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PROPERTIES OF SOLUTIONS OF COMPLEX DIFFERENTIAL EQUATIONS IN THE UNIT DISC

ZINELÂABIDINE LATREUCH AND BENHARRAT BELAÏDI*

Abstract. In this paper, we investigate the growth and oscillation of higher order differential polynomial with meromorphic coefficients in the unit disc $\Delta = \{z : |z| < 1\}$ generated by solutions of the linear differential equation

$$f^{(k)} + A(z) f = 0 \quad (k \geq 2),$$

where $A(z)$ is a meromorphic function of finite iterated $p-$order in Δ.

1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory on the complex plane and in the unit disc $\Delta = \{z : |z| < 1\}$ (see [14], [15], [18], [20], [22]). We need to give some definitions and discussions. Firstly, let us give two definitions about the degree of small growth order of functions in Δ as polynomials on the complex plane $\mathbb{C}$. There are many types of definitions of small growth order of functions in Δ (see [11], [12]).

Definition 1.1 Let f be a meromorphic function in Δ, and

$$D(f) := \limsup_{r\to 1^-} \frac{T(r,f)}{\log \frac{1}{1-r}} = b.$$

If $b < \infty$, we say that f is of finite b degree (or is non-admissible). If $b = \infty$, we say that f is of infinite degree (or is admissible), both defined by characteristic function $T(r,f)$.

Definition 1.2 Let f be an analytic function in Δ, and

$$D_M(f) := \limsup_{r\to 1^-} \frac{\log^+ M(r,f)}{\log \frac{1}{1-r}} = a < \infty \ \ (\text{or } a = \infty),$$

then we say that f is a function of finite a degree (or of infinite degree) defined by maximum modulus function $M(r,f) = \max_{|z|=r} |f(z)|$.

Now we give the definitions of iterated order and growth index to classify generally the functions of fast growth in Δ as those in $\mathbb{C}$ (see [5], [17], [18]). Let us define

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inductively, for $r \in [0,1)$, $\exp_1 r = e^r$ and $\exp_{p+1} r = \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large in $(0,1)$, $\log_1 r = \log r$ and $\log_{p+1} r = \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r = r$, $\log_0 r = r, \exp_{-1} r = \log_1 r, \log_{-1} r = \exp_1 r$.

Definition 1.3 [6] The iterated $p-$order of a meromorphic function f in Δ is defined as

$$\rho_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ T(r,f)}{\log \frac{1}{1-r}} \quad (p \geq 1).$$

For an analytic function f in Δ, we also define

$$\rho_{M,p}(f) = \limsup_{r \to 1^-} \frac{\log_{p+1}^+ M(r,f)}{\log \frac{1}{1-r}} \quad (p \geq 1).$$

Remark 1.1 It follows by M. Tsuji in [22] that if f is an analytic function in Δ, then

$$\rho_1(f) \leq \rho_{M,1}(f) \leq \rho_1(f) + 1.$$

However, it follows by Proposition 2.2.2 in [18]

$$\rho_{M,p}(f) = \rho_p(f) \quad (p \geq 2).$$

Definition 1.4 [6] The growth index of the iterated order of a meromorphic function $f(z)$ in Δ is defined as

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min\{j \in \mathbb{N} : \rho_j(f) < \infty\}, & \text{if } f \text{ is admissible} \\ & \text{and } \rho_j(f) < \infty \text{ for some } j \in \mathbb{N}, \\ +\infty, & \text{if } \rho_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

For an analytic function f in Δ, we also define

$$i_M(f) = \begin{cases} 0, & \text{if } f \text{ is of finite degree,} \\ \min\{j \in \mathbb{N} : \rho_{M,j}(f) < \infty\}, & \text{if } f \text{ is of infinite degree} \\ & \text{and } \rho_{M,j}(f) < \infty \text{ for some } j \in \mathbb{N}, \\ +\infty, & \text{if } \rho_{M,j}(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

Definition 1.5 ([13], [16]) The iterated $p-$type of a meromorphic function f of iterated $p-$order $\rho_p(f)$ $(0 < \rho_p(f) < \infty)$ in Δ is defined as

$$\tau_p(f) = \limsup_{r \to 1^-} (1-r)^{\rho_p(f)} \log_{p-1}^+ T(r,f).$$

Definition 1.6 [7] Let f be a meromorphic function in Δ. Then the iterated exponent of convergence of the sequence of zeros of $f(z)$ is defined as

$$\lambda_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ N\left(r, \frac{1}{f}\right)}{\log \frac{1}{1-r}},$$

where $N\left(r, \frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z \in \mathbb{C} : |z| < r\}$. Similarly, the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined as

$$\overline{\lambda}_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ \overline{N}\left(r, \frac{1}{f}\right)}{\log \frac{1}{1-r}},$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z \in \mathbb{C} : |z| < r\}$.

Definition 1.7 [7] The growth index of the convergence exponent of the sequence of zeros of a meromorphic $f(z)$ in Δ is defined as

$$i_\lambda(f) = \begin{cases} 0, & \text{if } N\left(r, \frac{1}{f}\right) = O\left(\log \frac{1}{1-r}\right), \\ \min\{j \in \mathbb{N} : \lambda_j(f) < \infty\}, & \text{if some } j \in \mathbb{N} \text{ with } \lambda_j(f) < \infty \text{ exists}, \\ +\infty, & \text{if } \lambda_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

Remark 1.2 Similarly, we can define the finiteness degree $i_{\overline{\lambda}}(f)$ of $\overline{\lambda}_p(f)$.

Consider for $k \geq 2$ the complex differential equation

$$f^{(k)} + A(z) f = 0 \tag{1.1}$$

and the differential polynomial

$$g_f = d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_1 f' + d_0 f, \tag{1.2}$$

where A and d_j $(j = 0, 1, \cdots, k)$ are meromorphic functions in Δ.

Let $\mathcal{L}(\mathbf{G})$ denote a differential subfield of the field $\mathcal{M}(\mathbf{G})$ of meromorphic functions in a domain $\mathbf{G} \subset \mathbb{C}$. Throughout this paper, we simply denote $\mathcal{L}$ instead of $\mathcal{L}(\Delta)$. Special case of such differential subfield

$$\mathcal{L}_{p+1,\rho} = \{g \text{ meromorphic in } \Delta\colon\ \rho_{p+1}(g) < \rho\},$$

where ρ is a positive constant. Recently, T. B. Cao, H. Y. Xu and C. X. Zhu [8], T. B. Cao, L. M. Li, J. Tu and H. Y. Xu [10] have studied the complex oscillation of differential polynomial generated by meromorphic and analytic solutions of second order linear differential equations with meromorphic coefficients and obtained the following results.

Theorem A [10] *Let $A(z)$ be an analytic function of infinite degree and of finite iterated order $\rho_{M,p}(A) = \rho > 0$ in the unit disc Δ, and let $f \not\equiv 0$ be a solution of the equation*

$$f'' + A(z) f = 0. \tag{1.3}$$

Moreover, let

$$P[f] = P\left(f, f', \cdots, f^{(m)}\right) = \sum_{j=0}^{m} p_j f^{(j)} \tag{1.4}$$

be a linear differential polynomial with analytic coefficients $p_j \in \mathcal{L}_{p+1,\rho}$, assuming that at least one of the coefficients p_j does vanish identically. If $\varphi(z) \in \mathcal{L}_{p+1,\rho}$ is a non-zero analytic function in Δ, and neither $P[f]$ nor $P[f] - \varphi$ vanishes identically, then we have

$$i_{\overline{\lambda}}(P[f] - \varphi) = i(f) = p + 1$$

and

$$\overline{\lambda}_{p+1}(P[f] - \varphi) = \rho_{M,p+1}(f) = \rho_{M,p}(A) = \rho.$$

Theorem B [8] *Let A be an admissible meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ $(1 \leq p < \infty)$ in the unit disc Δ such that $\delta(\infty, A) =$*

$\liminf\limits_{r\to 1^-}\frac{m(r,A)}{T(r,A)}=\delta>0$, *and let* f *be a non-zero meromorphic solution of equation* (1.3) *such that* $\delta(\infty,f)>0$. *Moreover, let be a linear differential polynomial* (1.4) *with meromorphic coefficients* $p_j\in\mathcal{L}_{p+1,\rho}$, *assuming that at least one of the coefficients* p_j *does not vanish identically. If* $\varphi\in\mathcal{L}_{p+1,\rho}$ *is a non-zero meromorphic function in* Δ, *and neither* $P[f]$ *nor* $P[f]-\varphi$ *vanishes identically, then we have*

$$i(f)=i_{\overline{\lambda}}(P[f]-\varphi)=p+1$$

and

$$\overline{\lambda}_p(P[f]-\varphi)=\rho_{p+1}(f)=\rho_p(A)=\rho$$

if $p>1$, *while*

$$\rho_p(A)\leq\overline{\lambda}_p(P[f]-\varphi)\leq\rho_{p+1}(f)\leq\rho_p(A)+1$$

if $p=1$.

Remark 1.3 The idea of the proofs of Theorems A-B is borrowed from the paper of Laine, Rieppo [19] with the modifications reflecting the change from the complex plane $\mathbb{C}$ to the unit disc $\boldsymbol{\Delta}$.

Before we state our results, we define the sequence of meromorphic functions $\alpha_{i,j}$ $(j=0,\cdots,k-1)$ in Δ by

$$\alpha_{i,j}=\begin{cases}\alpha'_{i,j-1}+\alpha_{i-1,j-1}, & \text{for all } i=1,\cdots,k-1,\\ \alpha'_{0,j-1}-A\alpha_{k-1,j-1}, & \text{for } i=0\end{cases}\tag{1.5}$$

and

$$\alpha_{i,0}=\begin{cases}d_i, & \text{for all } i=1,\cdots,k-1,\\ d_0-d_kA, & \text{for } i=0.\end{cases}\tag{1.6}$$

We define also h and $\psi(z)$ by

$$h=\begin{vmatrix}\alpha_{0,0} & \alpha_{1,0} & \cdot & \cdot & \alpha_{k-1,0}\\ \alpha_{0,1} & \alpha_{1,1} & \cdot & \cdot & \alpha_{k-1,1}\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \alpha_{0,k-1} & \alpha_{1,k-1} & \cdot & \cdot & \alpha_{k-1,k-1}\end{vmatrix},\tag{1.7}$$

$$\psi(z)=C_0\varphi+C_1\varphi'+\cdots+C_{k-1}\varphi^{(k-1)},\tag{1.8}$$

where C_j $(j=0,\cdots,k-1)$ are finite iterated $p-$order meromorphic functions in Δ depending on $\alpha_{i,j}$ and $\varphi\not\equiv 0$ is a meromorphic function in Δ with $\rho_p(\varphi)<\infty$.

The main purpose of this paper is to study the growth and oscillation of differential polynomial (1.2) generated by meromorphic solutions of equation (1.1) in the unit disc Δ.

Theorem 1.1 *Suppose that* $A(z)$ *is a meromorphic function of finite iterated* $p-$*order in* Δ *and that* $d_j(z)$ $(j=0,1,\cdots,k)$ *are finite iterated* $p-$*order meromorphic functions in* Δ *that are not all vanishing identically such that* $h\not\equiv 0$. *If* $f(z)$ *is an infinite iterated* $p-$*order meromorphic solution of* (1.1) *with* $\rho_{p+1}(f)=\rho$, *then the differential polynomial* (1.2) *satisfies*

$$\rho_p(g_f)=\rho_p(f)=\infty$$

and

$$\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho.$$

Furthermore, if f *is a finite iterated* $p-$*order meromorphic solution of* (1.1) *such that*

$$\rho_p(f) > \max\{\rho_p(A), \rho_p(d_j) \ \ (j = 0, 1, \cdots, k)\}, \tag{1.9}$$

then

$$\rho_p(g_f) = \rho_p(f).$$

Remark 1.4 In Theorem 1.1, if we do not have the condition $h \not\equiv 0$, then the conclusions of Theorem 1.1 cannot hold. For example, if we take $d_k = 1, d_0 = A$ and $d_j \equiv 0$ $(j = 1, \cdots, k-1)$, then $h \equiv 0$. It follows that $g_f \equiv 0$ and $\rho_p(g_f) = 0$. So, if $f(z)$ is an infinite iterated $p-$order meromorphic solution of (1.1), then $\rho_p(g_f) = 0 < \rho_p(f) = \infty$, and if f is a finite iterated $p-$order meromorphic solution of (1.1) such that (1.9) holds, then $\rho_p(g_f) = 0 < \rho_p(f)$.

Corollary 1.1 *Suppose that* $A(z)$ *is admissible meromorphic function in* Δ *such that* $i(A) = p$ $(1 \leq p < \infty)$ *and* $\delta(\infty, A) = \delta > 0$. *Let* $d_j(z)$ $(j = 0, 1, \cdots, k)$ *be finite iterated* $p-$*order meromorphic functions in* Δ *that are not all vanishing identically such that* $h \not\equiv 0$, *and let* f *be a nonzero meromorphic solution of* (1.1). *If* $\delta(\infty, f) > 0$, *then the differential polynomial* g_f *satisfies* $i(g_f) = p+1$ *and* $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A)$ *if* $p > 1$, *while*

$$\rho_p(A) \leq \rho_{p+1}(g_f) = \rho_{p+1}(f) \leq \rho_p(A) + 1$$

if $p = 1$.

Theorem 1.2 *Under the assumptions of Theorem 1.1, let* $\varphi(z) \not\equiv 0$ *be a meromorphic function with finite iterated* $p-$*order in* Δ *such that* $\psi(z)$ *is not a solution of* (1.1). *If* $f(z)$ *is an infinite iterated* $p-$*order meromorphic solution of* (1.1) *with* $\rho_{p+1}(f) = \rho$, *then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho.$$

Furthermore, if f *is a finite iterated* $p-$*order meromorphic solution of* (1.1) *such that*

$$\rho_p(f) > \max\{\rho_p(A), \rho_p(\varphi), \rho_p(d_j) \ \ (j = 0, 1, \cdots, k)\}, \tag{1.10}$$

then

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f).$$

Corollary 1.2 *Under the assumptions of Corollary 1.1, let* $\varphi(z) \not\equiv 0$ *be a meromorphic function with finite iterated* $p-$*order in* Δ *such that* $\psi(z) \not\equiv 0$. *Then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_p(A)$$

if $p > 1$, *while*

$$\rho_p(A) \leq \overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) \leq \rho_p(A) + 1$$

if $p = 1$.

Remark 1.5 The ideas of the proofs of Theorems 1.1 and 1.2 are from [21] with modification from the complex plane $\mathbb{C}$ to the unit disc $\mathbf{\Delta}$. For some papers related in the complex plane see $[19, 21, 4]$.

2. Auxiliary lemmas

Lemma 2.1 [9] *Let $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ be meromorphic functions in Δ, and let f be a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z) \tag{2.1}$$

such that

$$\max\{\rho_p(A_j) \quad (j = 0, 1, \cdots, k-1), \rho_p(F)\} < \rho_p(f) \leq +\infty.$$

Then

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f)$$

and

$$\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f).$$

Lemma 2.2 [6] *Let $p \geq 1$ be an integer, and let $A_0(z), \cdots, A_{k-1}(z)$ be analytic functions in Δ such that $i(A_0) = p$. If*

$$\max\{i(A_j) : j = 1, \cdots, k-1\} < p$$

or

$$\max\{\rho_p(A_j) : j = 1, \cdots, k-1\} < \rho_p(A_0),$$

then every solution $f \not\equiv 0$ of the equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{2.2}$$

satisfies $i(f) = p+1$ and $\rho_p(f) = \infty$, $\rho_p(A_0) \leq \rho_{p+1}(f) = \rho_{M,p+1}(f) \leq \max\{\rho_{M,p}(A_j) : j = 0, 1, \cdots, k-1\}$.

Lemma 2.3 [3] *Let f and g be meromorphic functions in the unit disc Δ such that $0 < \rho_p(f), \rho_p(g) < \infty$ and $0 < \tau_p(f), \tau_p(g) < \infty$. Then we have*
(i) *If $\rho_p(f) > \rho_p(g)$, then we obtain*

$$\tau_p(f+g) = \tau_p(fg) = \tau_p(f). \tag{2.3}$$

(ii) *If $\rho_p(f) = \rho_p(g)$ and $\tau_p(f) \neq \tau_p(g)$, then we get*

$$\rho_p(f+g) = \rho_p(fg) = \rho_p(f) = \rho_p(g). \tag{2.4}$$

Lemma 2.4 $[17, 2]$ *Let f be a meromorphic function in the unit disc for which $i(f) = p \geq 1$ and $\rho_p(f) = \beta < \infty$, and let $k \in \mathbb{N}$. Then for any $\varepsilon > 0$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left(\log\frac{1}{1-r}\right)^{\beta+\varepsilon}\right) \tag{2.5}$$

for all r outside a set $E \subset [0, 1)$ with $\int_E \frac{dr}{1-r} < \infty$.

Lemma 2.5 [8] *Let $A(z)$ be an admissible meromorphic function in Δ such that $i(A) = p$ $(1 \leq p < \infty)$ and $\delta(\infty, A) = \delta > 0$, and let f be a nonzero meromorphic* *solution of* (1.1). *If* $\delta(\infty, f) > 0$, *then* $i(f) = p+1$ *and* $\rho_{p+1}(f) = \rho_p(A)$ *if* $p > 1$, *while*

$$\rho_p(A) \le \rho_{p+1}(f) \le \rho_p(A) + 1$$

if $p = 1$.

Lemma 2.6 [1] *Let* $g : (0,1) \to \mathbb{R}$ *and* $h : (0,1) \to \mathbb{R}$ *be monotone increasing functions such that* $g(r) \le h(r)$ *holds outside of an exceptional set* $E_1 \subset [0,1)$ *for which* $\int_{E_1} \frac{dr}{1-r} < \infty$. *Then there exists a constant* $d \in (0,1)$ *such that if* $s(r) = 1 - d(1-r)$, *then* $g(r) \le h(s(r))$ *for all* $r \in [0,1)$.

3. Proofs of the Theorems and the Corollaries

Proof of Theorem 1.1 Suppose that f is an infinite iterated $p-$order meromorphic solution of (1.1) with $\rho_{p+1}(f) = \rho$. By (1.1) we have

$$f^{(k)} = -Af \tag{3.1}$$

which implies

$$\begin{aligned} g_f &= d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_0 f \\ &= d_{k-1} f^{(k-1)} + \cdots + (d_0 - d_k A) f. \end{aligned} \tag{3.2}$$

We can rewrite (3.2) as

$$g_f = \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i)}, \tag{3.3}$$

where $\alpha_{i,0}$ are defined in (1.6). Differentiating both sides of equation (3.3) and replacing $f^{(k)}$ with $f^{(k)} = -Af$, we obtain

$$\begin{aligned} g_f' &= \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,0} f^{(i)} \\ &= \alpha_{0,0}' f + \sum_{i=1}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} + \alpha_{k-1,0} f^{(k)} \\ &= \alpha_{0,0}' f + \sum_{i=1}^{k-1} \left(\alpha_{i,0}' + \alpha_{i-1,0}\right) f^{(i)} - \alpha_{k-1,0} A f \\ &= \sum_{i=1}^{k-1} \left(\alpha_{i,0}' + \alpha_{i-1,0}\right) f^{(i)} + \left(\alpha_{0,0}' - \alpha_{k-1,0} A\right) f. \end{aligned} \tag{3.4}$$

We can rewrite (3.4) as

$$g_f' = \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i)}, \tag{3.5}$$

where

$$\alpha_{i,1} = \begin{cases} \alpha_{i,0}' + \alpha_{i-1,0}, & \text{for all } i = 1, \cdots, k-1, \\ \alpha_{0,0}' - A\alpha_{k-1,0}, & \text{for } i = 0. \end{cases} \tag{3.6}$$

Differentiating both sides of equation (3.5) and replacing $f^{(k)}$ with $f^{(k)} = -Af$, we obtain

$$g_f'' = \sum_{i=0}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,1} f^{(i)}$$

$$= \alpha_{0,1}' f + \sum_{i=1}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,1} f^{(i)} + \alpha_{k-1,1} f^{(k)}$$

$$= \alpha_{0,1}' f + \sum_{i=1}^{k-1} \left(\alpha_{i,1}' + \alpha_{i-1,1}\right) f^{(i)} - \alpha_{k-1,1} A f$$

$$= \sum_{i=1}^{k-1} \left(\alpha_{i,1}' + \alpha_{i-1,1}\right) f^{(i)} + \left(\alpha_{0,1}' - \alpha_{k-1,1} A\right) f \tag{3.7}$$

which implies that

$$g_f'' = \sum_{i=0}^{k-1} \alpha_{i,2} f^{(i)}, \tag{3.8}$$

where

$$\alpha_{i,2} = \begin{cases} \alpha_{i,1}' + \alpha_{i-1,1}, \text{ for all } i = 1, \cdots, k-1, \\ \alpha_{0,1}' - A\alpha_{k-1,1}, \text{ for } i = 0. \end{cases} \tag{3.9}$$

By using the same method as above we can easily deduce that

$$g_f^{(j)} = \sum_{i=0}^{k-1} \alpha_{i,j} f^{(i)}, \ j = 0, 1, \cdots, k-1, \tag{3.10}$$

where

$$\alpha_{i,j} = \begin{cases} \alpha_{i,j-1}' + \alpha_{i-1,j-1}, \text{ for all } i = 1, \cdots, k-1, \\ \alpha_{0,j-1}' - A\alpha_{k-1,j-1}, \text{ for } i = 0 \end{cases} \tag{3.11}$$

and

$$\alpha_{i,0} = \begin{cases} d_i, \text{ for all } i = 1, \cdots, k-1, \\ d_0 - d_k A, \text{ for } i = 0. \end{cases} \tag{3.12}$$

By (3.3) – (3.12) we obtain the system of equations

$$\begin{cases} g_f = \alpha_{0,0} f + \alpha_{1,0} f' + \cdots + \alpha_{k-1,0} f^{(k-1)}, \\ g_f' = \alpha_{0,1} f + \alpha_{1,1} f' + \cdots + \alpha_{k-1,1} f^{(k-1)}, \\ g_f'' = \alpha_{0,2} f + \alpha_{1,2} f' + \cdots + \alpha_{k-1,2} f^{(k-1)}, \\ \cdots \\ g_f^{(k-1)} = \alpha_{0,k-1} f + \alpha_{1,k-1} f' + \cdots + \alpha_{k-1,k-1} f^{(k-1)}. \end{cases} \tag{3.13}$$

By Cramer's rule, and since $h \not\equiv 0$ we have

$$f = \frac{\begin{vmatrix} g_f & \alpha_{1,0} & . & . & \alpha_{k-1,0} \\ g_f' & \alpha_{1,1} & . & . & \alpha_{k-1,1} \\ . & . & . & . & . \\ . & . & . & . & . \\ g_f^{(k-1)} & \alpha_{1,k-1} & . & . & \alpha_{k-1,k-1} \end{vmatrix}}{h}. \tag{3.14}$$

So, we obtain

$$f = C_0 g_f + C_1 g_f' + \cdots + C_{k-1} g_f^{(k-1)}, \tag{3.15}$$

where C_j are finite iterated $p-$order meromorphic functions in Δ depending on $\alpha_{i,j}$, where $\alpha_{i,j}$ are defined in (3.11) and (3.12).

If $\rho_p(g_f) < +\infty$, then by (3.15) we obtain $\rho_p(f) < +\infty$, and this is a contradiction. Hence $\rho_p(g_f) = \rho_p(f) = +\infty$.

Now, we prove that $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$. By (3.2), we get $\rho_{p+1}(g_f) \leq \rho_{p+1}(f)$ and by (3.15) we have $\rho_{p+1}(f) \leq \rho_{p+1}(g_f)$. This yield $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$.

Furthermore, if f is a finite iterated $p-$order meromorphic solution of equation (1.1) such that

$$\rho_p(f) > \max\{\rho_p(A), \rho_p(d_j) \quad (j = 0, 1, \cdots, k)\}, \tag{3.16}$$

then

$$\rho_p(f) > \max\{\rho_p(\alpha_{i,j}) : i = 0, \cdots, k-1, j = 0, \cdots, k-1\}. \tag{3.17}$$

By (3.2) and (3.16) we have $\rho_p(g_f) \leq \rho_p(f)$. Now, we prove $\rho_p(g_f) = \rho_p(f)$. If $\rho_p(g_f) < \rho_p(f)$, then by (3.15) and (3.17) we get

$$\rho_p(f) \leq \max\{\rho_p(C_j) \quad (j = 0, \cdots, k-1), \rho_p(g_f)\} < \rho_p(f)$$

and this is a contradiction. Hence $\rho_p(g_f) = \rho_p(f)$.

Remark 3.1 From (3.15), it follows that the condition $h \not\equiv 0$ is equivalent to that $g_f, g_f', g_f'', ..., g_f^{(k-1)}$ are linearly independent over the field of meromorphic functions of finite iterated $p-$order in Δ.

Proof of Corollary 1.1 Suppose $f \not\equiv 0$ is a meromorphic solution of (1.1). Then, by Lemma 2.5, we have $i(f) = p+1$ and $\rho_{p+1}(f) = \rho_p(A)$ if $p > 1$, while

$$\rho_p(A) \leq \rho_{p+1}(f) \leq \rho_p(A) + 1$$

if $p = 1$. Thus, by Theorem 1.1 we obtain that the differential polynomial g_f satisfies $i(g_f) = p+1$ and $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A)$ if $p > 1$, while

$$\rho_p(A) \leq \rho_{p+1}(g_f) = \rho_{p+1}(f) \leq \rho_p(A) + 1$$

if $p = 1$.

Proof of Theorem 1.2 Suppose that f is an infinite iterated $p-$order meromorphic solution of equation (1.1) with $\rho_{p+1}(f) = \rho$. Set $w(z) = g_f - \varphi$. Since $\rho_p(\varphi) < \infty$, then by Theorem 1.1 we have $\rho_p(w) = \rho_p(g_f) = \infty$ and $\rho_{p+1}(w) = \rho_{p+1}(g_f) = \rho$. To prove $\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho$ we need to prove $\overline{\lambda}_p(w) = \lambda_p(w) = \infty$ and $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho$. By $g_f = w + \varphi$ and (3.15), we get

$$f = C_0 w + C_1 w' + \cdots + C_{k-1} w^{(k-1)} + \psi(z), \tag{3.18}$$

where

$$\psi(z) = C_0 \varphi + C_1 \varphi' + \cdots + C_{k-1} \varphi^{(k-1)}. \tag{3.19}$$

Substituting (3.18) into (1.1), we obtain

$$C_{k-1}w^{(2k-1)}+\sum_{i=0}^{2k-2}\phi_i w^{(i)}=-\left(\psi^{(k)}+A(z)\psi\right)=H, \tag{3.20}$$

where ϕ_i $(i=0,\cdots,2k-2)$ are meromorphic functions in Δ with finite iterated $p-$order. Since $\psi(z)$ is not a solution of (1.1), it follows that $H\not\equiv 0$. Then by Lemma 2.1, we obtain $\overline{\lambda}_p(w)=\lambda_p(w)=\infty$ and $\overline{\lambda}_{p+1}(w)=\lambda_{p+1}(w)=\rho$, i. e., $\overline{\lambda}_p(g_f-\varphi)=\lambda_p(g_f-\varphi)=\infty$ and $\overline{\lambda}_{p+1}(g_f-\varphi)=\lambda_{p+1}(g_f-\varphi)=\rho$.

Suppose that f is a finite iterated $p-$order meromorphic solution of equation (1.1) such that (1.10) holds. Set $w(z)=g_f-\varphi$. Since $\rho_p(\varphi)<\rho_p(f)$, then by Theorem 1.1 we have $\rho_p(w)=\rho_p(g_f)=\rho_p(f)$. To prove $\overline{\lambda}_p(g_f-\varphi)=\lambda_p(g_f-\varphi)=\rho_p(f)$ we need to prove $\overline{\lambda}_p(w)=\lambda_p(w)=\rho_p(f)$. Using the same reasoning as above, we get

$$C_{k-1}w^{(2k-1)}+\sum_{i=0}^{2k-2}\phi_i w^{(i)}=-\left(\psi^{(k)}+A(z)\psi\right)=F,$$

where C_{k-1}, ϕ_i $(i=0,\cdots,2k-2)$ are meromorphic functions in Δ with finite iterated $p-$order $\rho_p(C_{k-1})<\rho_p(w)$, $\rho_p(\phi_i)<\rho_p(w)$ $(i=0,\cdots,2k-2)$ and

$$\psi(z)=C_0\varphi+C_1\varphi'+\cdots+C_{k-1}\varphi^{(k-1)},\ \rho_p(F)<\rho_p(w).$$

Since $\psi(z)$ is not a solution of (1.1), it follows that $F\not\equiv 0$. Then by Lemma 2.1, we obtain $\overline{\lambda}_p(w)=\lambda_p(w)=\rho_p(f)$, i. e., $\overline{\lambda}_p(g_f-\varphi)=\lambda_p(g_f-\varphi)=\rho_p(f)$.

Proof of Corollary 1.2 Suppose that $f\not\equiv 0$ is a meromorphic solution of (1.1). Then, by Lemma 2.5, we have $i(f)=p+1$ and $\rho_{p+1}(f)=\rho_p(A)$ if $p>1$, while

$$\rho_p(A)\le\rho_{p+1}(f)\le\rho_p(A)+1$$

if $p=1$. Since $\psi\not\equiv 0$ and $\rho_p(\psi)<\infty$, then ψ cannot be a solution of equation (1.1). Thus, by Theorem 1.2 we obtain that the differential polynomial g_f satisfies

$$\overline{\lambda}_{p+1}(g_f-\varphi)=\lambda_{p+1}(g_f-\varphi)=\rho_{p+1}(f)=\rho_p(A)$$

if $p>1$, while

$$\rho_p(A)\le\overline{\lambda}_{p+1}(g_f-\varphi)=\lambda_{p+1}(g_f-\varphi)=\rho_{p+1}(f)\le\rho_p(A)+1$$

if $p=1$.

4. Discussions and applications

In this section, we consider the differential equation

$$f'''+A(z)f=0, \tag{4.1}$$

where $A(z)$ is a meromorphic function of finite iterated $p-$order in Δ. It is clear that the difficulty of the study of the differential polynomial generated by solutions lies in the calculation of the coefficients $\alpha_{i,j}$. We explain here that by using our method, the calculation of the coefficients $\alpha_{i,j}$ can be deduced easily. We study for example the growth of the differential polynomial

$$g_f=f'''+f''+f'+f. \tag{4.2}$$

We have

$$\left\{ \begin{array}{l} g_f = \alpha_{0,0} f + \alpha_{1,0} f' + \alpha_{2,0} f'', \\ g'_f = \alpha_{0,1} f + \alpha_{1,1} f' + \alpha_{2,1} f'', \\ g''_f = \alpha_{0,2} f + \alpha_{1,2} f' + \alpha_{2,2} f''. \end{array} \right. \tag{4.3}$$

By (1.6) we obtain

$$\alpha_{i,0} = \left\{ \begin{array}{c} 1, \text{ for all } i = 1,2, \\ 1 - A, \text{ for } i = 0. \end{array} \right. \tag{4.4}$$

Now, by (3.6) we get

$$\alpha_{i,1} = \left\{ \begin{array}{c} \alpha'_{i,0} + \alpha_{i-1,0}, \text{ for all } i = 1,2 \\ \alpha'_{0,0} - A\alpha_{2,0}, \text{ for } i = 0. \end{array} \right.$$

Hence

$$\left\{ \begin{array}{c} \alpha_{0,1} = \alpha'_{0,0} - A\alpha_{2,0} = -A' - A, \\ \alpha_{1,1} = \alpha'_{1,0} + \alpha_{0,0} = 1 - A, \\ \alpha_{2,1} = \alpha'_{2,0} + \alpha_{1,0} = 1. \end{array} \right. \tag{4.5}$$

Finally, by (3.11) we have

$$\alpha_{i,2} = \left\{ \begin{array}{c} \alpha'_{i,1} + \alpha_{i-1,1}, \text{ for all } i = 1,2, \\ \alpha'_{0,1} - A\alpha_{2,1}, \text{ for } i = 0. \end{array} \right.$$

So, we obtain

$$\left\{ \begin{array}{c} \alpha_{0,2} = \alpha'_{0,1} - A\alpha_{2,1} = -A'' - A' - A, \\ \alpha_{1,2} = \alpha'_{1,1} + \alpha_{0,1} = -2A' - A, \\ \alpha_{2,2} = \alpha'_{2,1} + \alpha_{1,1} = 1 - A. \end{array} \right. \tag{4.6}$$

Hence

$$\left\{ \begin{array}{c} g_f = (1-A) f + f' + f'', \\ g'_f = (-A' - A) f + (1-A) f' + f'', \\ g''_f = (-A'' - A' - A) f + (-2A' - A) f' + (1-A) f'' \end{array} \right. \tag{4.7}$$

and

$$h = \left| \begin{array}{lll} 1-A & 1 & 1 \\ -A' - A & 1-A & 1 \\ -A'' - A' - A & -2A' - A & 1-A \end{array} \right|$$

$$= 3A' - A - AA' - AA'' + A^2 - A^3 + 2(A')^2 + 1. \tag{4.8}$$

Suppose that $h \not\equiv 0$, by simple calculations we have

$$f = \frac{Ag''_f + (-1 - 2A') g'_f + \left(1 - A + 2A' + A^2\right) g_f}{h} \tag{4.9}$$

and by different conditions on the solution f we can ensure that

$$\rho_p \left(f''' + f'' + f' + f\right) = \rho_p (f).$$

Turning now to the problem of oscillation, for that we consider a meromorphic function $\varphi(z) \not\equiv 0$ of finite iterated $p-$order in Δ. From (4.9) we get

$$f = \frac{Aw'' + (-1 - 2A') w' + \left(1 - A + 2A' + A^2\right) w}{h} + \psi(z), \tag{4.10}$$

where $w = g_f - \varphi$ and

$$\psi(z) = \frac{A\varphi'' + (-1-2A')\varphi' + \left(1 - A + 2A' + A^2\right)\varphi}{h}. \tag{4.11}$$

Hence

$$f = \frac{A}{h}w'' + C_1 w' + C_0 w + \psi, \tag{4.12}$$

where

$$C_1 = -\frac{1+2A'}{h},\ C_0 = \frac{1 - A + 2A' + A^2}{h}.$$

Substituting (4.12) into (4.1), we obtain

$$\frac{A}{h}w^{(5)} + \sum_{i=0}^{4}\phi_i w^{(i)} = -\left(\psi^{(3)} + A(z)\psi\right),$$

where ϕ_i $(i = 0, \cdots, 4)$ are meromorphic functions in Δ with finite iterated $p-$order. Suppose that all meromorphic solutions $f \not\equiv 0$ of (4.1) are of infinite iterated $p-$order and $\rho_{p+1}(f) = \rho$. If $\psi \not\equiv 0$, then by Lemma 2.1 we obtain

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f) = +\infty \tag{4.13}$$

and

$$\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho. \tag{4.14}$$

Suppose that f is a meromorphic solution of (4.1) of finite iterated $p-$order such that

$$\rho_p(f) > \max\{\rho_p(A), \rho_p(\varphi)\}.$$

If $\psi^{(3)} + A(z)\psi \not\equiv 0$, then by Lemma 2.1 we obtain

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f).$$

Finally, we can state the following two results without the additional conditions $h \not\equiv 0$ and ψ is not a solution of (4.1).

Theorem 4.1 *Suppose that $A(z)$ is analytic function in Δ of finite iterated $p-$order $0 < \rho_p(A) < \infty$ and $0 < \tau_p(A) < \infty$, and that $d_j(z)$ $(j = 0, 1, 2, 3)$ are finite iterated $p-$order analytic functions in Δ that are not all vanishing identically such that*

$$\max\{\rho_p(d_j)\ \ (j = 0, 1, 2, 3)\} < \rho_p(A).$$

If f is a nontrivial solution of (4.1), then the differential polynomial

$$g_f = d_3 f^{(3)} + d_2 f'' + d_1 f' + d_0 f \tag{4.15}$$

satisfies

$$\rho_p(g_f) = \rho_p(f) = \infty$$

and

$$\rho_p(A) \le \rho_{p+1}(g_f) = \rho_{p+1}(f) \le \rho_{M,p}(A).$$

Theorem 4.2 *Under the assumptions of Theorem 4.1, let $\varphi(z) \not\equiv 0$ be an analytic function in Δ with finite iterated $p-$order. If f is a nontrivial solution of (4.1), then the differential polynomial $g_f = d_3 f^{(3)} + d_2 f'' + d_1 f' + d_0 f$ $(d_3 \not\equiv 0)$ satisfies*

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f) = \infty \tag{4.16}$$

and

$$\rho_p(A) \le \overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) \le \rho_{M,p}(A). \tag{4.17}$$

Remark 4.1 The results obtained in Theorems 4.1 and 4.2 and refinement of Corollaries 1.1 and 1.2 respectively.

Proof of Theorem 4.1 Suppose that f is a nontrivial solution of (4.1). Then by Lemma 2.2, we have

$$\rho_p(f) = \infty,\ \rho_p(A) \le \rho_{p+1}(f) \le \rho_{M,p}(A).$$

First, we suppose that $d_3 \not\equiv 0$. By the same reasoning as before we obtain that

$$h = \begin{vmatrix} H_0 & H_1 & H_2 \\ H_3 & H_4 & H_5 \\ H_6 & H_7 & H_8 \end{vmatrix},$$

where $H_0 = d_0 - d_3 A$, $H_1 = d_1$, $H_2 = d_2, H_3 = d_0' - (d_2 + d_3')A - d_3 A'$, $H_4 = d_0 + d_1' - d_3 A$, $H_5 = d_1 + d_2', H_6 = d_0'' - (d_1 + 2d_2' + d_3'')A - (d_2 + 2d_3')A' - d_3 A''$, $H_7 = 2d_0' + d_1'' - (d_2 + 2d_3')A - 2d_3 A'$, $H_8 = d_0 + 2d_1' + d_2'' - d_3 A$. Then

$$\begin{aligned}
h = (&3d_0d_1d_2 + 3d_0d_1d_3' + 3d_0d_2d_2' - 6d_0d_3d_1' + 3d_1d_2d_1' + 3d_1d_3d_0' \\
&+d_0d_2d_3'' - 2d_0d_3d_2'' + d_1d_2d_2'' + d_1d_3d_1'' + d_2d_3d_0'' + 2d_0d_2'd_3' + 2d_1d_1'd_3' - 4d_2d_0'd_3' \\
&+2d_2d_1'd_2' + 2d_3d_0'd_2' - d_1d_2'd_3'' + d_1d_3'd_2'' + d_2d_1'd_3'' - d_2d_1''d_3' - d_3d_1'd_2'' \\
&+d_3d_2'd_1'' - d_1^3 - 3d_0^2d_3 - 2d_1(d_2')^2 - 3d_1^2d_2' \ -2d_3(d_1')^2 - d_2^2d_1'' - d_1^2d_3'' - 3d_2^2d_0')\,A \\
&+ (2d_0d_2d_3' + 2d_0d_3d_2' - d_1d_2d_2' + 2d_1d_3d_1' - 4d_2d_3d_0' + d_1d_3d_2'' \\
&-d_2d_3d_1'' - 2d_1d_2'd_3' + 2d_2d_1'd_3' + 3d_0d_1d_3 + d_0d_2^2 - d_1^2d_2 + d_2^2d_1' - 2d_1^2d_3')\,A' \\
&+(d_2d_3d_1' + d_0d_2d_3 - d_1d_3d_2' - d_1^2d_3)A'' + (2d_2d_3d_3' - 3d_1d_3^2 + 2d_2^2d_3 - 2d_3^2d_2')AA' \\
&+ (d_2^3 - 3d_1d_2d_3 - 3d_1d_3d_3' - 3d_2d_3d_2' - d_2d_3d_3'' - 2d_3d_2'd_3' \\
&+3d_0d_3^2 + 3d_3^2d_1' + 2d_2(d_3')^2 + 3d_2^2d_3' + d_3^2d_2'')\,A^2 \\
&-d_3^3A^3 + 2d_2d_3^2(A')^2 - d_2d_3^2AA'' - 3d_0d_1d_0' - d_0d_1d_1'' - d_0d_2d_0'' - 2d_0d_0'd_2' \\
&+d_1d_0''d_2' + d_2d_0'd_1'' - d_2d_1'd_0'' + d_0^3 + 2d_0(d_1')^2 + 3d_0^2d_1' + 2d_2(d_0')^2 \\
&+d_1^2d_0'' + d_0^2d_2'' - 2d_1d_0'd_1' + d_0d_1'd_2'' - d_0d_2'd_1'' - d_1d_0'd_2''.
\end{aligned}$$

By $d_3 \not\equiv 0$, $A \not\equiv 0$ and Lemma 2.3, we have $\rho_p(h) = \rho_p(A)$, hence $h \not\equiv 0$. For the cases (i) $d_3 \equiv 0$, $d_2 \not\equiv 0$; (ii) $d_3 \equiv 0, d_2 \equiv 0$ and $d_1 \not\equiv 0$ by using a similar reasoning as above we get $h \not\equiv 0$. Finally, if $d_3 \equiv 0$, $d_2 \equiv 0$, $d_1 \equiv 0$ and $d_0 \not\equiv 0$, then we have $h = d_0^3 \not\equiv 0$. Hence $h \not\equiv 0$. By $h \not\equiv 0$, we obtain

$$f = \frac{1}{h}\begin{vmatrix} g_f & d_1 & d_2 \\ g_f' & d_0 + d_1' - d_3 A & d_1 + d_2' \\ g_f'' & 2d_0' + d_1'' - (d_2 + 2d_3')A - 2d_3 A' & d_0 + 2d_1' + d_2'' - d_3 A \end{vmatrix},$$

which we can write

$$f = \frac{1}{h}\left(D_0 g_f + D_1 g_f' + D_2 g_f''\right), \tag{4.18}$$

where

$$\begin{aligned}
D_0 = (&d_1d_2 - 2d_0d_3 + 2d_1d_3' + d_2d_2' - 3d_3d_1' - d_3d_2'' + 2d_2'd_3')\,A \\
&+ (2d_1d_3 + 2d_3d_2')\,A' + A^2d_3^2 + 3d_0d_1' - 2d_1d_0' + d_0d_2'' - d_1d_1''
\end{aligned}$$

$$-2d_0'd_2' + d_1'd_2'' - d_2'd_1'' + d_0^2 + 2(d_1')^2,$$
$$D_1 = \left(d_1 d_3 - 2d_2 d_3' - d_2^2\right) A + d_2 d_1'' - d_0 d_1 - 2d_1 d_1' + 2d_2 d_0' - d_1 d_2'',$$
$$D_2 = d_2 d_3 A + d_1^2 - d_2 d_1' + d_1 d_2' - d_0 d_2.$$

If $\rho_p(g_f) < +\infty$, then by (4.18) we obtain $\rho_p(f) < +\infty$, and this is a contradiction. Hence $\rho_p(g_f) = \rho_p(f) = +\infty$.

Now, we prove that $\rho_{p+1}(g_f) = \rho_{p+1}(f)$. By (4.15), we get $\rho_{p+1}(g_f) \leq \rho_{p+1}(f)$ and by (4.18) we have $\rho_{p+1}(f) \leq \rho_{p+1}(g_f)$. This yield $\rho_p(A) \leq \rho_{p+1}(g_f) = \rho_{p+1}(f) \leq \rho_{M,p}(A)$.

Proof of Theorem 4.2 By setting $w = g_f - \varphi$ in (4.18), we have

$$f = \frac{1}{h}\left(D_0 w + D_1 w' + D_2 w''\right) + \psi, \tag{4.19}$$

where

$$\psi = \frac{D_2\varphi'' + D_1\varphi' + D_0\varphi}{h}. \tag{4.20}$$

Since $d_3 \not\equiv 0$, then $h \not\equiv 0$. It follows by Theorem 4.1 that g_f is of infinite iterated $p-$order analytic function and $\rho_p(A) \leq \rho_{p+1}(g_f) \leq \rho_{M,p}(A)$. Since $\rho_p(\varphi) < \infty$, then we have $\rho_p(w) = \rho_p(g_f) = \rho_p(f) = \infty$ and $\rho_p(A) \leq \rho_{p+1}(w) = \rho_{p+1}(g_f) = \rho_{p+1}(f) \leq \rho_{M,p}(A)$. Substituting (4.19) into (4.1), we obtain

$$\frac{D_2}{h} w^{(5)} + \sum_{i=0}^{4} \phi_i w^{(i)} = -\left(\psi^{(3)} + A(z)\psi\right),$$

where ϕ_i $(i = 0, \cdots, 4)$ are meromorphic functions in Δ with finite iterated $p-$order. We prove first that $\psi \not\equiv 0$. Suppose that $\psi \equiv 0$, then (4.20) can be rewritten as

$$D_2\varphi'' + D_1\varphi' + D_0\varphi = 0 \tag{4.21}$$

and by Lemma 2.3, we have

$$\rho(D_0) > \max\{\rho(D_1), \rho(D_2)\}. \tag{4.22}$$

By (4.21) we obtain

$$D_0 = -\left(D_2\frac{\varphi''}{\varphi} + D_1\frac{\varphi'}{\varphi}\right).$$

Since $\rho_p(\varphi) = \beta < \infty$, then by Lemma 2.4 we have

$$T(r, D_0) \leq T(r, D_1) + T(r, D_2) + O\left(\exp_{p-2}\left(\log\frac{1}{1-r}\right)^{\beta+\varepsilon}\right), \ r \notin E,$$

where $E \subset [0,1)$ is a set with $\int_E \frac{dr}{1-r} < \infty$. Then, by using Lemma 2.6, we get

$$\rho_p(D_0) \leq \max\{\rho_p(D_1), \rho_p(D_2)\},$$

which is a contradiction. It is clear now that $\psi \not\equiv 0$ cannot be a solution of (4.1) because $\rho_p(\psi) < \infty$. Then, by Lemma 2.1 we obtain

$$\overline{\lambda}_p(w) = \lambda_p(w) = \overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f) = \infty$$

and

$$\rho_p(A) \leq \overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \overline{\lambda}_{p+1}(g_f - \varphi)$$
$$= \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) \leq \rho_{M,p}(A).$$

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Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (UMAB), B. P. 227 Mostaganem-(Algeria)

*Corresponding author