International Journal of Analysis and Applications
Volume 16, Number 6 (2018), 783-792
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-16-2018-783

ANALYTIC FUNCTIONS RELATED WITH MOCANU CLASS

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Abstract. In this article, we define a new class of analytic functions. This class generalizes the mocanu class. We obtain relationships of this class with other subclasses of analytic functions and derived many interesting results.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f analytic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = 0$ and $f^{'}(0) = 1$. So each $f \in \mathcal{A}$ has series representation of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A function $f \in \mathcal{A}$ is in class $\mathcal{S}$ if and only if $f(z_1) = f(z_2)$ implies $z_1 = z_2$, for all $z_1, z_2 \in \Delta$. An analytic function f is subordinate to an analytic function g (written as $f \prec g$) if and only if there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \Delta$ such that $f(z) = g(w(z))$.

Received 2018-01-24; accepted 2018-03-14; published 2018-11-02.

2010 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. convex functions; strongly starlike functions; subordination.

For $0 \leq \alpha \leq 1$, Mocanu [15] introduced the class $\mathcal{M}_\alpha$ of functions $f \in \mathcal{A}$ such that $\frac{f(z)f'(z)}{z} \neq 0$ for all $z \in \Delta$ and

$$\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right\} > 0, \ z \in \Delta. \tag{1.2}$$

Mocanu defined the class $\mathcal{M}_\alpha$ geometrically as a class of functions that maps the circle centered at the origin on $\alpha-$convex arcs and derived the condition (1.2).

The class $\mathcal{M}_\alpha$ was extensively studied in literature by several authors, for instance, see [4–6, 18–21]. For particular values of α, we obtain a number of interesting classes of analytic functions having nice geometry, for instance $\mathcal{M}_0 = \mathcal{S}^*$ and $\mathcal{M}_1 = \mathcal{C}$ are well known classes of starlike and convex univalent functions introduced by Alexander [1]. By $\mathcal{S}^*(\delta)$ and $\mathcal{C}(\delta)$, $0 \leq \delta < 1$, we mean the subclasses of starlike and convex function of order δ given by (1.3) and (1.4) respectively.

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta, \ z \in \Delta, \tag{1.3}$$

and

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta, \ z \in \Delta. \tag{1.4}$$

We denote the classes of uniformly starlike and uniformly convex functions by $\mathcal{UST}$ and $\mathcal{UCV}$, see [7,13,14]. A function $f \in \mathcal{S}$ is uniformly starlike if f maps every circular arc γ contained in Δ with center $\zeta \in \Delta$ onto a starlike arc with respect to $f(\zeta)$. A function $f \in \mathcal{S}$ is uniformly convex if f maps every circular arc γ contained in Δ with center $\zeta \in \Delta$ onto a convex arc.

In 1999, Kanas and Wisnoiska [9] introduced the class $k-\mathcal{UCV}$, $(k \geq 0)$ of k-uniformly convex functions as:

$$f \in k-\mathcal{UCV} \iff f \in \mathcal{A} \text{ and } \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in \Delta, \tag{1.5}$$

where its geometric definition and connections with the conic domains were considered. It is worth mentioning that $1-\mathcal{UCV} = \mathcal{UCV}$. In recent years many authors investigated interesting properties of these classes. For some details see [2,8–12,26,30] and references cited there in. Let $\mathcal{SS}^*(\lambda)$ denote the class of strongly starlike functions of order λ as:

$$\mathcal{SS}^*(\lambda) = \left\{f \in \mathcal{A} : \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\lambda\pi}{2}\right\}, \tag{1.6}$$

where $\lambda \in (0,1)$. This class of functions was introduced and discussed by [3,27].

In our current investigation, we extend the work of J. sokol [25] and introduced a new class of analytic function as:

Definition 1.1. *Let $f \in \mathcal{A}$ and $k \geq 0$, $0 \leq \alpha \leq 1$. Then $f \in k-\mathcal{UM}_\alpha$ if and only if*

$$\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right\} > k\left|\frac{(zf'(z))'}{f'(z)} - 1\right|, \ z \in \Delta. \tag{1.7}$$

For special values of parameters k and α, we obtain a number of known classes of analytic functions. Here we present few of them.

(i) $0-\mathcal{UM}_{\alpha}=\mathcal{M}_{\alpha}$, [15].

(ii) $0-\mathcal{UM}_{0}=\mathcal{S}^{*}$, [1].

(iii) $1-\mathcal{UM}_{1}=\mathcal{UCV}$, [7].

(iv) $k-\mathcal{UM}_{1}=k-\mathcal{UCV}$, [9].

Lemma 1.1. *Let $u=u_1+iu_2$ and $v=v_1+iv_2$ and let $\Psi(u,v):\mathbb{C}^2\times\Delta\to\mathbb{C}$ be a complex-valued function satisfying the conditions:*
(i) $\Psi(u,v)$ is continuous in a domain $D\subset\mathbb{C}^2$,
(ii) $(1,0)\in D$ and $\Psi(1,0)>0$.
(iii) $\operatorname{Re}\Psi(iu_2,v_1)\leq 0$ whenever $(iu_2,v_1)\in D$ and $v_1\leq -\frac{1}{2}\left(1+u_2^2\right)$.
If $h(z)=1+c_1z+c_2z^2+\cdots$ is a function that is analytic in Δ such that $(h(z),zh'(z))\in D$ and $\operatorname{Re}\Psi(h(z),zh'(z))>0$ for $z\in\Delta$, then $\operatorname{Re}h(z)>0$.

This result is due to Miller [16].

Lemma 1.2. *Let $\beta>0$ and $0\leq\gamma<1$. Let*

$$p(z)+\frac{\beta zp'(z)}{p(z)}\prec\frac{1+(1-2\gamma)}{1-z}.$$

Then

$$p(z)\prec\frac{1+(1-2\delta)}{1-z},$$

where

$$\delta=\frac{(2\gamma-\beta)+\sqrt{(2\gamma-\beta)^2+8\beta}}{4}. \tag{1.8}$$

Lemma 1.3. *[17] If $-1\leq B<A\leq 1$, $\lambda>0$ and the complex number γ satisfies $\operatorname{Re}\{\gamma\}\geq-\lambda(1-A)/(1-B)$, then the differential equation*

$$q(z)+\frac{zq'(z)}{\lambda q(z)+\gamma}=\frac{1+Az}{1+Bz},\quad z\in\Delta,$$

has a univalent solution in E given by

$$q(z)=\begin{cases}\dfrac{z^{\lambda+\gamma}(1+Bz)^{\lambda(A-B)/B}}{\lambda\int\limits_0^z t^{\lambda+\gamma-1}(1+Bt)^{\lambda(A-B)/B}dt}-\frac{\gamma}{\lambda}, & B\neq 0,\\[2ex] \dfrac{z^{\lambda+\gamma}e^{\lambda Az}}{\lambda\int\limits_0^z t^{\lambda+\gamma-1}e^{\lambda At}dt}-\frac{\gamma}{\lambda}, & B=0.\end{cases}$$

If $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$ *is analytic in* Δ *and satisfies*

$$h(z) + \frac{zh'(z)}{\lambda h(z) + \gamma} \prec \frac{1 + Az}{1 + Bz}, \quad z \in \Delta,$$

then

$$h(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz},$$

and $q(z)$ *is the best dominant.*

Lemma 1.4. *[29] Let* ε *be a positive measure on* $[0,1]$. *Let* g *be a complex-valued function defined on* $\Delta \times [0,1]$ *such that* $g(.,t)$ *is analytic in* Δ *for each* $t \in [0,1]$ *and* $g(z,.)$ *is* ε*-integrable on* $[0,1]$ *for all* $z \in \Delta$. *In addition, suppose that* $\operatorname{Re} g(z,t) > 0$, $g(-r,t)$ *is real and* $\operatorname{Re}\{1/g(z,t)\} \geq 1/g(-r,t)$ *for* $|z| \leq r < 1$ *and* $t \in [0,1]$. *If* $g(z) = \int_0^1 g(z,t)\, d\varepsilon(t)$, *then* $\operatorname{Re}\{1/g(z)\} \geq 1/g(-r)$.

Lemma 1.5. *[28, Chapter 14] Let* a, b *and* $c \neq 0, -1, -2 \ldots$ *be complex numbers. Then, for* $\operatorname{Re} c > \operatorname{Re} b > 0$,

$$\begin{aligned}
(i) \quad {}_2F_1(a,b,c;z) &= \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt. \\
(ii) \quad {}_2F_1(a,b,c;z) &= {}_2F_1(b,a,c;z). \\
(iii) \quad {}_2F_1(a,b,c;z) &= (1-z)^{-a}\ {}_2F_1\left(a, c-b, c; \frac{z}{z-1}\right).
\end{aligned}$$

Lemma 1.6. *Let* p *be analytic in* Δ *of the form*

$$p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n, \ c_m \neq 0,$$

with $p(z) \neq 0$ *in* Δ. *If there exists a point* $z_0, |z_0| < 1$ *such that* $|\arg p(z)| < \frac{\pi\varphi}{2} \quad (|z| < |z_0|)$ *and* $|\arg p(z_0)| = \frac{\pi\varphi}{2}$ *for some* $\varphi > 0$, *then we have* $\frac{z_0 p'(z_0)}{p(z_0)} = il\varphi$, *where*

$$\begin{cases} l \geq \frac{m}{2}\left(a + \frac{1}{a}\right), & \text{when } \arg p(z_0) = \frac{\pi\varphi}{2}, \\ l \leq -\frac{m}{2}\left(a + \frac{1}{a}\right), & \text{when } \arg p(z_0) = -\frac{\pi\varphi}{2}, \end{cases}$$

where $(p(z_0))^{1/\varphi} = \pm ia \quad (a > 0)$.

This result is generalization of the Nunokawa's lemma [23].

2. Results and Discussion

Theorem 2.1. *Let* $f \in k - \mathcal{UM}_\alpha$. *Then* $f \in \mathcal{S}^*(\delta)$, *where*

$$\delta = \frac{(2\gamma - \beta) + \sqrt{(2\gamma - \beta)^2 + 8\beta}}{4}, \tag{2.1}$$

with $\beta = \frac{\alpha + k}{1 + k}$ *and* $\gamma = \frac{k}{1+k}$.

Proof. Let

$$\frac{zf'(z)}{f(z)} = p(z),$$

where p is analytic in $\cdot$ with $p(0) = 1$. We obtain

$$\begin{aligned} \operatorname{Re}\left\{(1-\alpha)p(z) + \alpha\left(p(z) + \frac{zp'(z)}{p(z)}\right)\right\} &> k\left|p(z) + \frac{zp'(z)}{p(z)} - 1\right| \\ &= k\left|1 - p(z) - \frac{zp'(z)}{p(z)}\right| \\ \operatorname{Re}\left\{(1-\alpha)p(z) + \alpha\left(p(z) + \frac{zp'(z)}{p(z)}\right)\right\} &> \operatorname{Re} k\left\{(1-p(z)) - \frac{zp'(z)}{p(z)}\right\}, \end{aligned}$$

hence

$$\operatorname{Re}\left\{p(z) + \frac{zp'(z)}{\frac{1}{\beta}p(z)}\right\} > \gamma, \tag{2.2}$$

where $\beta = \frac{\alpha+k}{1+k}$ and $\gamma = \frac{k}{1+k}$. The above relation can be written in the following Briot-Bouquet differential subordination

$$p(z) + \frac{zp'(z)}{\frac{1}{\beta}p(z)} \prec \frac{1+(1-2\gamma)z}{1-z}.$$

Now using Lemma 2, we have $p \in \mathcal{S}^*(\delta)$, where δ is given by (2.1). □

Special Cases

(i) For $\alpha = 0$, $k = 1$,we have $\beta = \gamma = \frac{1}{2}$. Then for $f \in 1-\mathcal{UM}_0$, we have $f \in \mathcal{S}^*(\delta)$, where $\delta \simeq 0.64$.

(ii) For $\alpha = 0$, we have $\beta = \frac{k}{1+k}$ and $\gamma = \frac{k}{1+k}$. Then

$$\delta_1 = \frac{k + \sqrt{9k^2+8k}}{4(k+1)}.$$

In other words for $f \in k-\mathcal{UM}_0$, we have $f \in \mathcal{S}^*(\delta_1)$.

(iii) If $\alpha = 1,\ \beta = 1,\ \gamma = \frac{k}{1+k}$, then

$$\delta_2 = \frac{(k-1) + \sqrt{(k-1)^2 + 8(k+1)^2}}{4(k+1)}.$$

In other words $f \in k-\mathcal{UCV}$ implies $f \in \mathcal{S}^*(\delta_2)$.

Theorem 2.2. *Let $f \in k-\mathcal{UM}_\alpha$ and of the form*

$$f(z) = z + \sum_{n=m+1}^{\infty} a_n z^n,\ a_{m+1} \neq 0.$$

Then f is strongly starlike of order β_0, where

$$\beta_0 = \min_{\beta\in(0,1)} \left\{ \begin{array}{c} 1 - 2a^\beta \cos\beta\frac{\pi}{2} + \left(k^2-1\right)\left(a^\beta\cos\beta\frac{\pi}{2}\right)^2 + \\ \left(\frac{\beta m\left(a^2+1\right)}{2a} + a^\beta \sin\beta\frac{\pi}{2}\right)^2 \end{array} \right\}. \tag{2.3}$$

Proof. Let

$$\frac{zf'(z)}{f(z)} = p(z).$$

Then p is the form

$$p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n, \ c_m \neq 0.$$

Now using the definition of the class $k - \mathcal{UM}_\alpha$, we have

$$\operatorname{Re}\left\{(1-\alpha)p(z) + \alpha\left(p(z) + \frac{zp'(z)}{p(z)}\right)\right\} > k\left|p(z) + \frac{zp'(z)}{p(z)} - 1\right|. \tag{2.4}$$

If there exists a point $z_0 \in \Delta$ such that

$$|\arg p(z)| < \frac{\beta\pi}{2}, \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\beta\pi}{2},$$

then from Lemma 1.6, we have $\frac{z_0 p'(z_0)}{p(z_0)} = il\beta$, where $(p(z_0))^{1/\beta} = \pm ia \quad (a > 0)$ and

$$l : \begin{cases} l \geq \frac{m}{2}\left(a + \frac{1}{a}\right), & \text{when } \arg p(z_0) = \frac{\beta\pi}{2}, \\ l \leq -\frac{m}{2}\left(a + \frac{1}{a}\right), & \text{when } \arg p(z_0) = -\frac{\beta\pi}{2}. \end{cases} \tag{2.5}$$

For the case $\arg p(z_0) = \frac{\beta\pi}{2}$, we obtain

$$\operatorname{Re}\left\{p(z_0) + \frac{\alpha z_0 p'(z_0)}{p(z_0)}\right\} = \operatorname{Re}\left\{(ia)^\beta + i\alpha l\beta\right\} = a^\beta \cos\frac{\beta\pi}{2}. \tag{2.6}$$

Also, we have

$$\begin{aligned} k\left|p(z_0) - 1 + \frac{z_0 p'(z_0)}{p(z_0)}\right| &= k\left|(ia)^\beta + il\beta - 1\right| \\ &= k\left|a^\beta \cos\frac{\beta\pi}{2} - 1 + i\left(a^\beta \sin\frac{\beta\pi}{2} + l\beta\right)\right| \\ &= k\sqrt{\left(a^\beta \cos\frac{\beta\pi}{2} - 1\right)^2 + \left(a^\beta \sin\frac{\beta\pi}{2} + l\beta\right)^2}. \end{aligned} \tag{2.7}$$

Then from $l \geq m\left(a^2+1\right)/2a$ for $\beta \geq \beta_0$, we have

$$0 \leq 1 - 2a^\beta \cos\beta\frac{\pi}{2} + \left(k^2 - 1\right)\left(a^\beta \cos\beta\frac{\pi}{2}\right)^2 + \left(\frac{\beta m\left(a^2+1\right)}{2a} + a^\beta \sin\beta\frac{\pi}{2}\right)^2. \tag{2.8}$$

Therefore

$$0 \leq 1 - 2a^{\beta_0} \cos\beta_0\frac{\pi}{2} + \left(k^2 - 1\right)\left(a^{\beta_0} \cos\beta_0\frac{\pi}{2}\right)^2 + \left(\frac{\beta_0 m\left(a^2+1\right)}{2a} + a^{\beta_0} \sin\beta_0\frac{\pi}{2}\right)^2. \tag{2.9}$$

By using (2.6) and (2.7), we have

$$\operatorname{Re}\left\{(1-\alpha)p(z_o) + \alpha\left\{p(z_o) + \frac{zp'(z_o)}{p(z_o)}\right\}\right\} < k\left|p(z_0) + \frac{zp'(z_0)}{p(z_0)} - 1\right|.$$

which is contradiction, therefore $|\arg p(z)| < \frac{\beta\pi}{2}$ for $|z| < 1$.

Similarly we can prove the case $\arg p(z_0) = -\frac{\beta\pi}{2}$ by using the same method as the above we will get a contradiction. This proves that f is strongly starlike of order β_0. □

Corollary 2.1. *Let $f \in \mathcal{UCV}$ and of the form*

$$f(z) = z + \sum_{n=m+1}^{\infty} a_n z^n, \ a_{m+1} \neq 0.$$

Then f is strongly starlike of order β_0, where

$$\beta_0 = \min_{\beta\in(0,1)} \left\{ 1 - 2a^{\beta}\cos\beta\frac{\pi}{2} + \left(\frac{\beta m\left(a^2+1\right)}{2a} + a^{\beta}\cos\beta\frac{\pi}{2} \right)^2 \right\}.$$

This result is due to Nunokawa and Sokol [24].

Theorem 2.3. *If $f \in k - \mathcal{UM}_{\alpha}$, then*

$$\frac{zf'(z)}{f(z)} \prec q(z) = \frac{1}{g(z)}, \tag{2.10}$$

where $g(z) = \left[{}_2F_1\left(\frac{1}{\beta}(1-\gamma), 1; \frac{1}{\beta}+1; \frac{z}{z-1}\right)\right]$ with $\beta = \frac{k+\alpha}{1+k}$ and $\gamma = \frac{k}{1+k}$.

Proof. Let

$$\frac{zf'(z)}{f(z)} = p(z),$$

where p is analytic in Δ with $p(0) = 1$. Now using (1.7), we obtain

$$\begin{aligned} \operatorname{Re}\left\{(1-\alpha)p(z) + \alpha\left(p(z) + \frac{zp'(z)}{p(z)}\right)\right\} &> k\left|p(z) + \frac{zp'(z)}{p(z)} - 1\right| \\ &= k\left|1 - p(z) - \frac{zp'(z)}{p(z)}\right| \\ \operatorname{Re}\left\{(1-\alpha)p(z) + \alpha\left(p(z) + \frac{zp'(z)}{p(z)}\right)\right\} &> \operatorname{Re} k\left\{(1-p(z)) - \frac{zp'(z)}{p(z)}\right\}. \end{aligned}$$

This implies that

$$\operatorname{Re}\left\{p(z) + \frac{zp'(z)}{\frac{1}{\beta}p(z)}\right\} > \gamma,$$

where $\beta = \frac{k+\alpha}{1+k}$ and $\gamma = \frac{k}{1+k}$. The above relation can be written in the following Briot-Bouquet differential subordination

$$p(z) + \frac{zp'(z)}{\frac{1}{\beta}p(z)} \prec \frac{1+(1-2\gamma)z}{1-z}. \tag{2.11}$$

Using Lemma 1.3 for $\lambda = \frac{1}{\beta}$ and $\gamma = 0$, we have

$$\begin{aligned} p(z) \prec q(z) = \frac{1}{g(z)} &= \frac{1}{1/\beta \int_0^1 t^{1/\beta-1}\left(\frac{1-tz}{1-z}\right)^{-2(\gamma-1)/\beta} dt} \\ &= \left\{ {}_2F_1\left(\frac{2}{\beta}(1-\gamma), 1; \frac{1}{\beta}+1; \frac{z}{z-1}\right)\right\}^{-1}. \end{aligned}$$

which is the required result. □

Theorem 2.4. *If $f \in k-\mathcal{UM}_{\alpha}$. Then*

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{g(-1)} = \gamma_0 = \left\{ {}_2F_1\left(\frac{2}{\beta}(1-\gamma), 1; \frac{1}{\beta}+1; \frac{1}{2}\right)\right\}^{-1}.$$

In other words $k-\mathcal{UM}_{\alpha} \subset \mathcal{S}^*(\gamma_0)$, where

$$\gamma_0 = \left\{ {}_2F_1\left(\frac{2}{\beta}(1-\gamma), 1; \frac{1}{\beta}+1; \frac{1}{2}\right)\right\}^{-1}. \tag{2.12}$$

Proof. To prove $k-\mathcal{UM}_{\alpha} \subset \mathcal{S}^*(\gamma_0)$, we show that $\inf_{|z|<1}\{\operatorname{Re} q(z)\} = q(-1)$. Now for $a = \frac{2}{\beta}(1-\gamma)$, $b = \frac{1}{\beta}$, $c = \frac{1}{\beta}+1$, we have

$$\begin{aligned} g(z) &= (1+Bz)^a \int_0^1 t^{b-1}(1+Btz)^{-a}\,dt. \\ &= \frac{\Gamma(b)}{\Gamma(c)}\, {}_2F_1\left(1, a, c; \frac{z}{z-1}\right). \end{aligned} \tag{2.13}$$

To prove that $\inf_{|z|<1}\{\operatorname{Re} q(z)\} = q(-1)$, we need to show that

$$\operatorname{Re}\{1/g(z)\} \geq 1/g(-1).$$

Now by using Lemma 1.4, (2.13) it can easily follows that

$$g(z) = \int_0^1 g(z,t)\,d\varepsilon(t),$$

where

$$\begin{aligned} g(z,t) &= \frac{1-z}{1-(1-t)z}, \quad (0 \leq t \leq 1), \\ d\varepsilon(t) &= \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} t^{a-1}(1-t)^{c-a-1}\,dt, \end{aligned}$$

which is a positive measure on $[0,1]$. It is clear that $\operatorname{Re} g(z,t) > 0$ and $g(-r,t)$ is real for $0 \leq |z| \leq r < 1$ and $t \in [0,1]$. Also

$$\operatorname{Re}\left\{\frac{1}{g(z,t)}\right\} = \operatorname{Re}\left\{\frac{1-(1-t)z}{1-z}\right\} \geq \frac{1+(1-t)r}{1+r} = \frac{1}{g(-r,t)}$$

for $|z| \leq r < 1$. Therefore using Lemma 1.4, we have

$$\operatorname{Re}\{1/g(z)\} \geq 1/g(-r).$$

Now letting $r \to 1^-$, it follows

$$\operatorname{Re}\{1/g(z)\} \geq 1/g(-1).$$

Therefore $k-\mathcal{UM}_{\alpha} \subset \mathcal{S}^*(\gamma_0)$. □

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