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GENERALIZED r-CONVEX FUNCTIONS AND INTEGRAL INEQUALITIES

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Abstract. In this paper, we introduce and investigate a new class of generalized convex functions, known as generalized r-convex function. Some new Hermite-Hadamard integral inequalities via generalized r-convex functions have been established. Results proved in this paper can be viewed as significant new contributions in this area of research.

1. Introduction

In order to obtain some desirable results in general framework related to pure and applied sciences, the concept of convexity has been extended and generalized in several directions, see [1, 2, 4, 5, 7, 11–17]. Several branches of mathematical and engineering sciences has been developed by using the crucial and significant concepts of convex analysis and hence it becomes one of the most interesting and useful concept of mathematics for last few decades.

Hermite-Hadamard inequality is one of the most important inequality related to convex function, see [9, 10]. In recent years, much attention has been given to derive the Hermite-Hadamard type inequalities for various types of convex functions, see [8–10, 17, 18, 22]. Gill et al [8] introduced and investigated the concept of r-convex functions. They established some new Hermite-Hadamard integral inequalities for r-convex functions. For the applications of r-convex functions in statistics and probability theory, see Pecaric et al. [25–27, 30, 31] and the references therein.

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Gordji et al. [7] has introduced an other class of convex functions, which is called generalized convex(φ-convex) functions. The generalized convex functions are nonconvex functions. For recent developments, see [3, 6, 18–21, 25–31] and the references therein. Inspired and motivated by the ongoing research, we introduce a new class of generalized convex functions, which is called generalized r-convex functions. We derive some new Hermite-Hadamard integral inequalities for these nonconvex functions. Some special cases are also discussed, which can be obtained from our main results. The ideas and techniques of this may be starting point for further research in this field.

2. Preliminaries

Let $I = [a, b]$ be an interval in real line $\mathbb{R}$. Let $f : I \to \mathbb{R}$ be a continuous function and $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous bifunction. We need the following well known results and concepts.

Definition 2.1. *[7] A function $f : I = [a, b] \to \mathbb{R}$ is said to be generalized convex function with respect to a bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f((1-t)a + tb) \leq (1-t)f(a) + t[f(a) + \eta(f(b), f(a))], \forall a, b \in I, t \in [0, 1].$$

Definition 2.2. *A function $f : I = [a, b] \to \mathbb{R}$ is said to be generalized r-convex function with respect to a bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if $\forall a, b \in I$, $t \in [0, 1]$*

$$f((1-t)a + tb) \leq \begin{cases} \left\{(1-t)[f(a)]^r + t[f(a) + \eta(f(b), f(a))]^r\right\}^{\frac{1}{r}}, & r \neq 0, \\ [f(a)]^{1-t}[f(a) + \eta(f(b), f(a))]^t, & r = 0. \end{cases}$$

1-convex is called generalized convex functions and 0-convex is called generalized log convex functions.

If $t = \frac{1}{2}$, then

$$f\left(\frac{a+b}{2}\right) \leq \begin{cases} \left\{\frac{[f(a)]^r + [f(a)+\eta(f(b),f(a))]^r}{2}\right\}^{\frac{1}{r}}, & r \neq 0, \\ \sqrt{[f(a)][f(a) + \eta(f(b), f(a))]}, & r = 0. \end{cases}$$

The function f is called generalized Jensen r-convex function.

If $\eta(f(b) - f(a)) = f(b) - f(a)$, then

Definition 2.3. *[8] A function $f : I = [a, b] \to \mathbb{R}$ is said to be r-convex function, if*

$$f((1-t)a + tb) \leq \begin{cases} \left\{(1-t)[f^r(a)] + t[f^r(b)]\right\}^{\frac{1}{r}}, & r \neq 0, \\ [f(a)]^{1-t}[f(b)]^t, & r = 0. \end{cases}$$

Note that for $r = 1$, we have classical convex functions and for $r = 0$, we have log convex functions. For the recent applications of convex functions and their generalizations, see [1–9, 12–15, 17, 18, 22–27, 31].

The generalized logarithmic means of order r of positive numbers a, b is defined by

$$L_r(a,b) = \begin{cases} \frac{r}{r+1}\frac{a^{r+1}-b^{r+1}}{a^r-b^r}, & r \neq 0, -1, a \neq b, \\ \frac{a-b}{\log a - \log b}, & r = 0, a \neq b, \\ ab\frac{\log a - \log b}{a-b}, & r = -1, a \neq b, \\ a, & a = b. \end{cases}$$

Definition 2.4. *The beta function, also called the Euler integral of the first kind, is defined as*

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0$$

where $\Gamma(.)$ is a Gamma function.

3. Main results

In this section, we establish several new integral inequalities of Hermite-Hadamard type for generalized r-convex functions.

Theorem 3.1. *Let $f : I \to \mathbb{R}$ be generalized r-convex function on I. Then for $0 < r \leq 1$, we have*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \left(\frac{r}{r+1}\right)\left\{\left([f(a)]^r + [f(a) + \eta(f(b), f(a))]^r\right)\right\}^{\frac{1}{r}}.$$

Proof. Let f be generalized r-convex function on I. Then, $\forall a, b \in I, t \in [0,1]$, we have

$$f((1-t)a + tb) \leq \left\{(1-t)[f(a)]^r + t[f(a) + \eta(f(b), f(a))]^r\right\}^{\frac{1}{r}}.$$

Using Minkowski's inequality and the fact that f is generalized r-convex function, we have

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x &= \int_0^1 f((1-t)a+tb)\mathrm{d}t \\ &\leq \int_0^1 \left\{(1-t)[f(a)]^r + t[f(a)+\eta(f(b),f(a))]^r\right\}^{\frac{1}{r}}\mathrm{d}t \\ &\leq \left\{\left(\int_0^1 (1-t)^{\frac{1}{r}}[f(a)]\mathrm{d}t\right)^r + \left(\int_0^1 t^{\frac{1}{r}}[f(a)+\eta(f(b),f(a))]\mathrm{d}t\right)^r\right\}^{\frac{1}{r}} \\ &= \left\{\left(\frac{r}{r+1}\right)^r\left([f(a)]^r + [f(a)+\eta(f(b),f(a))]^r\right)\right\}^{\frac{1}{r}} \\ &= \left(\frac{r}{r+1}\right)\left\{\left([f(a)]^r + [f(a)+\eta(f(b),f(a))]^r\right)\right\}^{\frac{1}{r}}, \end{aligned}$$

which is the required result. □

Corollary 3.1. *[29] If $\eta(f(b), f(a)) = f(b) - f(a)$, then, under the assumptions of Theorem 3.7, we have*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \left(\frac{r}{r+1}\right)\left\{\left([f^r(a) + f^r(b)]\right)\right\}^{\frac{1}{r}}.$$

Theorem 3.2. *Let $f : I \to \mathbb{R}$ be generalized r-convex function on I. Then for $0 < r \leq 1$, we have*

$$f^r(\frac{a+b}{2}) - \frac{1}{2(b-a)}\int_a^b \eta(f(a+b-x), f(x)]^r\mathrm{d}x \leq \frac{1}{2(b-a)}\int_a^b f^r(x)\mathrm{d}x$$
$$\leq \quad \left\{\frac{\{[f(a)]^r + [f(b)]^r\}}{4} + \frac{1}{4}\Big([f(a) + \eta(f(b), f(a))]^r + [f(b) + \eta(f(a), f(b))]^r\Big)\right\}.$$

Proof. Let f be generalized r-convex function on I. Then for $t = \frac{1}{2}$, we have

$$f^r(\frac{x+y}{2}) \leq \left\{\frac{[f(x)]^r + [f(x) + \eta(f(y), f(x))]^r}{2}\right\}.$$

This implies that

$$f^r(\frac{a+b}{2})$$
$$\leq \left\{\frac{[f((1-t)a+tb)]^r + [f((1-t)a+tb) + \eta(f(ta+(1-t)b), f((1-t)a+tb))]^r}{2}\right\}.$$

Integrating the above inequality with respect to t on [0,1], we have

$$\begin{aligned} f^r(\frac{a+b}{2}) \quad &\leq \quad \frac{1}{2}\int_0^1 \Big\{[f((1-t)a+tb)]^r + [f((1-t)a+tb) \\ &\quad +\eta(f(ta+(1-t)b), f((1-t)a+tb))]^r\Big\}\mathrm{d}t \\ &= \quad \frac{1}{2(b-a)}\int_a^b \Big\{[f(x)]^r + [f(x) + \eta(f(a+b-x), f(x))]^r\Big\}\mathrm{d}x. \end{aligned}$$

This implies

$$\begin{aligned} &f^r(\frac{a+b}{2}) - \frac{1}{2(b-a)}\int_a^b [f(x) + \eta(f(a+b-x), f(x))]^r\mathrm{d}x \\ &\leq \frac{1}{2(b-a)}\int_a^b f^r(x)\mathrm{d}x. \end{aligned} \tag{3.1}$$

Consider,

$$f^r((1-t)a+tb) \leq \Big\{(1-t)[f(a)]^r + t[f(a) + \eta(f(b), f(a))]^r\Big\}, \tag{3.2}$$

$$f^r((1-t)b+ta) \leq \Big\{(1-t)[f(b)]^r + t[f(b) + \eta(f(a), f(b))]^r\Big\}. \tag{3.3}$$

Adding (3.2) and (3.3), we have

$$\begin{aligned} f^r((1-t)a+tb) + f^r((1-t)b+ta) \quad &\leq \quad \Big\{(1-t)[f(a)]^r + t[f(a) + \eta(f(b), f(a))]^r\Big\} \\ &\quad +\Big\{(1-t)[f(b)]^r + t[f(b) + \eta(f(a), f(b))]^r\Big\}. \end{aligned}$$

Integrating the above inequality with respect to t on [0,1], we have

$$\frac{2}{b-a}\int_a^b f^r(x)\mathrm{d}x \leq \int_0^1 \Big\{(1-t)\big([f(a)]^r+[f(b)]^r\big)+t\Big([f(a)+\eta(f(b),f(a))]^r +[f(b)+\eta(f(a),f(b))]^r\Big)\Big\}\mathrm{d}t,$$

which implies that

$$\frac{1}{b-a}\int_a^b f^r(x)\mathrm{d}x \leq \Big\{\frac{\{[f(a)]^r+[f(b)]^r\}}{4}+\frac{1}{4}\Big([f(a)+\eta(f(b),f(a))]^r +[f(b)+\eta(f(a),f(b))]^r\Big)\Big\}. \tag{3.4}$$

Combining (3.1) and (3.4), we have

$$f^r(\frac{a+b}{2})-\frac{1}{2(b-a)}\int_a^b [f(x)+\eta(f(a+b-x),f(x))]^r\mathrm{d}x \leq \frac{1}{2(b-a)}\int_a^b f^r(x)\mathrm{d}x \leq \Big\{\frac{\{[f(a)]^r+[f(b)]^r\}}{4}+\frac{1}{4}\Big([f(a)+\eta(f(b),f(a))]^r+[f(b)+\eta(f(a),f(b))]^r\Big)\Big\},$$

which is the required result. □

Corollary 3.2. *[29] If $\eta(f(b),f(a))=f(b)-f(a)$, then, under the assumptions of Theorem 3.2, we have*

$$f^r(\frac{a+b}{2})\leq \frac{1}{b-a}\int_a^b f^r(x)\mathrm{d}x \leq \Big\{\frac{[f^r(a)+f^r(b)]}{2}\Big\}.$$

Theorem 3.3. *Let $f: I\to\mathbb{R}$ be generalized r-convex function on I and $r\geq 0$, then*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \begin{cases}\Big(\frac{r}{r+1}\Big)\Big\{\frac{\eta(f^{r+1}(b),f^{r+1}(a))}{\eta(f^r(b),f^r(a))}\Big\}, & r\neq 0\\ \Big\{\frac{[f(a)+\eta(f(b),f(a))]-[f(a)]}{\log[f(a)+\eta(f(b),f(a))]-\log[f(a)]}\Big\}, & r=0.\end{cases}$$

Proof. First, let $r>0$ and f be generalized r-convex function on I. Then, $\forall a,b\in I, t\in[0,1]$, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x &= \int_0^1 f((1-t)a+tb)\mathrm{d}t\\ &\leq \int_0^1\Big\{(1-t)[f(a)]^r+t[f(a)+\eta(f(b),f(a))]^r\Big\}^{\frac{1}{r}}\mathrm{d}t.\end{aligned} \tag{3.5}$$

Subsituting $u=[(1-t)[f(a)]^r+t[f(a)+\eta(f(b),f(a))]^r]$ in (3.5), we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x &\leq \frac{1}{[f(a)+\eta(f(b),f(a))]^r-[f(a)]^r}\int_{[f(a)]^r}^{[f(a)+\eta(f(b),f(a))]^r} u^{\frac{1}{r}}\mathrm{d}u\\ &= \Big(\frac{r}{r+1}\Big)\Big\{\frac{\big([f(a)+\eta(f(b),f(a))]^{r+1}-[f(a)]^{r+1}\big)}{[f(a)+\eta(f(b),f(a))]^r-[f(a)]^r}\Big\}\end{aligned}$$

For $r = 0$, we have

$$f((1-t)a+tb) \leq \left\{[f(a)]^{1-t}[f(a)+\eta(f(b),f(a))]^t\right\}.$$

Hence we have,

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x &= \int_0^1 f((1-t)a+tb)\mathrm{d}t \\ &\leq \int_0^1 \left\{[f(a)]^{1-t}[f(a)+\eta(f(b),f(a))]^t\right\}\mathrm{d}t \\ &= [f(a)]\int_0^1 \left\{\frac{[f(a)+\eta(f(b),f(a))]}{[f(a)]}\right\}^t \mathrm{d}t \\ &= \left\{\frac{[f(a)+\eta(f(b),f(a))]-[f(a)]}{\log[f(a)+\eta(f(b),f(a))]-\log[f(a)]}\right\},\end{aligned}$$

which is the required result. □

Corollary 3.3. *[8] If $\eta(f(b),f(a)) = f(b)-f(a)$, then, under the assumptions of Theorem 3.3, we have*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \begin{cases}\left(\frac{r}{r+1}\right)\left\{\frac{(f^{r+1}(b)-f^{r+1}(a))}{(f^r(b)-f^r(a))}\right\} = L_r\big(f(a),f(b)\big), & r\neq 0\\ \left\{\frac{[f(b)-f(a)]}{\log[f(b)]-\log[f(a)]}\right\} = L\big(f(a),f(b)\big), & r=0.\end{cases}$$

Theorem 3.4. *Let $f : I \to \mathbb{R}$ be generalized r-convex function on I and $r \geq 0$, then*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \begin{cases}\left\{[f^r(a)]+\left(\frac{r}{r+1}\right)^r\eta(f^r(a),f^r(a))]\right\}^{\frac{1}{r}}, r\neq 0, f(a)=f(b),\\ \frac{1}{\eta(f^{-1}(b),f^{-1}(a))}\big[\log[f^{-1}(a)+\eta(f^{-1}(b),f^{-1}(a))]-\log[f^{-1}(a)]\big], \quad r=-1.\end{cases}$$

Proof. First, let $r > 0$, f be generalized r-convex function on I and $f(a) = f(b)$. Then, $\forall a, b \in I, t \in [0,1]$, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x &= \int_0^1 f((1-t)a+tb)\mathrm{d}t \\ &\leq \int_0^1 \left\{(1-t)[f(a)]^r + t[f(a)+\eta(f(a),f(a))]^r\right\}^{\frac{1}{r}}\mathrm{d}t. \\ &= f(a)\end{aligned}$$

For $r = -1$ and $f(a) \neq f(b)$, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x &= \int_0^1 f((1-t)a+tb)\mathrm{d}t \\ &\le \int_0^1 \left\{(1-t)[f(a)]^{-1}+t[f(a)+\eta(f(b),f(a))]^{-1}\right\}^{-1}\mathrm{d}t \\ &= \frac{1}{[f(a)+\eta(f(b),f(a))]^{-1}-[f(a)]^{-1}}\int_{[f(a)]^{-1}}^{[f(a)+\eta(f(b),f(a))]^{-1}} \frac{1}{u}\mathrm{d}u \\ &= \frac{\left[\log[f(a)+\eta(f(b),f(a))]^{-1}-\log[f(a)]^{-1}\right]}{[f(a)+\eta(f(b),f(a))]^{-1}-[f(a)]^{-1}},\end{aligned}$$

which is the required result. □

Corollary 3.4. *[29] If $\eta(f(b),f(a)) = f(b)-f(a)$, then, under the assumptions of Theorem 3.4, we have*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \begin{cases} f(a), r\neq 0, f(a)=f(b), \\ f(a)f(b)\frac{\log[f(b)]-\log[f(a)]}{f(b)-f(a)} = L_{-1}(f(a),f(b), \quad r=-1. \end{cases}$$

Theorem 3.5. *Let $f,g : I \to \mathbb{R}$ be generalized r_1-convex function and generalized r_2-convex function respectively on I. Then for $r_1 > 0, r_2 > 0 >$, we have*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x &\le \left(\frac{r}{r+1}\right)\left\{\left([f(a)]^{r_1}+[f(a)+\eta(f(b),f(a))]^{r_1}\right)\right\}^{\frac{2}{r_1}} \\ &\quad +\left(\frac{r}{r+1}\right)\left\{\left([g(a)]^{r_2}[g(a)+\eta(g(b),g(a))]^{r_2}\right)\right\}^{\frac{2}{r_2}}.\end{aligned}$$

Proof. Let $f,g : I \to \mathbb{R}$ be generalized r_1-convex function and generalized r_2-convex function respectively on I with $(r_1 > 0, r_2 > 0)$. Then, $\forall a,b \in I, t\in[0,1]$, we have

$$f((1-t)a+tb) \le \left\{(1-t)[f(a)]^{r_1}+t[f(a)+\eta(f^{(}b),f(a))]^{r_1}\right\}^{\frac{1}{r_1}},$$

$$g((1-t)a+tb) \le \left\{(1-t)[g(a)]^{r_2}+t[g(a)+\eta(g(b),g(a))]^{r_2}\right\}^{\frac{1}{r_2}}.$$

Using Cauchy's and Minkowski's inequalities and the fact that f and g are generalized r_1 and r_2-convex functions, we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x &= \int_0^1 f((1-t)a+tb)g((1-t)a+tb)\mathrm{d}t \\
&\leq \int_0^1 \left\{(1-t)[f(a)]^{r_1}+t[f(a)+\eta(f^(b),f(a))]^{r_1}\right\}^{\frac{1}{r_1}} \\
&\quad \left\{(1-t)[g(a)]^{r_2}+t[g(a)+\eta(g(b),g(a))]^{r_2}\right\}^{\frac{1}{r_2}}\mathrm{d}t \\
&\leq \frac{1}{2}\int_0^1 \left\{(1-t)[f(a)]^{r_1}+t[f(a)+\eta(f(b),f(a))]^{r_1}\right\}^{\frac{2}{r_1}}\mathrm{d}t \\
&\quad +\frac{1}{2}\int_0^1 \left\{(1-t)[g(a)]^{r_2}+t[g(a)+\eta(g(b),g(a))]^{r_2}\right\}^{\frac{2}{r_2}}\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\Bigg\{\left(\int_0^1 (1-t)^{\frac{2}{r_1}}[f(a)]^2\mathrm{d}t\right)^{\frac{r_1}{2}} \\
&\quad +\left(\int_0^1 t^{\frac{2}{r_1}}[f(a)+\eta(f(b),f(a))]^2\mathrm{d}t\right)^{\frac{r_1}{2}}\Bigg\}^{\frac{2}{r_1}} \\
&\quad +\frac{1}{2}\Bigg\{\left(\int_0^1 (1-t)^{\frac{2}{r_2}}[g(a)]^2\mathrm{d}t\right)^{\frac{r_2}{2}} \\
&\quad +\left(\int_0^1 t^{\frac{2}{r_2}}[g(a)+\eta(g(b),g(a))]^2\mathrm{d}t\right)^{\frac{r_2}{2}}\Bigg\}^{\frac{2}{r_2}} \\
&= \left\{\left(\frac{r}{r+1}\right)^{\frac{r_1}{2}}\left([f(a)]^{r_1}+[f(a)+\eta(f(b),f(a))]^{r_1}\right)\right\}^{\frac{2}{r_1}} \\
&\quad +\left\{\left(\frac{r}{r+1}\right)^{\frac{r_2}{2}}\left([g(a)]^{r_2}[g(a)+\eta(g(b),g(a))]^{r_2}\right)\right\}^{\frac{2}{r_2}} \\
&= \left(\frac{r}{r+1}\right)\left\{\left([f(a)]^{r_1}+[f(a)+\eta(f(b),f(a))]^{r_1}\right)\right\}^{\frac{2}{r_1}} \\
&\quad +\left(\frac{r}{r+1}\right)\left\{\left([g(a)]^{r_2}[g(a)+\eta(g(b),g(a))]^{r_2}\right)\right\}^{\frac{2}{r_2}},
\end{aligned}$$

which is the required result. □

Corollary 3.5. *[29] If $\eta(f(b),f(a))=f(b)-f(a)$, then, under the assumptions of Theorem 3.5, we have*

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x &\leq \left(\frac{r}{r+1}\right)\left\{\left([f^{r_1}(a)+f^{r_1}(b)]\right)\right\}^{\frac{2}{r_1}} \\
&\quad +\left(\frac{r}{r+1}\right)\left\{\left([g^{r_2}(a)+g^{r_2}(b)]\right)\right\}^{\frac{2}{r_2}}.
\end{aligned}$$

Theorem 3.6. *Let $f, g : I \to \mathbb{R}$ be generalized r_1-convex function and generalized r_2-convex function respectively on I. Then for $r_1 > 0, r_2 > 0$ and $\frac{1}{r_1} + \frac{1}{r_2} = 1$, we have*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{1}{2}\Bigg\{\Big([f(a)]^{r_1} + [f(a)+\eta(f(b),f(a))]^{r_1}\Big)^{\frac{1}{r_1}} \Big([g(a)]^{r_2} + [g^{r_2}(a)+\eta(g(b),g(a))]^{r_2}\Big)^{\frac{1}{r_1}}\Bigg\}.$$

Proof. Let $f, g : I \to \mathbb{R}$ be generalized r_1-convex function and generalized r_2-convex function respectively on I with $(r_1 > 0, r_2 > 0)$. Then, $\forall a, b \in I, t \in [0,1]$, we have

$$f((1-t)a+tb) \leq \Big\{(1-t)[f(a)]^{r_1} + t[f(a)+\eta(f^{(}b),f(a))]^{r_1}\Big\}^{\frac{1}{r_1}}$$

$$g((1-t)a+tb) \leq \Big\{(1-t)[g(a)]^{r_2} + t[g(a)+\eta(g(b),g(a))]^{r_2}\Big\}^{\frac{1}{r_2}}.$$

Using Holder's inequality and the fact that f and g are generalized r_1 and r_2-convex functions, we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x &= \int_0^1 f((1-t)a+tb)g((1-t)a+tb)\mathrm{d}t \\
&\leq \int_0^1 \Big\{(1-t)[f(a)]^{r_1} + t[f(a)+\eta(f^{(}b),f(a))]^{r_1}\Big\}^{\frac{1}{r_1}} \\
&\quad \Big\{(1-t)[g(a)]^{r_2} + t[g(a)+\eta(g(b),g(a))]^{r_2}\Big\}^{\frac{1}{r_2}}\mathrm{d}t \\
&\leq \Big\{\int_0^1 (1-t)[f(a)]^{r_1} + t[f(a)+\eta(f(b),f(a))]^{r_1}\mathrm{d}t\Big\}^{\frac{1}{r_1}} \\
&\quad \Big\{\int_0^1 (1-t)[g(a)]^{r_2} + t[g(a)+\eta(g(b),g(a))]^{r_2}\mathrm{d}t\Big\}^{\frac{1}{r_2}} \\
&= \Big\{\Big([f(a)]^{r_1}\int_0^1 (1-t)\mathrm{d}t + [f(a)+\eta(f(b),f(a))]^{r_1}\int_0^1 t\mathrm{d}t\Big)\Big\}^{\frac{1}{r_1}} \\
&\quad \Big\{\Big([g(a)]^{r_2}\int_0^1 (1-t)\mathrm{d}t + [g(a)+\eta(g(b),g(a))]^{r_2}\int_0^1 t\mathrm{d}t\Big)\Big\}^{\frac{1}{r_2}} \\
&= \frac{1}{2}\Bigg\{\Big([f(a)]^{r_1} + [f(a)+\eta(f(b),f(a))]^{r_1}\Big)^{\frac{1}{r_1}} \\
&\quad \Big([g(a)]^{r_2} + [g^{r_2}(a)+\eta(g(b),g(a))]^{r_2}\Big)^{\frac{1}{r_1}}\Bigg\},
\end{aligned}$$

which is the required result. □

Corollary 3.6. *[29] If $\eta(f(b),f(a)) = f(b) - f(a)$, then, under the assumptions of Theorem 3.6, we have*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{\Big\{\Big([f^{r_1}(a)+f^{r_1}(b)]\Big)^{\frac{1}{r_1}}\Big([g^{r_2}(a)+g^{r_2}(b)\Big)^{\frac{1}{r_1}}\Big\}}{2}.$$

Theorem 3.7. *Let $f, g : I \to \mathbb{R}$ be generalized r-convex function on I. Then for $r > 0$, we have*

$$\left(\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\right)^r \leq \left\{M(a,b)\left(\frac{r}{r+2}\right)^r + N(a,b)\big(\beta(\frac{1}{r}+1,\frac{1}{r}+1)\big)^r\right\}.$$

where

$$M(a,b) = \left([f^r(a)g^r(a)] + [f(a)+\eta(f(b),f(a))]^r[g(a)+\eta(g(b),g(a))]^r\right)$$
$$N(a,b) = \left([f^r(a)][g(a)+\eta(g(b),g(a))]^r + [g^r(a)][f(a)+\eta(f(b),f(a))]^r\right),$$

and $\beta(\cdot,\cdot)$ is the Beta function.

Proof. Let f, g be two generalized r-convex functions on I. Then $\forall a, b \in I, t \in [0,1]$, we have

$$f((1-t)a+tb) \leq \left\{(1-t)[f(a)]^r + t[f(a)+\eta(f(b),f(a))]^r\right\}^{\frac{1}{r}}$$
$$g((1-t)a+tb) \leq \left\{(1-t)[g(a)]^r + t[g(a)+\eta(g(b),g(a))]^r\right\}^{\frac{1}{r}}.$$

Using Minkowski's inequality and the fact that f and g are generalized r-convex functions, we have

$$\begin{aligned}
& \left(\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\right)^r \\
= & \left(\int_0^1 f((1-t)a+tb)g((1-t)a+tb)\mathrm{d}t\right)^r \\
\leq & \left\{\int_0^1 \left((1-t)[f(a)]^r + t[f(a)+\eta(f(b),f(a))]^r\right.\right. \\
& \left.\left.(1-t)[g(a)]^r + t[g(a)+\eta(g(b),g(a))]^r\right)^{\frac{1}{r}}\mathrm{d}t\right\}^r \\
= & \left\{\int_0^1 \left((1-t)^2[f^r(a)g^r(a)] + t(1-t)\right.\right. \\
& \left([f^r(a)][g(a)+\eta(g(b),g(a))]^r + [g^r(a)][f(a)+\eta(f(b),f(a))]^r\right) \\
& \left.\left.+t^2\left([f(a)+\eta(f(b),f(a))]^r[g(a)+\eta(g(b),g(a))]^r\right)\right\}^r
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \Bigg\{[f^r(a)g^r(a)]\Bigg(\int_0^1 (1-t)^{\frac{2}{r}}\mathrm{d}t\Bigg)^r \\
& +\Bigg([f^r(a)][g(a)+\eta(g(b),g(a))]^r+[g^r(a)][f(a)+\eta(f(b),f(a))]^r\Bigg) \\
& \Bigg(\int_0^1 [t(1-t)]^{\frac{1}{r}}\mathrm{d}t\Bigg)^r \\
& +\Bigg([f(a)+\eta(f(b),f(a))]^r[g(a)+\eta(g(b),g(a))]^r\Bigg)\Bigg(\int_0^1 t^{\frac{2}{r}}\mathrm{d}t\Bigg)^r \\
=\ & \Bigg([f^r(a)g^r(a)]+[f(a)+\eta(f(b),f(a))]^r[g(a)+\eta(g(b),g(a))]^r\Bigg) \\
& \Bigg(\int_0^1 t^{\frac{2}{r}}\mathrm{d}t\Bigg)^r \\
& +\Bigg([f^r(a)][g^r(a)+\eta(g^r(b),g^r(a))]+[g^r(a)][f^r(a)+\eta(f^r(b),f^r(a))]\Bigg) \\
& \Bigg(\int_0^1 [t(1-t)]^{\frac{1}{r}}\mathrm{d}t\Bigg)^r \\
=\ & \Bigg\{M(a,b)\Bigg(\frac{r}{r+2}\Bigg)^r+N(a,b)\big(\beta(\frac{1}{r}+1,\frac{1}{r}+1)\big)^r\Bigg\},
\end{aligned}
$$

which is the required result. □

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