\bar{x},x,t)}{t}\geq 0, \tag{3.3}$$

holds for each $x\in K$, then $\bar{x}$ is the optimal solution of f on K.

Proof. For any $x\in k$, from Theorem 3.4, we have

$$\langle f'(x),\frac{\bar{x}x}{\bar{x}-x}\rangle-\lim_{t\to 0^+}\frac{F(\bar{x},x,t)}{t}\leq f(x)-f(\bar{x}). \tag{3.4}$$

From (3.3) and (3.4), we have $f(x)-f(\bar{x})\geq 0$. So, $\bar{x}$ is an optimal solution of f on K. □

Next, we apply the above results to the nonlinear programming problem with inequality constraints:

$$(P_g)\quad \begin{array}{ll} \min & f(x) \\ \text{s.t} & g_i(x)\leq 0,\ i\in I=\{1,2,3,...,m\} \\ & x\in\mathbb{R}^n \end{array}$$

Denote the feasible set of (P_g) by $S_g=\{x\in\mathbb{R}^n|g_i(x)\leq 0,\ i\in I\}$.

Definition 3.5. *Let K be a nonempty harmonic convex set in $\mathbb{R}^n$. The function $f:K\to\mathbb{R}$ is said to be generalized strongly harmonic pseudo convex on K with respect to $F:K\times K\times[0,1]\to\mathbb{R}$, if for each $x,y\in K$ and $t\in(0,1)$, from $\langle f'(y),\frac{xy}{y-x}\rangle+\lim_{t\to 0}\frac{F(y,x,t)}{t}\geq 0$ one can have $f(x)\geq f(y)$.*

Theorem 3.9. (Karush-Kuhn-Tucker Sufficient Conditions) *The function f is differentiable generalized strongly harmonic pseudo convex with respect to $F(.,.,.):K\times K\times[0,1]\to\mathbb{R}$, $g_i(x)$ $(i\in I)$ are differentiable and generalized strongly harmonic convex with respect to $F:K\times K\times[0,1]\to\mathbb{R}$. Assume that $\bar{x}\in S_g$ is a KKT point of (P_g), that is, there exist multiplier $\lambda_i\geq 0$ $(i\in I)$ such that*

$$\langle f'(\bar{x}),\frac{\bar{x}x}{\bar{x}-x}\rangle+\sum_{i\in I(\bar{x})}\lambda_i\langle g'(\bar{x}),\frac{\bar{x}x}{\bar{x}-x}\rangle=0,\quad \lambda_i\langle g'(\bar{x}),\frac{\bar{x}x}{\bar{x}-x}\rangle=0. \tag{3.5}$$

If

$$\lim_{t\to 0}\frac{F(x,\bar{x},t)}{t}\geq\sum_{i\in I}\lambda_i\lim_{t\to 0}\frac{F(x,\bar{x},t)}{t},\ \forall x\in S_g. \tag{3.6}$$

Then $\bar{x}$ is an optimal solution of the problem (P_g).

Proof. For any $x\in S_g$, we have $g_i(x)\leq 0$, $g_i(\bar{x})=0$, $i\in I(\bar{x})=\{i\in I\mid g_i(\bar{x})=0\}$. Therefore, from the generalized strongly harmonic convexity of $g_i(x)$ and Theorem 3.4, we obtain

$$\langle g'(\bar{x}),\frac{\bar{x}x}{\bar{x}-x}\rangle-\lim_{t\to 0^+}\frac{F(\bar{x},x,t)}{t}\leq 0,\ \text{for } i\in I(\bar{x}). \tag{3.7}$$

From (3.5), one has

$$\langle f'(\bar{x}),\frac{\bar{x}x}{\bar{x}-x}\rangle=-\sum_{i\in I(\bar{x})}\lambda_i\langle g'(\bar{x}),\frac{\bar{x}x}{\bar{x}-x}\rangle.$$

In view of (3.6) and from (3.7), we have

$$\begin{aligned}\langle f'(\bar{x}),\frac{\bar{x}x}{\bar{x}-x}\rangle+\lim_{t\to 0^+}\frac{F(\bar{x},x,t)}{t} &\geq -\sum_{i\in I(\bar{x})}\lambda_i\langle g'(\bar{x}),\frac{\bar{x}x}{\bar{x}-x}\rangle+\sum_{i\in I(\bar{x})}\lambda_i\lim_{t\to 0^+}\frac{F(\bar{x},x,t)}{t}\\ &= -\sum_{i\in I(\bar{x})}\lambda_i\left[\langle g'(\bar{x}),\frac{\bar{x}x}{\bar{x}-x}\rangle-\lim_{t\to 0^+}\frac{F(\bar{x},x,t)}{t}\right]\\ &\geq 0.\end{aligned}$$

So,

$$\langle f'(\bar{x}),\frac{\bar{x}x}{\bar{x}-x}\rangle+\lim_{t\to 0^+}\frac{F(\bar{x},x,t)}{t}\quad\geq\quad 0.$$

From the generalized strongly harmonic pseudo convexity of $f(x)$, we have $f(x)\geq f(\bar{x})$. Therefore $\bar{x}$ is an optimal solution of the problem (P_g). □

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References

[1] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, J. Math. Anal. Appl., 335(2007), 1294-1308.

[2] M. S. Bazaraa, D. Hanif, C. M. Shetty, Nonlinear Programming Theory and Algorithms John Wiley and Sons, New York, 1993.

[3] C.R. Bector, C. Singh, B-vex functions, J. Optim. Theory Appl., 71 (1991), 439-453.

[4] M. T. Chao, J. B. Jian and D. Y. Liang, Sub-b-convex functions and sub-F-convex programming, Oper. Res. Trans., 16(2012), 1-8.

[5] I. Iscan, Hermite-Hadamard type inequalities for harmonically convex functions. Hacet, J. Math. Stats., 43(6)(2014), 935-942.

[6] J. Liao and T. Du, On some characterizations of sub-b-s-convex functions, Filomat, 30(14)(2016), 3885-3895.

[7] C. P. Niculescu and L. E. Persson, Convex Functions and Their Applications, Springer-Verlag, New York, (2006).

[8] M. A. Noor, Some developments in general variational inequalities, Appl. Math. Comput. 251(2004), 199-277.

[9] M. A. Noor and K. I. Noor, Harmonic variational inequalities, Appl. Math. Inf. Sci., 10(5)(2016), 1811-1814.

[10] M. A. Noor and K. I. Noor, Some implicit methods frsolving harmonic variational inequalites, Inter. J. Anal. Appl. 12(1)(2016), 10-14.

[11] M. A. Noor, K. I. Noor and S. Iftikhar, Hermite-Hadamard inequalities for strongly harmonic convex functions, J. Inequ. Special Func.,7(3)(2016), 99-113.

[12] M. A. Noor, K. I. Noor and S. Iftikhar, Integral inequalities for differentiable relative harmonic preinvex functions(survey), TWMS J. Pure Appl. Math. 7(1)(2016), 3-19.

[13] M. A. Noor, K. I. Noor and S. Iftikhar, Hermite-Hadamard inequalities for harmonic preinvex functions, Saussurea 6(1)(2016), 34-53.

[14] M. A. Noor, K. I. Noor and S. Iftikhar, Integral inequalities of Hermite-Hadamard type for harmonic (h, s)-convex functions, Int. J. Anal. Appl., 11(1)(2016), 61-69.

[15] M. A. Noor, K. I. Noor, S. Iftikhar and F. Safdar, Integral inequalities for relative harmonic (s, η)-convex functions, Appl. Math. Comput. Sci. 1(1)(2016), 27-34.

[16] M. A. Noor, K. I. Noor, S. Iftikhar and C. Ionescu, Hermite-Hadamard inequalities for co-ordinated harmonic convex functions, U.P.B. Sci. Bull., Ser: A, 79(1)(2017), 25-34.

[17] M. A. Noor, K. I. Noor and S. Iftikhar, Some characterizations of harmonic convex functions, Int. J. Anal. Appl. 15(2)(2017), 179-187.

[18] M. A. Noor, B. Bin-Mohsin, K. I. Noor and S. Iftikhar, Relative strongly harmonic convex functions and their characterizations, J. Nonlinear Sci. Appl. in press.

[19] J. Pecaric, F. Proschan, and Y. L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Acdemic Press, New York, (1992).

[20] T.-Y. Zhang, Ai-P. Ji, F. Qi, Integral inequalities of Hermite-Hadamard type for harmonically quasi-convex functions, Proc. Jangjeon. Math. Soc., 16(3)(2013), 399-407.