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GENERALIZED (h, r)-HARMONIC CONVEX FUNCTIONS AND INEQUALITIES

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Abstract. The main aim of this paper is to introduce a new class of harmonic convex functions with respect to non-negative function h, which is called generalized (h, r)-harmonic convex functions. We derive some new Fejer-Hermite-Hadamard type inequalities for generalized harmonic convex functions. Some special cases are also discussed. The ideas and techniques of this paper may stimulate further research.

1. Introduction

Convexity theory has become a rich source of inspiration in pure and applied sciences. This theory had not only stimulated new and deep results in many branches of mathematical and engineering sciences, but also provided us a unified and general framework for studying a wide class of unrelated problems. For recent applications, generalizations and other aspects of convex functions, see [1, 2, 4, 6, 8, 10, 11, 13–16, 18–23, 25, 27–29, 32, 35] and the references therein. Varosanec [31] introduced the class of h-convex functions with respect to an arbitrary non-negative function h, which is quite flexible and unifying one. Pearce et. al [18] generalized the Hermite-Hadamard inequality to a r-convex positive functions. Gordji et al. [3, 4] considered a new class of convex functions, which is called the generalized convex(φ-convex) functions. For some properties of the generalized convex functions, see [3–5]. Anderson et al. [1] and Iscan [9] introduced and studied the harmonic convex functions, which can be viewed as an important and siginificant generaliztion

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of convex functions. Noor et. al. [24] introduced and investigated new class of convex functions, which is called relative harmonic (s, η)-convex functions. They discussed some basic results of harmonic (s, η)-convex functions and also derived the Hermite-Hadamard and Fejer type inequalities for this class of functions. Noor et al. [17–23, 25, 27–29] have derived various error estimates for different classes of generalized convex functions.

Inspired and motivated by the ongoing research, we introduce the concept of (h, r)-harmonic convex functions with respect to an arbitrary nonnegative function h and $r \geq 0$. This class is more general and contains several new classes of harmonic r-convex functions functions as special cases. We discuss some properties of generalized harmonic r-convex function. We establish several Hermite-Hadamard inequalities for generalized harmonic r-convex function. Our results represent a significant refinement of the known results.

2. Preliminaries

In this section, we recall some basic concepts. Let $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous bifunction.

Definition 2.1. [13] *A set $I = [a, b] \subset \mathbb{R}$ is said to be a convex set, if*

$$(1-t)x + ty \in I, \qquad \forall x, y \in I, t \in [0, 1].$$

Definition 2.2. [13] *A function $f : I = [a, b] \to \mathbb{R}$ is said to be a convex function, if*

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0, 1].$$

Definition 2.3. [31] *Let $h : J = [0, 1] \to \mathbb{R}$ be a nonnegative function. A function $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be an h-convex function, if*

$$f((1-t)x + ty) \leq h(1-t)f(x) + h(t)f(y), \qquad \forall x, y \in I, t \in [0, 1].$$

Definition 2.4. [18] *A function $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ is r-convex, if f is positive and for all $x, y \in I$ and $t \in [0, 1]$, we have*

$$f((1-t)x + ty) = \left\{ \begin{array}{ll} ((1-t)[f(x)]^r + t[f(y)]^r)^{\frac{1}{r}} & , r \neq 0 \\ (f(x))^{1-t}(f(y))^t & , r = 0 \end{array} \right\}.$$

It is clear that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions. Ngoc et. al [12] obtained the Hermite-Hadamard inequality for r-convex function. Hap and Vinh [7] established a Hermite-Hadamard inequality for (h, r)-convex functions.

Gordi et al [4] introduced another class of convex functions, which is called the generalized convex functions.

Definition 2.5. [3,4] *A function $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be generalized convex (ϕ-convex) function, if and only if,*

$$f((1-t)x + ty) \leq f(x) + t\eta(f(y), f(x)), \qquad \forall x, y \in I, t \in [0,1].$$

Noor [14] introuced and studied the genralized r-convex functions.

Definition 2.6. [14] *A function $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be generalized r-convex, if f is positive and for all $x, y \in I$ and $t \in [0,1]$, we have*

$$f((1-t)x + ty) = \left\{ \begin{array}{ll} ((1-t)[f(x)]^r + t[f(x) + \eta(f(y), f(x))]^r)^{\frac{1}{r}} & , r \neq 0 \\ (f(x))^{1-t}(f(x) + \eta(f(y), f(x)))^t & , r = 0 \end{array} \right\}.$$

It is clear that generalized 0-convex functions are simply generalized log-convex functions [29] and generalized 1-convex functions are generalized convex (ϕ-convex) functions, see [3].

Definition 2.7. [33]. *A set $I = [a, b] \subset \mathbb{R} \setminus \{0\}$ is said to be a harmonic convex set, if*

$$\frac{xy}{tx + (1-t)y} \in I, \qquad \forall x, y \in I, t \in [0,1].$$

Definition 2.8. [9]. *A function $f : I = [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic convex function, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0,1].$$

In particular, it has been shown that f is a harmonic convex function, if and only if,

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2}\, dx \leq \frac{f(a) + f(b)}{2}, \qquad x \in [a, b],$$

which is called Hermite-Hadamard inequality for harmonic convex function.

Definition 2.9. [26] *Let $r \neq 0$ be a real number and $h : J \to \mathbb{R}$ be a nonnegative function. We say that $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is harmonic (h, r)-convex function, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq [h(1-t)[f(x)]^r + h(t)[f(y)]^r]^{\frac{1}{r}}, \qquad \forall x, y \in I, t \in [0,1].$$

It is clear that harmonic $(h, 0)$-convex functions are simply harmonic logarithmic h-convex functions and harmonic $(h, 1)$-convex functions are harmonic h-convex functions [17].

We now introduce some new concepts. Throughout this paper, we take $r \neq 0$, a real number, unless otherwise specified.

Definition 2.10. *Let $h : J = [0, 1] \to \mathbb{R}$ be a nonnegative function. A function $f : I = [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be generalized (h, r)-harmonic convex function, or f belongs to the class $HR(h, r)$, if and only if,*

$$f\left(\frac{xy}{tx + (1-t)y}\right) = \left\{ \begin{array}{ll} [h(1-t)[f(x)]^r + h(t)[f(x) + \eta(f(y), f(x))]^r]^{\frac{1}{r}} & , r \neq 0 \\ (f(x))^{h(1-t)}(f(x) + \eta(f(y), f(x)))^{h(t)} & , r = 0 \end{array} \right\}. \tag{2.1}$$

The function f is said to be generalized (h, r)-harmonic concave function, if and only if, -f is generalized (h, r)-harmonic convex function.

For $t = \frac{1}{2}$, we have

$$f\left(\frac{2xy}{x+y}\right) = \left\{ \begin{array}{ll} \left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}} \left([f(x)]^r + [f(x) + \eta(f(y), f(x))]^r\right)^{\frac{1}{r}} & , r \neq 0 \\ \sqrt{f(x)(f(x) + \eta(f(y), f(x)))} & , r = 0 \end{array} \right\}. \tag{2.2}$$

The function f is called Jensen generalized (h, r)-harmonic convex function.

We now discuss some special cases of generalized (h, r)-harmonic convex function, which appear to be new ones.

I. If $h(t) = t$ in Definition 2.10, then it reduces to the Definition of generalized r-harmonic convex functions.

Definition 2.11. *A function $f : I = [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is r-harmonic convex function, if and only if,*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq [(1-t)[f(x)]^r + t[f(x) + \eta(f(y), f(x))]^r]^{\frac{1}{r}}, \qquad \forall x, y \in I, t \in [0, 1].$$

II. If $r = 1$ in Definition 2.10, then it reduces to the Definition of generalized h- harmonic convex functions.

III. If $h(t) = t^s$ in Definition 2.10, then it reduces to the Definition of Breckner type of generalized (s, r)-harmonic convex functions.

Definition 2.12. *A function $f : I = [a, b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is (s, r)-harmonic convex function, where $s \in (0, 1)$, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq [(1-t)^s[f(x)]^r + t^s[f(x) + \eta(f(y), f(x))]^r]^{\frac{1}{r}}, \qquad \forall x, y \in I, t \in [0, 1].$$

IV. If $h(t) = t^{-s}$ in Definition 2.10, then it reduces to the Definition of Godunova-Levin type of generalized (s, r)-harmonic convex functions.

Definition 2.13. *A function $f : I = [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is Godunova-Levin type of generalized (s, r)-harmonic convex functions where $s \in (0, 1)$, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq [(1-t)^{-s}[f(x)]^r + t^{-s}[f(x) + \eta(f(y), f(x))]^r]^{\frac{1}{r}}, \qquad \forall x, y \in I, t \in (0, 1).$$

If $s = 1$, then Godunova-Levin type of generalized r-harmonic convex functions reduces to Godunova-Levin type of generalized $(1, r)$-harmonic convex functions.

Lemma 2.1. *Suppose that $a, b, c \in \mathbb{R}$. Then*

(1) $\min\{a, b\} \leq \frac{a+b}{2}$.

(2) *if* $c \geq 0$, $c.\min\{a,b\} = \min\{ca, cb\}$.

Remark 2.1. *If* $I = [a,b] \subset \mathbb{R} \setminus \{0\}$ *and if we consider the function* $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ *defined by* $g(t) = f\left(\frac{1}{t}\right)$, *then* f *is generalized* r*-harmonic convex on* $[a,b]$*, if and only if,* g *is generalized* r*-convex in the usual sense on* $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Generalized logarithmic means of order r of positive numbers x, y is defined by:

$$L_r(x,y) = \begin{cases} \frac{r}{r+1}\left(\frac{x^{r+1}-y^{r+1}}{x^r-y^r}\right), & r \neq \{-1,0\}, x \neq y \\ \frac{x-y}{\ln x - \ln y}, & r = 0, x \neq y \\ xy\frac{\ln x - \ln y}{x-y}, & r = -1, x \neq y \\ x, & x = y. \end{cases}$$

Minkowskis Inequality is stated as follows: Let

$$r \geq 1, \quad 0 < \int_a^b [f(x)]^r dx < \infty, \quad 0 < \int_a^b [g(x)]^r dx < \infty.$$

Then

$$\left(\int_a^b [f(x)+g(x)]^r dx\right)^{\frac{1}{r}} \leq \left(\int_a^b [f(x)]^r dx\right)^{\frac{1}{r}} + \left(\int_a^b [g(x)]^r dx\right)^{\frac{1}{r}}.$$

3. Main results

In this section, we obtain several new Hermite-Hadamard type inequalities for generalized (h,r)-harmonic convex functions.

Theorem 3.1. *Let* $f : I = [a,b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be a generalized* (h,r)*-harmonic convex function. If* $f \in L[a,b]$*, then*

$$\begin{aligned} \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \quad &\leq \quad \min\Big\{[[f(a)]^r + [f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}}\left(\int_0^1 [h(t)]^{\frac{1}{r}}\mathrm{d}t\right), \\ &\quad\quad [[f(b)]^r + [f(b)+\eta(f(a),f(b))]^r]^{\frac{1}{r}}\left(\int_0^1 [h(t)]^{\frac{1}{r}}\mathrm{d}t\right)\Big\} \\ &\leq \quad \frac{1}{2}\Big\{[[f(a)]^r + [f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}} \\ &\quad\quad +[[f(b)]^r + [f(b)+\eta(f(a),f(b))]^r]^{\frac{1}{r}}\Big\}\left(\int_0^1 [h(t)]^{\frac{1}{r}}\mathrm{d}t\right). \end{aligned}$$

Proof. Let f be a generalized (h,r)-harmonic convex function. Then

$$f\Big(\frac{ab}{ta+(1-t)b}\Big) \leq [h(1-t)[f(a)]^r + h(t)[f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}},$$

and

$$f\Big(\frac{ab}{(1-t)a+tb}\Big) \leq [h(1-t)[f(b)]^r + h(t)[f(b)+\eta(f(a),f(b))]^r]^{\frac{1}{r}}, \qquad \forall x,y \in I, t \in [0,1].$$

Thus, we have

$$\begin{aligned} & f(\frac{ab}{ta+(1-t)b})+f(\frac{ab}{(1-t)a+tb}) \\ \leq\ & [h(1-t)[f(a)]^r+h(t)[f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}} \\ & +[h(1-t)[f(b)]^r+h(t)[f(b)+\eta(f(a),f(b))]^r]^{\frac{1}{r}}, \end{aligned} \tag{3.1}$$

Integrating (3.1) over the interval $[0,1]$ and using Minkowskis inequality, we have

$$\begin{aligned} & \int_0^1 f(\frac{ab}{ta+(1-t)b})\mathrm{d}t+\int_0^1 f(\frac{ab}{(1-t)a+tb})\mathrm{d}t \\ \leq\ & \left[\left(\int_0^1 [h(1-t)]^{\frac{1}{r}} f(a)\mathrm{d}t\right)^r+\left(\int_0^1 [h(t)]^{\frac{1}{r}}[f(a)+\eta(f(b),f(a))]\mathrm{d}t\right)^r\right]^{\frac{1}{r}} \\ & +\left[\left(\int_0^1 [h(1-t)]^{\frac{1}{r}} f(b)\mathrm{d}t\right)^r+\left(\int_0^1 [h(t)]^{\frac{1}{r}}[f(b)+\eta(f(a),f(b))]\mathrm{d}t\right)^r\right]^{\frac{1}{r}} \\ =\ & \Big\{[[f(a)]^r+[f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}} \\ & +[[f(b)]^r+[f(b)+\eta(f(a),f(b))]^r]^{\frac{1}{r}}\Big\}\left(\int_0^1 [h(t)]^{\frac{1}{r}}\mathrm{d}t\right). \end{aligned}$$

This implies

$$\begin{aligned} \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq\ & \frac{1}{2}\Big\{[[f(a)]^r+[f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}} \\ & +[[f(b)]^r+[f(b)+\eta(f(a),f(b))]^r]^{\frac{1}{r}}\Big\}\left(\int_0^1 [h(t)]^{\frac{1}{r}}\mathrm{d}t\right), \end{aligned}$$

which is the required result. □

Corollary 3.1. *Under the assumptions of Theorem 3.1 with $r=1$, we have*

$$\begin{aligned} \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq\ & \min\Big\{f(a)\int_0^1 [h[(1-t)]+h(t)]\mathrm{d}t \\ & +\eta(f(b),f(a))\int_0^1 h(t)\mathrm{d}t, f(b)\int_0^1 [h[(1-t)]+h(t)]\mathrm{d}t \\ & +\eta(f(a),f(b))\int_0^1 h(t)\mathrm{d}t\Big\} \\ \leq\ & [f(a)+f(b)]\int_0^1 h(t)\mathrm{d}t \\ & +\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{2}\int_0^1 h(t)\mathrm{d}t. \end{aligned}$$

Theorem 3.2. *Let $f : I = [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a generalized (h,r)-harmonic convex function. If $f \in L[a,b]$, then*

$$\begin{aligned} & f\Big(\frac{2ab}{a+b}\Big) \\ \leq\ & \min\Bigg\{\Big[h\Big(\frac{1}{2}\Big)\Big]^{\frac{1}{r}}\Bigg(\Big[f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big]^r \\ & +\Big[f\Big(\frac{ab}{ta+(1-t)b}\Big)+\eta\Big(f\Big(\frac{ab}{(1-t)a+tb}\Big),f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big)\Big]^r\Bigg)^{\frac{1}{r}}, \\ & \Big[h\Big(\frac{1}{2}\Big)\Big]^{\frac{1}{r}}\Bigg(\Big[f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big]^r \\ & +\Big[f\Big(\frac{ab}{(1-t)a+tb}\Big)+\eta\Big(f\Big(\frac{ab}{ta+(1-t)b}\Big),f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big)\Big]^r\Bigg)^{\frac{1}{r}}\Bigg\}. \end{aligned} \tag{3.2}$$

Proof. Let f be a generalized (h,r)-harmonic convex function. Then, taking $x = \frac{ab}{ta+(1-t)b}$ and $y = \frac{ab}{(1-t)a+tb}$ in (2.2), we have

$$\begin{aligned} f\Big(\frac{2ab}{a+b}\Big) \leq\ & \Big[h\Big(\frac{1}{2}\Big)\Big]^{\frac{1}{r}}\Bigg(\Big[f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big]^r \\ & +\Big[f\Big(\frac{ab}{ta+(1-t)b}\Big)+\eta\Big(f\Big(\frac{ab}{(1-t)a+tb}\Big),f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big)\Big]^r\Bigg)^{\frac{1}{r}}, \end{aligned}$$

and

$$\begin{aligned} f\Big(\frac{2ab}{a+b}\Big) \leq\ & \Big[h\Big(\frac{1}{2}\Big)\Big]^{\frac{1}{r}}\Bigg(\Big[f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big]^r \\ & +\Big[f\Big(\frac{ab}{(1-t)a+tb}\Big)+\eta\Big(f\Big(\frac{ab}{ta+(1-t)b}\Big),f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big)\Big]^r\Bigg)^{\frac{1}{r}}. \end{aligned}$$

Thus,

$$\begin{aligned} & f\Big(\frac{2ab}{a+b}\Big) \\ \leq\ & \min\Bigg\{\Big[h\Big(\frac{1}{2}\Big)\Big]^{\frac{1}{r}}\Bigg(\Big[f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big]^r \\ & +\Big[f\Big(\frac{ab}{ta+(1-t)b}\Big)+\eta\Big(f\Big(\frac{ab}{(1-t)a+tb}\Big),f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big)\Big]^r\Bigg)^{\frac{1}{r}}, \\ & \Big[h\Big(\frac{1}{2}\Big)\Big]^{\frac{1}{r}}\Bigg(\Big[f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big]^r \\ & +\Big[f\Big(\frac{ab}{(1-t)a+tb}\Big)+\eta\Big(f\Big(\frac{ab}{ta+(1-t)b}\Big),f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big)\Big]^r\Bigg)^{\frac{1}{r}}\Bigg\}. \end{aligned}$$

The required result. □

Corollary 3.2. *Under the assumptions of Theorem 3.3 with $r = 1$, we have*

$$\begin{aligned} f\Big(\frac{2ab}{a+b}\Big) &\leq \min\Big\{h\Big(\frac{1}{2}\Big)\Big[2f\Big(\frac{ab}{ta+(1-t)b}\Big) \\ &+\eta\Big(f\Big(\frac{ab}{(1-t)a+tb}\Big), f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big)\Big], \\ &h\Big(\frac{1}{2}\Big)\Big[2f\Big(\frac{ab}{(1-t)a+tb}\Big) \\ &+\eta\Big(f\Big(\frac{ab}{ta+(1-t)b}\Big), f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big)\Big]\Big\}. \end{aligned}$$

Theorem 3.3. *Let $f : I = [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a generalized (h, r)-harmonic convex function. If $f \in L[a, b]$, then*

$$\begin{aligned} &\frac{2^{\frac{1-r}{r}}}{h(\frac{1}{2})}\Big(f\Big(\frac{2ab}{a+b}\Big)\Big)^r - \Big(\frac{ab}{b-a}\int_a^b \frac{f(x)+\eta(f(\frac{abx}{(a+b)x-ab}), f(x))}{x^2}\mathrm{d}x\Big)^r \\ \leq\ &\Big(\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big)^r \\ \leq\ &\frac{1}{2^r}\Big([[f(a)]^r + [f(a)+\eta(f(b), f(a))]^r]^{\frac{1}{r}} \\ &+[[f(b)]^r + [f(b)+\eta(f(a), f(b))]^r]^{\frac{1}{r}}\Big)^r\Big(\int_0^1 [h(t)]^{\frac{1}{r}}\mathrm{d}t\Big)^r. \end{aligned}$$

Proof. Let f be a generalized (h, r)-harmonic convex function. From inequality (3.2) and Lemma 2.1, we have

$$\begin{aligned} &\frac{2}{[h(\frac{1}{2})]^{\frac{1}{r}}} f\Big(\frac{2ab}{a+b}\Big) \\ \leq\ &\Big[\Big(\int_0^1 f\Big(\frac{ab}{ta+(1-t)b}\Big)\mathrm{d}t\Big)^r + \Big(\int_0^1 f\Big(\frac{ab}{(1-t)a+tb}\Big)\mathrm{d}t\Big)^r \\ &+\Big(\int_0^1 f\Big(\frac{ab}{ta+(1-t)b}\Big)\mathrm{d}t + \int_0^1 \eta\Big(f\Big(\frac{ab}{(1-t)a+tb}\Big), f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big)\mathrm{d}t\Big)^r \\ &+\Big(\int_0^1 f\Big(\frac{ab}{(1-t)a+tb}\Big)\mathrm{d}t + \int_0^1 \eta\Big(f\Big(\frac{ab}{ta+(1-t)b}\Big), f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big)\mathrm{d}t\Big)^r\Big]^{\frac{1}{r}} \\ =\ &\Big[2\Big(\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big)^r \\ &+2\Big(\frac{ab}{b-a}\int_a^b \frac{f(x)+\eta(f(\frac{abx}{(a+b)x-ab}), f(x))}{x^2}\mathrm{d}x\Big)^r\Big]^{\frac{1}{r}}. \end{aligned}$$

This implies

$$\begin{aligned}
&\frac{2^{\frac{1-r}{r}}}{h(\frac{1}{2})}\Big(f(\frac{2ab}{a+b})\Big)^r-\Big(\frac{ab}{b-a}\int_a^b\frac{f(x)+\eta(f(\frac{abx}{(a+b)x-ab}),f(x))}{x^2}\mathrm{d}x\Big)^r\\
\leq\ &\Big(\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\Big)^r\\
\leq\ &\frac{1}{2^r}\Big([[f(a)]^r+[f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}}\\
&+[[f(b)]^r+[f(b)+\eta(f(a),f(b))]^r]^{\frac{1}{r}}\Big)^r\Big(\int_0^1[h(t)]^{\frac{1}{r}}\mathrm{d}t\Big)^r,
\end{aligned}$$

which is the required result. □

Corollary 3.3. *Under the assumptions of Theorem 3.3 with $r=1$, we have*

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})}f(\frac{2ab}{a+b})-\frac{ab}{2(b-a)}\int_a^b\frac{\eta(f(\frac{abx}{(a+b)x-ab}),f(x))}{x^2}\mathrm{d}x\\
\leq\ &\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\\
\leq\ &[f(a)+f(b)]\int_0^1h(t)\mathrm{d}t+\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{2}\int_0^1h(t)\mathrm{d}t.
\end{aligned}$$

One can also obtain the Hermite-Hadamard inequality for generalized (h,r)-harmonic convex functions as:

$$\begin{aligned}
&\frac{1}{h(\frac{1}{2})}f^r\Big(\frac{2ab}{a+b}\Big)-\frac{ab}{b-a}\int_a^b\frac{[f(x)+\eta(f(\frac{abx}{(a+b)x-ab}),f(x))]^r}{x^2}\mathrm{d}x\\
\leq\ &\frac{ab}{b-a}\int_a^b\frac{f^r(x)}{x^2}\mathrm{d}x\\
\leq\ &[[f(a)]^r+[f(a)+\eta(f(b),f(a))]^r]\Big(\int_0^1[h(t)]^{\frac{1}{r}}\mathrm{d}t\Big)^r.
\end{aligned}$$

We now obtain some Fejer type integral inequalities for generalized (h,r)-harmonic convex functions.

Theorem 3.4. *Let $f,g:I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be generalized (h,r)-harmonic convex functions. If $fg\in L[a,b]$, the*

$$\begin{aligned}
&\int_a^b\frac{f(x)g(x)}{x^2}\mathrm{d}x\\
&\leq\frac{ab}{2(b-a)}\int_a^b[h\Big(\frac{a(b-x)}{x(b-a)}\Big)[f(a)]^r+h\Big(\frac{b(x-a)}{x(b-a)}\Big)[f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}}]\frac{g(x)}{x^2}\mathrm{d}x\\
&+\frac{ab}{2(b-a)}\int_a^b[h\Big(\frac{a(b-x)}{x(b-a)}\Big)[f(b)]^r+h\Big(\frac{b(x-a)}{x(b-a)}\Big)[f(b)+\eta(f(a),f(b))]^r]^{\frac{1}{r}}]\frac{g(x)}{x^2}\mathrm{d}x,
\end{aligned}$$

where $g:[a,b]\subset\mathbb{R}\setminus\{0\}$ is symmetric, nonnegative, integrable and satisfies

$$g(x)=g\big(\frac{abx}{[a+b]x-ab}\big),\quad\forall x\in[a,b].$$

Proof. Let f be a generalized harmonic r-convex function. Then, multiplying inequality (3.1) with $g\left(\frac{ab}{ta+(1-t)b}\right)$ and integrating over t, we have

$$\begin{aligned}
&\int_0^1 [f(\frac{ab}{ta+(1-t)b}) + f(\frac{ab}{(1-t)a+tb})]g(\frac{ab}{ta+(1-t)b})\mathrm{d}t \\
\leq\ &\int_0^1 [h(1-t)[f(a)]^r + h(t)[f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}}]g(\frac{ab}{ta+(1-t)b})\mathrm{d}t \\
&+\int_0^1 [h(1-t)[f(b)]^r + h(t)[f(b)+\eta(f(a),f(b))]^r]^{\frac{1}{r}}]g(\frac{ab}{ta+(1-t)b})\mathrm{d}t.
\end{aligned}$$

Since g is symmetric, we have

$$\begin{aligned}
&\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \\
\leq\ &\frac{1}{2}\int_a^b [h\left(\frac{a(b-x)}{x(b-a)}\right)[f(a)]^r + h\left(\frac{b(x-a)}{x(b-a)}\right)[f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}}]\frac{g(x)}{x^2}\mathrm{d}x \\
&+\frac{1}{2}\int_a^b [h\left(\frac{a(b-x)}{x(b-a)}\right)[f(b)]^r + h\left(\frac{b(x-a)}{x(b-a)}\right)[f(b)+\eta(f(a),f(b))]^r]^{\frac{1}{r}}]\frac{g(x)}{x^2}\mathrm{d}x,
\end{aligned}$$

the required result. □

Corollary 3.4. *Under the assumptions of Theorem 3.4 with $r = 1$, we have*

$$\begin{aligned}
&\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \\
\leq\ &\frac{f(a)+f(b)}{2}\int_a^b [h\left(\frac{a(b-x)}{x(b-a)}\right) + h\left(\frac{b(x-a)}{x(b-a)}\right)]\frac{g(x)}{x^2}\mathrm{d}x \\
&+\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{2}\int_a^b h\left(\frac{b(x-a)}{x(b-a)}\right)\frac{g(x)}{x^2}\mathrm{d}x.
\end{aligned}$$

Theorem 3.5. *Let $f, g : I = [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be generalized (h,r)-harmonic convex functions. If $fg \in L[a,b]$, then*

$$\begin{aligned}
f(\frac{2ab}{a+b})\int_a^b \frac{g(x)}{x^2}\mathrm{d}x \leq\ &\int_a^b \frac{g(x)}{x^2}\min\left\{\left[h(\frac{1}{2})\right]^{\frac{1}{r}}\left([f(x)]^r\right.\right. \\
&\left.+[f(x)+\eta(f\left(\frac{abx}{(a+b)x-ab}\right),f(x))]^r\right)^{\frac{1}{r}}, \\
&\left[h(\frac{1}{2})\right]^{\frac{1}{r}}\left(\left[f\left(\frac{abx}{(a+b)x-ab}\right)\right]^r\right. \\
&\left.\left.+[f\left(\frac{abx}{(a+b)x-ab}\right)+\eta(f(x),f\left(\frac{abx}{(a+b)x-ab}\right))]^r\right)^{\frac{1}{r}}\right\}\mathrm{d}x.
\end{aligned}$$

where $g : [a,b] \subset \mathbb{R}\setminus\{0\}$ is symmetric, nonnegative, integrable and satisfies

$$g(x) = g(\frac{abx}{[a+b]x-ab}), \quad \forall x \in [a,b].$$

Proof. Let f, g be generalized (h, r)-harmonic convex functions. Then multiplying (3.2) with $g\left(\frac{ab}{ta+(1-t)b}\right)$ and integrating over t, we have

$$\begin{aligned}
& f\Big(\frac{2ab}{a+b}\Big)\int_0^1 g\Big(\frac{ab}{ta+(1-t)b}\Big)\mathrm{d}t \\
\leq\ & \int_0^1 g\Big(\frac{ab}{ta+(1-t)b}\Big)\min\left\{\left[h\Big(\frac{1}{2}\Big)\right]^{\frac{1}{r}}\left(\left[f\Big(\frac{ab}{ta+(1-t)b}\Big)\right]^r\right.\right. \\
& \left.+\left[f\Big(\frac{ab}{ta+(1-t)b}\Big)+\eta\Big(f\Big(\frac{ab}{(1-t)a+tb}\Big), f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big)\right]^r\right)^{\frac{1}{r}}, \\
& \left[h\Big(\frac{1}{2}\Big)\right]^{\frac{1}{r}}\left(\left[f\Big(\frac{ab}{(1-t)a+tb}\Big)\right]^r\right. \\
& \left.\left.+\left[f\Big(\frac{ab}{(1-t)a+tb}\Big)+\eta\Big(f\Big(\frac{ab}{ta+(1-t)b}\Big), f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big)\right]^r\right)^{\frac{1}{r}}\right\}\mathrm{d}t.
\end{aligned}$$

By the symmetry of g on $[a, b]$, we have

$$\begin{aligned}
f\Big(\frac{2ab}{a+b}\Big)\int_a^b \frac{g(x)}{x^2}\mathrm{d}x \leq\ & \int_a^b \frac{g(x)}{x^2}\min\left\{\left[h\Big(\frac{1}{2}\Big)\right]^{\frac{1}{r}}\Big([f(x)]^r\right. \\
& \left.+\left[f(x)+\eta\left(f\left(\frac{abx}{(a+b)x-ab}\right), f(x)\right)\right]^r\Big)^{\frac{1}{r}}, \\
& \left[h\Big(\frac{1}{2}\Big)\right]^{\frac{1}{r}}\left(\left[f\left(\frac{abx}{(a+b)x-ab}\right)\right]^r\right. \\
& \left.\left.+\left[f\left(\frac{abx}{(a+b)x-ab}\right)+\eta\left(f(x), f\left(\frac{abx}{(a+b)x-ab}\right)\right)\right]^r\right)^{\frac{1}{r}}\right\}\mathrm{d}x,
\end{aligned}$$

which is the required result. ∎

Corollary 3.5. *Under the assumptions of Theorem 3.5 with $r = 1$, we have*

$$\begin{aligned}
& \frac{1}{2h(\frac{1}{2})} f\Big(\frac{2ab}{a+b}\Big)\int_a^b \frac{g(x)}{x^2}\mathrm{d}x \\
\leq\ & \int_a^b \frac{g(x)}{x^2}\min\left\{f(x)+\frac{1}{2}\eta\left(f\left(\frac{abx}{(a+b)x-ab}\right), f(x)\right),\right. \\
& \left. f\left(\frac{abx}{(a+b)x-ab}\right)+\frac{1}{2}\eta\left(f(x), f\left(\frac{abx}{(a+b)x-ab}\right)\right)\right\}\mathrm{d}x \\
\leq\ & \int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x+\frac{1}{2}\int_a^b \frac{g(x)}{x^2}\left[\eta\Big(f(x), f\Big(\frac{abx}{(a+b)x-ab}\Big)\Big)\right]\mathrm{d}x.
\end{aligned}$$

Theorem 3.6. *Let $f : I = [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be generalized r-harmonic convex function. If $fg \in L[a, b]$, then*

$$\begin{aligned}
& \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\
\leq & \begin{cases}
\frac{r}{r+1}\left(\frac{[f(a)]^{r+1}-[f(a)+\eta(f(b),f(a))]^{r+1}}{[f(a)]^{r}-[f(a)+\eta(f(b),f(a))]^{r}}\right), & r \neq \{-1, 0\}, f(a) \neq f(b) \\
\frac{\eta(f(b),f(a))}{\ln[f(a)+\eta(f(b),f(a))]-\ln f(a)}, & r = 0, f(a) \neq f(b) \\
f(a)[f(a)+\eta(f(b), f(a))]^{\frac{\ln[f(a)+\eta(f(b),f(a))]-\ln f(a)}{\eta(f(b),f(a))}}, & r = -1, f(a) \neq f(b) \\
f(a), & f(a) = f(b).
\end{cases}
\end{aligned}$$

Proof. Let f be a harmonic r-convex functions. Then

$$f(\frac{ab}{ta+(1-t)b}) \leq [(1-t)[f(a)]^r + [f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}},$$

I. The case $r \neq \{-1,0\}, f(a) \neq f(b)$.

$$\begin{aligned}
&\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\
= &\int_0^1 f(\frac{ab}{ta+(1-t)b})\mathrm{d}t \\
\leq &\int_0^1 [(1-t)[f(a)]^r + t[f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}}\mathrm{d}t \\
= &\frac{r}{r+1}\left(\frac{[f(a)]^{r+1}-[f(a)+\eta(f(b),f(a))]^{r+1}}{[f(a)]^r-[f(a)+\eta(f(b),f(a))]^r}\right).
\end{aligned}$$

II. The case $r = 0, f(a) \neq f(b)$.

$$\begin{aligned}
\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x &= \int_0^1 f(\frac{ab}{ta+(1-t)b})\mathrm{d}t \\
&\leq \int_0^1 [f(a)]^{1-t} + [f(a)+\eta(f(b),f(a))]^t\mathrm{d}t \\
&= \frac{\eta(f(b),f(a))}{\ln[f(a)+\eta(f(b),f(a))]-\ln f(a)}.
\end{aligned}$$

III. The case $r = -1, f(a) \neq f(b)$.

$$\begin{aligned}
&\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\
= &\int_0^1 f(\frac{ab}{ta+(1-t)b})\mathrm{d}t \\
\leq &\int_0^1 [(1-t)[f(a)]^{-1} + t[f(a)+\eta(f(b),f(a))]^{-1}]^{-1}\mathrm{d}t \\
= &\frac{f(a)[f(a)+\eta(f(b),f(a))]}{\eta(f(b),f(a))}\int_{f(a)}^{f(a)+\eta(f(b),f(a))}\frac{1}{u}\mathrm{d}u \\
= &f(a)[f(a)+\eta(f(b),f(a))]\frac{\ln[f(a)+\eta(f(b),f(a))]-\ln f(a)}{\eta(f(b),f(a))}.
\end{aligned}$$

IV. The case $f(a) = f(b)$ is obvious. This completes the proof. □

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