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ON g-β-IRRESOLUTE FUNCTIONS ON GENERALIZED TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce and investigate a new kind of function namely g-β-irresolute function along with its two weak and strong forms in generalized topological spaces. Several characterizations and interesting properties of these functions are discussed.

1. Introduction

Concepts of generalized topological spaces (GTS), generalized open sets and generalized continuity (= (g, g')-continuous functions) were introduced by A. Császár [8,11,14]. Since then, several research works devoted to generalize the existing notions of topological spaces to generalized topological spaces have appeared. In [22], [23], W. K. Min introduced the notions of weak (g, g')-continuity and almost (g, g')-continuity on generalized topological spaces. The concept of g-α-irresolute functions on generalized topological spaces was introduced by Bai and Zuo [4]. In 2013, Bayhan et al. [7] investigated some functions between generalized topological spaces. Recently, Acikgoz et al. [2] also studied some functions between GTS's.

On the other hand, Abd El-Monsef et al. [1] introduced the notions of β-open sets and β-continuity in topological spaces early in 1983. Andrijevic [3] introduced the notion of semi-preopen sets which are equivalent to β-open sets. Since then, β-open sets [1] played a significant role in the theory of generalized open sets in topological spaces. In [21], Mahmoud and El-Monsef defined and studied β-irresolute functions.

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T. Noiri [26] studied some weak and strong forms of β-irresolute functions in 2003. This work is concerned with the extension various forms of β-irresolute functions to generalized topological spaces.

2. Preliminaries

A collection g of subsets of X is called a generalized topology (briefly GT) on X [11] if and only if $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $G = \bigcup_{i \in I} G_i \in g$. A set X with a GT g on X is called a generalized topological space (GTS) and is denoted by (X, g). By a space X or (X, g), we will always mean a GTS. A GT g on X is called a strong GT [13] if $X \in g$. For a space (X, g), the elements of g are called g-open sets and the complements of g-open sets are called g-closed sets.

For $A \subset X$, the g-closure of A, denoted by cA is the intersection of all g-closed sets containing A and the g-interior of A, denoted by iA is the union of all g-open sets contained in A. It was pointed out in [14] that each of the operations iA and cA are monotonic i.e. if $A \subset B \subset X$, then $iA \subset iB$ and $cA \subset cB$, idempotent [16], i.e. if $A \subset X$, then $i(iA) = iA$ and $c(cA) = cA$, iA is restricting [16], i.e. if $A \subset X$, then $iA \subset A$, cA is enlarging [16], i.e., if $A \subset X$, then $A \subset cA$. In a space (X, g), for $A \subset X$, $x \in iA$ if and only if there exists an g-open set V containing x such that $V \subset A$ and $x \in cA$ if and only if $V \cap A \neq \emptyset$ for every g-open set V containing x [9]. In a space (X, g), $A \subset X$ is g-open if and only if $A = iA$ and is g-closed if and only if $A = cA$ [8] and $cA = X \setminus i(X \setminus A)$.

A subset A of a topological space is called β-open [1] if $A \subset cl(int(cl(A)))$. The complement of a β-open set is called β-closed. For a subset A of a topological space (X, τ), the β-closure of A, denoted by $\beta cl(A)$ is the intersection of all β-open sets containing A and the β-interior of A, denoted by $\beta int(A)$ is the union of all β-open sets contained in A.

In a GTS (X, g), a subset A of X is said to be g-β-open (resp. g-α-open, g-preopen, g-semiopen) [14] if $A \subset cicA$ (resp. $A \subset iciA$, $A \subset icA$, $A \subset ciA$). We denote by $\beta(g_X)$ (resp. $\alpha(g_X)$, $\pi(g_X)$, $\sigma(g_X)$) the class of all g-β-open (resp. resp. g-α-open, g-preopen, g-semiopen) sets of (X, g). From [14], it is clear that, $g \subset \alpha(g_X) \subset \sigma(g_X) \subset \beta(g_X)$, $\alpha(g_X) \subset \pi(g_X) \subset \beta(g_X)$ and each of $\beta(g_X)$ (resp. $\alpha(g_X)$, $\pi(g_X)$, $\sigma(g_X)$) forms a GT on X. The complements of a g-β-open sets (resp. g-α-open, g-preopen, g-semiopen) is called g-β-closed (resp. g-α-closed, g-preclosed, g-semiclosed) set. We denote by $\beta(g_X, x)$, the set of all g-β-open sets of (X, g) containing $x \in X$ and by $\beta c(g_X)$ the class of all g-β-closed sets of (X, g). For $A \subset X$, we denote by βcA the intersection of all g-β-closed sets containing A and by βiA the union of all g-β-open sets contained in A.

3. g-β-regular sets and g-β-θ-open sets

We first state a lemma which will be used in the sequel. Proofs can be checked easily and therefore omitted.

Lemma 3.1. *The following hold for a subset A of GTS (X, g):*

(i) Arbitrary union of g-β-open sets is g-β-open.

(ii) Arbitrary intersection of g-β-closed sets is g-β-closed.

(iii) $\beta iA = A \cap cicA$.

(iv) $\beta cA = A \cup iciA$.

(v) $x \in \beta cA$ *if* $A \cap U \neq \emptyset$ *for every g-β-open set U of X containing x.*

(vi) $\beta c(X \setminus A) = X \setminus \beta iA$.

(vii) A is g-β-closed if and only if $A = \beta cA$.

(viii) βiA is g-β-open and βcA is g-β-closed.

Lemma 3.2. *[22] In a GTS (X, g), X is both g-semiopen and g-β-open.*

Definition 3.1. *A subset A of a space X is said to be g-β-regular if it is both g-β-open and g-β-closed. The family of all g-β-regular sets of a space X is denoted by $\beta r(X)$ and those of containing a point x of X by $\beta r(X, x)$.*

Theorem 3.1. *For a subset A of a GTS (X, g),*

(i) $A \in \beta(g_X)$ *if and only if* $\beta cA \in \beta r(X)$.

(ii) $A \in \beta c(g_X)$ *if and only if* $\beta iA \in \beta r(X)$.

Proof: (i) First suppose, $A \in \beta(g_X)$. Then $A \subset cicA$ and therefore, $\beta c(cA) \subset \beta c(cicA) = cicA \subset cic(\beta cA)$ i.e. βcA is g-β-open. Since βcA is g-β-open and g-β-closed, $\beta cA \in \beta r(X)$. Next suppose, $\beta cA \in \beta r(X)$. Then $A \subset \beta cA \subset cic(\beta cA) \subset cic(cA) = cicA$. Hence $A \in \beta(g_X)$.

(ii) Follows from (i) and Lemma 3.1 (vi).

Theorem 3.2. *The following are equivalent for a subset A of a GTS (X, g).*

(i) $A \in \beta r(X)$;

(ii) $A = \beta i\beta cA$;

(iii) $A = \beta c\beta iA$.

Proof: Proofs of (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are obvious.

Proofs of (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) follow from Lemma 3.1 and Theorem 3.1.

Definition 3.2. *In a GTS (X, g), a point $x \in X$ is said to be in the g-β-θ-closure of A, denoted by β-θ-cA, if $A \cap \beta cV \neq \emptyset$ for every g-β-open set V of X containing x.*

If β-θ-$cA = A$, then A is said to be g-β-θ-closed. The complement of a g-β-θ-closed set is said to be g-β-θ-open. For a subset A of X, union of all g-β-θ-open sets contained in A is said to be g-β-θ-interior of A, denoted by β-θ-iA.

Lemma 3.3. *For a subset A of a space (X, g),*

β-θ-$cA = \cap\{V : A \subset V$ *and* V *is* g-β-θ-*closed* $\} = \cap\{V : A \subset V$ *and* $V \in \beta r(X)\}$

Proof: We give a proof of the first equality, because that of the other is quite similar. Suppose that, $x \notin \beta$-θ-cA. Then there exists, g-β-open set V containing x such that $\beta cV \cap A = \emptyset$. Therefore by Theorem 3.1, $X \setminus \beta cV$ is g-β-regular and so g-β-θ-closed set containing A such that $x \notin X \setminus \beta cV$. Hence, $x \notin \cap\{V : A \subset V$ and V is g-β-θ-closed set$\}$.

Conversely, suppose that, $x \notin \cap\{V : A \subset V$ and V is g-β-θ-closed set$\}$. Then there exist, a g-β-θ-closed set V containing A and $x \notin V$. Also, there exists a $U \in \beta(g_X)$ such that $x \in U \subset \beta cU \subset X \setminus V$. Then we have, $\beta cU \cap A \subset \beta cU \cap V = \emptyset$ and so $x \notin \beta$-θ-cA.

Lemma 3.4. *Let A and B be any subset of a GTS (X, g). Then the following properties hold:*

(i) $x \in \beta$-θ-cA if and only if $A \cap V \neq \emptyset$ for every $V \in \beta r(X, x)$.

(ii) If $A \subset B$ then β-θ-$cA \subset \beta$-θ-cB.

(iii) β-θ-$c(\beta$-θ-$cA) = \beta$-θ-cA.

(iv) intersection of an arbitrary family of g-β-θ-closed sets in X is g-β-θ-closed in X.

(v) A is g-β-θ-open if and only if for each $x \in A$, there exists $V \in \beta r(X, x)$, such that $x \in V \subset A$.

(vi) If $A \in \beta(g)$ then $\beta cA = \beta$-θ-cA.

(vii) If $A \in \beta r(X)$ then A is g-β-θ-closed.

(viii) $A \in \beta r(X)$ if and only if A is g-β-θ-open and g-β-θ-closed.

Proof: We give only the proofs of (iii) and (iv). Others proofs are obvious.

(iii) We have β-θ-$cA \subset \beta$-θ-$c(\beta$-θ-$cA)$. Now, if $x \notin \beta$-θ-cA, there exits $V \in \beta r(X, x)$ such that $A \cap V = \emptyset$. Since $V \in \beta r(X, x)$, we have β-θ-$cA \cap V = \emptyset$. This implies $x \notin \beta$-$\theta c(\beta$-$\theta cA)$ and so β-θ-$c(\beta$-θ-$cA) \subset \beta$-θ-cA.

(iv) Let A_α be a g-β-θ-closed for each $\alpha \in \Delta$. Then for each $\alpha \in \Delta$, we have $A_\alpha = \beta$-θ-cA_α. Therefore β-θ-$c(\cap_{\alpha\in\Delta} A_\alpha) \subset \cap_{\alpha\in\Delta}\beta$-$\theta$-$cA_\alpha$=$\cap_{\alpha\in\Delta} A_\alpha \subset \beta$-$\theta$-$c(\cap_{\alpha\in\Delta} A_\alpha)$. Hence, β-θ-$c(\cap_{\alpha\in\Delta} A_\alpha) = \cap_{\alpha\in\Delta} A_\alpha$. Therefore, $\cap_{\alpha\in\Delta} A_\alpha$ is g-β-θ-closed.

Corollary 3.1. *For a subset A of a GTS (X, g), the following properties hold:*

(i) A is g-β-θ-open in X if and only if for each $x \in A$ there exists $V \in \beta r(X, x)$ such that $x \in V \subset A$.

(ii) β-θ-cA is g-β-θ-closed and β-θ-iA is g-β-θ-open.

(iii) Arbitrary union of g-β-θ-open sets is g-β-θ-open.

Theorem 3.3. *For a subset A of a GTS (X, g), the following properties hold:*

(i) If $A \in \beta(g_X)$ then $\beta cA = \beta$-θ-cA.

(ii) $A \in \beta r(X)$ if and only if A is g-β-θ-open and g-β-θ-closed.

Proof: (i) Let $A \in \beta(g_X)$ and $x \notin \beta cA$. Then, there exists $V \in \beta(g_X, x)$ such that $A \cap V = \emptyset$. Now, since $A \in \beta(g_X)$ we have $A \cap \beta cV = \emptyset$. This implies $x \notin \beta$-θ-cA and so β-θ-$cA \subset \beta$-cA. Also, for every subset A of X, we have β-$cA \subset \beta$-θ-cA. Hence, $\beta cA = \beta$-θ-cA.

(ii) Suppose $A \in \beta r(X)$. Then $A = \beta cA$=β-θ-cA. Hence A is g-β-θ-closed. Again, since $X \setminus A \in \beta r(X)$, we get $X \setminus A$ is g-β-θ-closed and so A is β-θ-open. The converse part is obvious.

Remark 3.1. *It is clear that in a GTS (X, g), g-β-regular $\Rightarrow$ g-β-θ-open $\Rightarrow$ g-β-open. But the converses are not necessarily true.*

Example 3.1. *Let $X = \{a, b, c, d\}$ and $g = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ be a GT on X. Then the subsets $\{a, b\}$, $\{a, b, c\}$ and $\{a, b, d\}$ of X are g-β-θ-open but not g-β-regular.*

Example 3.2. *Let $X = \{a, b, c, d\}$ and $g = \{\emptyset, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ be a GT on X. Then the subset $\{b, c\}$ of X is g-β-open but not g-β-θ-open.*

4. g-β-irresolute functions

Definition 4.1. *Let g_X and g_Y be generalized topologies on X and Y respectively. Then a function $f : (X, g_X) \to (Y, g_Y)$ is defined to be generalized continuous or more properly (g_X, g_Y)-continuous [11] if $f^{-1}(V) \in g_X$ for each $V \in g_Y$.*

Definition 4.2. *A function $f : (X, g_X) \to (Y, g_Y)$ is called (β, g_Y)-continuous [24] if $f^{-1}(V) \in \beta(g_X)$ for each $V \in g_Y$.*

Definition 4.3. *[4] A function $f : (X, g_X) \to (Y, g_Y)$ is called g-α-irresolute if $f^{-1}(V)$ is g-α-open in X for every g-α-open set V of Y.*

Definition 4.4. *A function $f : (X, g_X) \to (Y, g_Y)$ is called g-β-irresolute if the inverse image of each g-β-open set of Y is g-β-open in X.*

Definition 4.5. *A function $f : (X, g_X) \to (Y, g_Y)$ is said to be g-β-irresolute at $x \in X$ if for each $V \in \beta(g_Y, f(x))$, there exists $U \in \beta(g_X, x)$ such that $f(U) \subset V$.*

Definition 4.6. *A function $f : (X, g_X) \to (Y, g_Y)$ is called weakly g-β-irresolute (resp. strongly g-β-irresolute) if for each point $x \in X$ and each g-β-open set V of Y containing $f(x)$, there exists a g-β-open set U of X containing x such that $f(U) \subset \beta cV$ (resp. $f(\beta cU) \subset V$).*

Remark 4.1. *From the above definitions we have the following implications:*

Strongly g-β-irresolute $\Rightarrow$ g-β-irresolute $\Rightarrow$ weakly g-β-irresolute and g-α-irresolute $\Rightarrow$ g-β-irresolute $\Rightarrow$ (β, g_Y)-continuous.

We now state basic properties of a g-β-irresolute function. Some results of the following Theorem may be analogous to Theorem 3.18 of [24] in terms of other terminologies.

Theorem 4.1. *Let $f:(X,g_X)\to(Y,g_Y)$ be a function. Then the following are equivalent:*
(i) f is g-β-irresolute;
(ii) $f^{-1}(F)$ is g-β-closed in X for every g-β-closed subset F of Y;
(iii) $f(\beta cA)\subset\beta c(f(A))$ for every subset A of X;
(iv) $\beta c(f^{-1}(B))\subset f^{-1}(\beta cB)$ for every subset B of Y;
(v) $f^{-1}(\beta iV)\subset\beta i(f^{-1}(V))$ for every subset V of Y;
(vi) for every $x\in X$ and for every g-β-open set V containing $f(x)$, there exists a g-β-open set U of X containing x such that $f(U)\subset V$;

Proof: (i) $\Rightarrow$ (ii): Obvious.
(ii) $\Rightarrow$ (iii): Let A be any subset of X. Then since $f^{-1}(\beta c(f(A)))$ is a g-β-closed set we get $\beta cA\subset\beta c(f^{-1}(f(A)))\subset\beta c(f^{-1}(\beta c(f(A))))=f^{-1}(\beta c(f(A)))$. Hence $f(\beta cA)\subset\beta c(f(A))$.
(iii) $\Rightarrow$ (iv): For any subset V of Y, using (iii) we get $f(\beta c(f^{-1}(V)))\subset\beta c(ff^{-1}(V))\subset\beta cV$. Therefore, $\beta c(f^{-1}(V))\subset f^{-1}f(\beta c(f^{-1}(V))\subset f^{-1}(\beta cV)$.
(iv) $\Rightarrow$ (v): For any subset V of Y, using (iv) we get, $f^{-1}(\beta c(Y\setminus V))\supset\beta c(f^{-1}(Y\setminus V))=\beta c(X\setminus f^{-1}(V))$. Now by Lemma 3.1, $f^{-1}(\beta iV)=f^{-1}(Y\setminus\beta c(Y\setminus V))=X\setminus f^{-1}(\beta c(Y\setminus V))\subset X\setminus\beta c(X\setminus f^{-1}(V))=\beta i(f^{-1}(V))$.
(v) $\Rightarrow$ (i): Let V be any g-β-open subset of Y. Then $f^{-1}(V)=f^{-1}(\beta iV)\subset\beta i(f^{-1}(V))\subset f^{-1}(V)$. This implies $f^{-1}(V)=\beta i(f^{-1}(V))$ i.e. $f^{-1}(V)$ is g-β-open set of X. Hence f is g-β-irresolute.
(i) $\Rightarrow$ (vi): Let f be g-β-irresolute. Also let $x\in X$ and $V\in\beta(g_Y,f(x))$. Then $x\in f^{-1}(V)=\beta i(f^{-1}(V))$. If we set $U=f^{-1}(V)$, then $U\in\beta(g_X)$ and $f(U)\subset V$. Hence f is g-β-irresolute for each $x\in X$.
(vi) $\Rightarrow$ (i): Let $V\in\beta(g_Y)$ and $x\in f^{-1}(V)$. Then $f(x)\in V$. So, there exists $U\in\beta(g_X,x)$ such that $f(U)\subset V$. Then $x\in U\subset f^{-1}f(U)\subset f^{-1}(V)$ and $x\in U=\beta iU\subset\beta i(f^{-1}(V))$. Therefore $f^{-1}(V)\subset\beta i(f^{-1}(V))$ and so $f^{-1}(V)=\beta i(f^{-1}(V))$. Hence f is g-β-irresolute.

Theorem 4.2. *Let $f:(X,g_X)\to(Y,g_Y)$ be a bijective function. Then f is g-β-irresolute if and only if $\beta i(f(U))\subset f(\beta iU)$ for every subset U of X.*

Proof: Let V be any subset of X. Then by above Theorem 4.1, $f^{-1}(\beta i(f(V)))\subset\beta i(f^{-1}f(V))=\beta iV$. Therefore, $ff^{-1}(\beta i(f(V)))\subset f(\beta iV)$ so $\beta i(f(V))\subset f(\beta iV)$.

Conversely, let V be any g-β-open set of Y. Then $V=\beta iV=\beta i(ff^{-1}(V))\subset f(\beta i(f^{-1}(V)))$ i.e. $f^{-1}(V)\subset f^{-1}f(\beta i(f^{-1}(V))$. Since f is bijective, this implies $f^{-1}(V)\subset f^{-1}f(\beta i(f^{-1}(V)))=\beta i(f^{-1}(V))$ i.e $f^{-1}(V)=\beta i(f^{-1}(V))$. Therefore $f^{-1}(V)$ is g-β-open set of X and so f is g-β-irresolute.

Definition 4.7. *[10] A GTS (X, g) is called β-compact if each cover of X by g-β-open sets of X, has a finite subcover.*

Theorem 4.3. *Let $f : (X, g_X) \to (Y, g_Y)$ be a g-β-irresolute function. If (X, g_X) is β-compact then so is (Y, g_Y).*

Proof: Let $\{U_\alpha : \alpha \in \Lambda\}$ be a g-β-open cover of Y. Then since f is g-β-irresolute, $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a g-β-open cover of X. Now, since (X, g_X) is β-compact, there exists a finite subcover, say $\{f^{-1}(U_{\alpha_1}), f^{-1}(U_{\alpha_2}), ..., f^{-1}(U_{\alpha_n})\}$ such that $\{ff^{-1}(U_{\alpha_1}), ff^{-1}(U_{\alpha_2}), ..., ff^{-1}(U_{\alpha_n})\} \subset \{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n}\}$ is a finite subcover of (Y, g_Y). Hence (Y, g_Y) is β-compact.

5. Properties of weakly g-β-irresolute functions

Theorem 5.1. *For a function $f : (X, g_X) \to (Y, g_Y)$, the following are equivalent:*
(i) f is weakly g-β-irresolute;
(ii) $f^{-1}(V) \subset \beta i(f^{-1}(\beta cV))$ for every g-β-open set V of Y;
(iii) $\beta c(f^{-1}(V)) \subset f^{-1}(\beta cV)$ for every g-β-open set V of Y.

Proof: (i) $\Rightarrow$ (ii): Let V be any g-β-open set of Y and $x \in f^{-1}(V)$. Then $f(U) \subset \beta cV$ for some $U \in \beta(g_X, x)$. This implies $U \subset f^{-1}(\beta cV)$ and $x \in U \subset \beta i(f^{-1}(\beta cV))$. Hence $f^{-1}(V) \subset \beta i(f^{-1}(\beta cV))$.
(ii) $\Rightarrow$ (iii): Let $V \in \beta(g_Y)$ and $x \notin f^{-1}(\beta cV)$. Then $f(x) \notin \beta cV$. So, there exists $W \in \beta(g_Y, f(x))$ such that $V \cap W = \emptyset$. Now, since V is g-β-open, we have $V \cap \beta cW = \emptyset$ and so $f^{-1}(V) \cap \beta i(f^{-1}(\beta cW)) = \emptyset$. As $x \in f^{-1}(W) \subset \beta i(f^{-1}(\beta cW)) \in \beta(g_X)$, we have $x \notin \beta c(f^{-1}(V))$. Hence, $\beta c(f^{-1}(V)) \subset f^{-1}(\beta cV)$.
(iii) $\Rightarrow$ (i): For $x \in X$, suppose $V \in \beta(g_Y, f(x))$. Then by Lemma 3.1, $\beta cV \in \beta r(Y)$ and $x \notin f^{-1}(\beta c(Y \backslash \beta cV)$. Since $Y \setminus \beta cV$ is a g-β-open set of Y, we get $x \notin \beta c(f^{-1}(Y \setminus \beta cV))$. So there exists $U \in \beta(g_X, x)$ such that $f^{-1}(Y \setminus \beta cV) \cap U = \emptyset$. This implies $f(U) \cap (Y \setminus \beta cV) = \emptyset$ and so $f(U) \subset \beta cV$ i.e. f is weakly g-β-irresolute.

Theorem 5.2. *For a function $f : (X, g_X) \to (Y, g_Y)$, the following are equivalent:*
(i) f is weakly g-β-irresolute;
(ii) $\beta c(f^{-1}(V)) \subset f^{-1}(\beta\text{-}\theta\text{-}cV)$ for every subset V of Y;
(iii) $f(\beta cU) \subset \beta\text{-}\theta\text{-}c(f(U))$ for every subset U of X;
(iv) $f^{-1}(F) \in \beta c(g_X)$ for every g-β-θ-closed set F of Y;
(v) $f^{-1}(G) \in \beta(g_X)$ for every g-β-θ-open set G of Y.

Proof: (i) $\Rightarrow$ (ii): Let V be any subset of Y and $x \notin f^{-1}(\beta\text{-}\theta\text{-}cV)$. Then $f(x) \notin \beta\text{-}\theta\text{-}cV$ and so there exists $Q \in \beta(g_Y, f(x))$ such that $V \cap \beta cQ = \emptyset$. Since f is weakly g-β-irresolute, there exists $P \in \beta(g_X, x)$ such that $f(P) \subset \beta cQ$. Hence $f(P) \cap V = \emptyset$ and so $P \cap f^{-1}(V) = \emptyset$. Therefore, $x \notin \beta c(f^{-1}(V))$ and consequently $\beta c(f^{-1}(V)) \subset f^{-1}(\beta\text{-}\theta\text{-}cV)$.

(ii) $\Rightarrow$ (iii) : For any subset U of X, we have $\beta cU \subset \beta c(f^{-1}(f(U))) \subset f^{-1}(\beta\text{-}\theta\text{-}f(U))$ and so $f(\beta cU) \subset \beta\text{-}\theta\text{-}c(f(U))$.

(iii) $\Rightarrow$ (iv): For any g-β-θ-closed set F of Y, $f(\beta c(f^{-1}(F))) \subset \beta\text{-}\theta\text{-}c(f(f^{-1}(F))) \subset \beta\text{-}\theta\text{-}cF = F$. This implies $\beta c(f^{-1}(F)) \subset f^{-1}(F)$ and hence $\beta c(f^{-1}(F)) = f^{-1}(F)$. Therefore $f^{-1}(F) \in \beta c(g_X)$.

(iv) $\Rightarrow$ (v): Obvious.

(v) $\Rightarrow$ (i): For any $x \in X$, let $Q \in \beta(g_Y, f(x))$. Then by Theorem 3.1 (i) and Theorem 3.3 (ii), we get βcQ is g-β-θ-open in Y. If we set $P = f^{-1}(\beta cQ)$, then $P \in \beta(g_X, x)$ and $f(P) \subset \beta cQ$. Hence f is weakly g-β-irresolute.

Theorem 5.3. *For a function $f : (X, g_X) \to (Y, g_Y)$, the following are equivalent:*

(i) f is weakly g-β-irresolute;

(ii) for each $x \in X$ and $V \in \beta(g_Y, f(x))$, there exists $U \in \beta(g_X, x)$ such that $f(\beta cU) \subset \beta cV$;

(iii) $f^{-1}(R) \in \beta r(X)$ for every $R \in \beta r(Y)$.

Proof: (i) $\Rightarrow$ (ii): For any $x \in X$, let $V \in \beta(g_Y, f(x))$. Then by Theorem 3.1 and 3.3, βcV is g-β-θ-open and g-β-θ-closed in Y. If we set $U = f^{-1}(\beta cV)$, then by Theorem 5.2, we have $U \in \beta r(X)$ and so $U \in \beta(g_X, x)$. Also we have $f(\beta cU) \subset \beta cV$.

(ii) $\Rightarrow$ (iii): Let $R \in \beta r(Y)$ and $x \in f^{-1}(R)$. Then we have $f(x) \in R$ and there exists $U \in \beta(g_X, x)$ such that $f(\beta cU) \subset R$. This implies $x \in U \subset \beta cU \subset f^{-1}(R)$ and so $f^{-1}(R) \in \beta(g_X)$. Again since $Y \setminus R \in \beta r(Y)$ $f^{-1}(Y \setminus R) = X \setminus f^{-1}(R) \in \beta(g_X)$. Thus $f^{-1}(R) \in \beta c(g_X)$ and consequently $f^{-1}(R) \in \beta r(X)$.

(iii) $\Rightarrow$ (i): For any $x \in X$, suppose $V \in \beta(g_Y, f(x))$. Then by Theorem 3.1, we get $\beta cV \in \beta r(Y, f(x))$ and $f^{-1}(\beta cV) \in \beta r(X, x)$. If we take, $U = f^{-1}(\beta cV)$, then $U \in \beta(g_X, x)$ and $f(U) \subset \beta cV$. Hence f is weakly g-β-irresolute.

Theorem 5.4. *For a function $f : (X, g_X) \to (Y, g_Y)$, the following are equivalent:*

(i) f is weakly g-β-irresolute;

(ii) $f^{-1}(V) \subset \beta\text{-}\theta\text{-}i(f^{-1}(\beta\text{-}\theta\text{-}cV))$ for every g-β-open set V of Y;

(iii) $\beta\text{-}\theta\text{-}c(f^{-1}(V)) \subset f^{-1}(\beta\text{-}\theta\text{-}cV)$ for every g-β-open set V of Y.

Proof: The proof is quite similar to the Proof of Theorem 5.1, if we observe that every g-β-closed set is g-β-θ-closed.

Theorem 5.5. *For a function $f : (X, g_X) \to (Y, g_Y)$, the following are equivalent:*

(i) f is weakly g-β-irresolute;

(ii) $\beta\text{-}\theta\text{-}c(f^{-1}(V)) \subset f^{-1}(\beta\text{-}\theta\text{-}cV)$ for every subset V of Y;

(iii) $f(\beta\text{-}\theta\text{-}cU) \subset \beta\text{-}\theta\text{-}c(f(U))$ for every subset U of X;

(iv) $f^{-1}(F)$ *is g-β-θ-closed in* X *for every g-β-θ-closed set* F *of* Y*;*

(v) $f^{-1}(G)$ *is g-β-θ-open set in* X *for every g-β-θ-open set* G *of* Y*.*

Proof: The proof is quite similar to Proof of Theorem 5.2 and hence omitted.

Definition 5.1. *A GTS* (X, g) *is said to be g-β-regular if for each* $F \in \beta c(g_X)$ *and each* $x \notin F$*, there exist disjoint g-β-open sets* U *and* V *such that* $x \in U$ *and* $F \subset V$*.*

Lemma 5.1. *The following properties are equivalent in a GTS* (X, g)*:*

(i) X *is g-β-regular;*

(ii) For each $U \in \beta(g_X)$ *and each* $x \in U$*, there exists* $V \in \beta(g_X)$ *such that* $x \in V \subset \beta cV \subset U$*;*

(iii) For each $U \in \beta(g_X)$ *and each* $x \in U$*, there exists* $V \in \beta r(X)$ *such that* $x \in V \subset U$

Proof: Follows from Theorem 3.1.

Theorem 5.6. *A function* $f : (X, g_X) \to (Y, g_Y)$ *is g-β-irresolute if and only if it is weakly g-β-irresolute and* (Y, g_Y) *is g-β-regular.*

Proof: Suppose that f is weakly g-β-irresolute. Let V be any g-β-open set of Y and $x \in f^{-1}(V)$, then $f(x) \in V$. Now, since Y is g-β-regular, by above Lemma 5.1, there exists $W \in \beta(g_X)$ such that $f(x) \in W \subset \beta cW \subset V$. Again since f is weakly g-β-irresolute, there exists $U \in \beta(g_X, x)$ such that $f(U) \subset \beta cW$. This implies $x \in U \subset f^{-1}(V)$ and $f^{-1}(V) \in \beta(g_X)$. Hence f is g-β-irresolute. The converse part is obvious.

Proposition 5.1. *[28] Let* (X, g_X) *and* (Y, g_Y) *be generalized topological spaces and let* $\mathcal{U} = \{U \times V : U \in g_X, V \in g_Y\}$*. Then* $\mathcal{U}$ *generates a generalized topology* $g_{X\times Y}$ *on* $X \times Y$*, called the generalized product topology on* $X \times Y$*, i.e.* $g_{X\times Y} = \{$*all possible union of members of* $\mathcal{U}\}$

Proposition 5.2. *[28] Let* (X, g_X) *and* (Y, g_Y) *be generalized topological spaces,* $g_{X\times Y}$ *be the generalized topology on* $X \times Y$*,* $A \subset X$*,* $B \subset Y$ *and* $K \subset X \times Y$*. Then the following hold:*

(i) K *is* $g_{X\times Y}$*-open if and only if for each* $(x, y) \in K$*, there exist* $U_x \in g_X$ *and* $V_y \in g_Y$ *such that* $(x, y) \in U_x \times V_y \subset K$*.*

(ii) $c(A \times B) = cA \times cB$*.*

(iii) $i(A \times B) = iA \times iB$*.*

Proposition 5.3. *Let* (X, g_X) *and* (Y, g_Y) *be generalized topological spaces,* $g_{X\times Y}$ *be the generalized topology on* $X \times Y$*,* $A \subset X$*,* $B \subset Y$ *and* $K \subset X \times Y$*. Then the following hold:*

(ii) $\beta c(A \times B) = \beta cA \times \beta cB$*.*

(iii) $\beta i(A \times B) = \beta iA \times \beta iB$*.*

Theorem 5.7. *A function $f : (X, g_X) \to (Y, g_Y)$ is weakly g-β-irresolute if the graph function, defined by $G(f) = (x, f(x))$ for each $x \in X$, is weakly g-β-irresolute.*

Proof: Let $x \in X$ and $V \in \beta(g_X)$. Then using Lemma 3.2, we get $X \times V$ is g-β-open set of $X \times Y$ containing $G(f)$. Since G is weakly g-β-irresolute, there exists $U \in \beta(g_X, x)$ such that $G(U) \subset \beta c(X \times V) \subset X \times \beta cV$. Hence $f(U) \subset \beta cV$ i.e. f is weakly g-β-irresolute.

Definition 5.2. *[27] A GTS (X, g) is said to be g-β-T_2 if and only if for each pair of distinct points $x, y \in X$, there exits disjoint g-β-open sets containing x and y respectively.*

Lemma 5.2. *A GTS (X, g) is g-β-T_2 if and only if for each pair of distinct points $x, y \in X$, there exist $U \in \beta(g_X, x)$ and $V \in \beta(g_X, y)$ such that $\beta cU \cap \beta cV = \emptyset$.*

Proof: Follows from Theorem 3.1.

Theorem 5.8. *If Y is a g-β-T_2 space and $f : (X, g_X) \to (Y, g_Y)$ is a weakly g-β-irresolute injection, then X is g-β-T_2.*

Proof: Let x, y be any two distinct points of X, then since f is an injection, we have $f(x) \neq f(y)$. Now Y being g-β-T_2, by Lemma 5.1, there exists $U \in \beta(g_Y, f(x))$ and $V \in \beta(g_Y, f(y))$ such that $\beta cU \cap \beta cV = \emptyset$. Again since f is weakly g-β-irresolute, there exist $P \in \beta(g_X, x)$ and $Q \in \beta(g_X, y)$ such that $f(P) \subset \beta cU$ and $f(Q) \subset \beta cV$. This implies $P \cap Q = \emptyset$. Therefore X is g-β-T_2.

Definition 5.3. *A function $f : (X, g_X) \to (Y, g_Y)$ is said to have strongly g-β-closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \beta(g_X, x)$ and $V \in \beta(g_Y, y)$ such that $(\beta cU \times \beta cV) \cap G(f) = \emptyset$.*

Theorem 5.9. *If a function $f : (X, g_X) \to (Y, g_Y)$ is weakly g-β-irresolute, where Y is g-β-T_2, then $G(f)$ is strongly g-β-closed.*

Proof: Let $(x \times y) \in (X \times Y) \setminus G(f)$. Then since $y \neq f(x)$, by Lemma 5.1, there exists, $U \in \beta(g_X, f(x))$ and $V \in \beta(g_Y, y)$ such that $\beta cU \cap \beta cV = \emptyset$. Again since f is weakly g-β-irresolute, by Theorem 5.3, there exists $W \in \beta(g_X, x)$ such that $f(\beta cW) \subset \beta cU$. This implies $f(\beta cW) \cap \beta cV = \emptyset$ and so $(\beta cW \times \beta cV) \cap G(f) = \emptyset$. Hence $G(f)$ is strongly g-β-closed in $X \times Y$.

Theorem 5.10. *If a function $f : (X, g_X) \to (Y, g_Y)$ is weakly g-β-irresolute injection and $G(f)$ is strongly g-β-closed, then X is g-β-T_2.*

Proof: Let $x, y \in X$ and $x \neq y$. Since f is an injection $f(x) \neq f(y)$ and $(x, f(y)) \notin G(f)$. Again since $G(f)$ is strongly g-β-closed, there exists $U \in \beta(g_X, x)$ and $V \in \beta(g_Y, f(y))$ such that $f(\beta cU) \cap \beta cV = \emptyset$. Also, f being weakly g-β-irresolute, there exists $W \in \beta(g_X, y)$ such that $f(W) \subset \beta cV$. Hence $f(\beta cU) \cap f(W) = \emptyset$ and so $U \cap W = \emptyset$. Therefore X is g-β-T_2.

Definition 5.4. *A GTS (X, g) is said to be connected [29] if there are no nonempty disjoint sets $A, B \in g$ such that $A \cup B = X$. A GTS (X, g) is said to be β-connected [29] if $(X, \beta(g_X))$ is connected.*

Theorem 5.11. *If a function $f : (X, g_X) \to (Y, g_Y)$ is a weakly g-β-irresolute surjection and X is β-connected, then Y is β-connected.*

Proof: If possible, suppose that Y is not β-connected. Then there exists nonempty disjoint sets $A, B \in \beta(g_Y)$ such that $Y = A \cup B$. This implies $A, B \in \beta r(Y)$ by Lemma 3.2. Now, since f is weakly g-β-irresolute, by Lemma 3.2 and Theorem 5.3, we get $f^{-1}(A), f^{-1}(B) \in \beta r(X)$. Moreover f being a surjection, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nonempty sets. Therefore X is not β-connected.

6. Properties of strongly g-β-irresolute functions

Theorem 6.1. *For a function $f : (X, g_X) \to (Y, g_Y)$, the following are equivalent:*
(i) f is strongly g-β-irresolute;
(ii) for each $x \in X$, and each $V \in \beta(g_Y, f(x))$, there exists $U \in \beta(g_X, x)$ such that $f(\beta\text{-}\theta\text{-}cU) \subset V$;
(iii) for each $x \in X$ and each $V \in \beta(g_Y, f(x))$, there exists $U \in \beta r(X, x)$ such that $f(U) \subset V$;
(iv) for each $x \in X$ and for each $V \in \beta(g_Y, f(x))$, there exist an g-β-open set U in X containing x such that $f(U) \subset V$;
(v) $f^{-1}(G)$ is g-β-θ-open in X for every $G \in \beta(g_Y)$;
(vi) $f^{-1}(F)$ is g-β-θ-closed in X for every $F \in \beta c(g_Y)$;
(vii) $f(\beta\text{-}\theta\text{-}cA) \subset \beta c(f(A))$ for every subset A of X;
(viii) $\beta\text{-}\theta\text{-}c(f^{-1}(B)) \subset f^{-1}(\beta cB)$ for every subset B of Y.

Proof: We first observe that (i) to (iv) are equivalent from Theorem 3.1 and Theorem 3.3.
(iv) $\Rightarrow$ (v): Let $G \in \beta(g_Y)$ and $x \in f^{-1}(G)$. Then we have $f(x) \in G$ and there exists a g-β-θ-open set U in X containing x such that $f(U) \subset G$. Therefore, $x \in U \subset f^{-1}(G)$. Hence by using Corollary 3.1, $f^{-1}(G)$ is g-β-θ-open in X.
(v) $\Rightarrow$ (vi): Obvious.
(vi) $\Rightarrow$ (vii): Let A be any subset of X. Then $f^{-1}(\beta c(f(A)))$ is g-β-θ-closed in X and so we get $\beta\text{-}\theta\text{-}cA \subset \beta\text{-}\theta\text{-}c(f^{-1}(f(A))) \subset \beta\text{-}\theta\text{-}c(f^{-1}(\beta c(f(A)))) = f^{-1}(\beta c(f(A)))$. Hence $f(\beta\text{-}\theta\text{-}cA) \subset \beta\text{-}c(f(A))$.
(vii) $\Rightarrow$ (viii): Let B be any subset of Y. Then we have $f(\beta\text{-}\theta\text{-}c(f^{-1}(B))) \subset \beta c(f(f^{-1}(B))) \subset \beta cB$. Hence $\beta\text{-}\theta\text{-}c(f^{-1}(B)) \subset f^{-1}(\beta cB)$.
(viii) $\Rightarrow$ (i): Let $x \in X$ and $V \in \beta(g_Y, f(x))$. Since $Y \setminus V \in \beta c(g_Y)$, we have $\beta\text{-}\theta\text{-}c(f^{-1}(Y \setminus V)) \subset f^{-1}(\beta c(Y \setminus V)) = f^{-1}(Y \setminus V)$. This implies $f^{-1}(Y \setminus V)$ is g-β-θ-closed in X and so $f^{-1}(V)$ is a β-θ-open set containing x. Then there exists $U \in \beta(g_X, x)$ such that $\beta cU \subset f^{-1}(V)$ i.e. $f(\beta cU) \subset V$. Therefore f is strongly g-β-irresolute.

Theorem 6.2. *A g-β-irresolute function $f : (X, g_X) \to (Y, g_Y)$ is strongly g-β-irresolute if and only if X is a g-β-regular space.*

Proof: First let, every g-β-irresolute function be strongly g-β-irresolute. The identity function $i_d : (X, g_X) \to (X, g_X)$ is g-β-irresolute and therefore strongly g-β-irresolute. Therefore, for any $P \in \beta(g_X)$ and any point $x = i_d(x) \in P$, there exists $Q \in \beta(g_X, x)$ such that $i_d(\beta cQ) \subset P$. This implies $x \in Q \subset \beta cQ \subset P$. Hence by Lemma 5.1, X is g-β-regular.

Conversely, let $f : (X, g_X) \to (Y, g_Y)$ be g-β-irresolute and X be g-β-regular. Then for any $x \in X$ and any $Q \in \beta(g_Y, f(x))$, we get $f^{-1}(Q) \in \beta(g_X, x)$. Now, since X is g-β-regular, there exists $P \in \beta(g_X, x)$ such that $x \in P \subset \beta cP \subset f^{-1}(Q)$ i.e. $f(\beta cP) \subset Q$. Hence f is strongly g-β-irresolute.

Corollary 6.1. *Let X be a g-β-regular space. Then a function $f : (X, g_X) \to (Y, g_Y)$ is strongly g-β-irresolute if and only if it is g-β-irresolute.*

Theorem 6.3. *Let $f : (X, g_X) \to (Y, g_Y)$ be a function and $G(f) : X \to X \times Y$ be the graph of f. If $G(f)$ is strongly g-β-irresolute, then f is strongly g-β-irresolute and X is g-β-regular.*

Proof: Suppose $G(f)$ is strongly g-β-irresolute. To show, f is strongly g-β-irresolute, let $x \in X$ and $Q \in \beta(g_Y, f(x))$. Now by Lemma 3.2, we have $X \times Q$ is a g-β-open set of $X \times Y$ containing $G(f)$. Since $G(f)$ is strongly g-β-irresolute, there exists $P \in \beta(g_X, x)$ such that $G(\beta cP) \subset X \times Q$. This implies $f(\beta cP) \subset Q$ and so f is strongly g-β-irresolute. To show X is g-β-regular, let $P \in \beta(g_X, x)$. Then since $G(f) \in P \times Y$, using Lemma 3.2 we get, $P \times Y$ is g-β-open set in $X \times Y$. Hence there exists $S \in \beta(g_X, x)$ such that $G(\beta cS) \subset P \times Y$. Therefore we obtain, $x \in S \subset \beta cS \subset P$. So by Lemma 5.1, X is g-β-regular.

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