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A PERTURBED VERSION OF GENERAL WEIGHTED OSTROWSKI TYPE INEQUALITY AND APPLICATIONS

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Abstract. The main purpose of this paper is to derive some new generalizations of weighted Ostrowski type inequalities. The new established inequalities are carried out for a twice differentiable mapping in different L_p spaces. Applications throught considering Grüss type inequality and numerical integration are also provided.

1. Introduction

The Ostrowski's inequality [1] can be considered as a very powerful tool for enhancement of numerical integration rules. It provides convenient potintial window for establishing bounds for the well known Newton-Cotes rules. To illustrate this point, consider $f : [a, b] \to \mathbb{R}$ to be a bounded function such that $b - a$ is small, then

$$I = \int_a^b f\ (x)\, dx,$$

can be, simply, approximated by sampling at one point as $I^*(x) = (b - a)\, f(x)$ for some $x \in [a, b]$. Now, if $f\,'$ exists and is bounded, the inequality of Ostrowski may be stated as follows

$$|\ I^*(x) - I| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b - a}\right)^2\right] \|f'\|_\infty\,, \tag{1.1}$$

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where

$$\|f'\|_{\infty} = \sup_{x\in[a,b]} |f'(x)|.$$

Consequently, over the past few decades, there have been many studies on obtaining sharp bounds of (1.1) by considering the mappings and their derivatives in various Lebesgue spaces. Further, the new bounds have been carried out by implementing weighted and non-weighted Peano kernel. Several weighted and non-weighted versions of (1.1) have been derived. Applications in both numerical integration and probability are also presented in this regards. For instance, Roumeliotis et. al [2] proved a weighted integral inequality of Ostrowski's type for mappings whose second derivatives are bounded. Cerone [3] obtained bounds for the deviation of a function from a combination of integral means over the end intervals covering the entire interval. Qayyum et. al [4] established a new Ostrowski's type inequality using weight function which generalizes the inequality in [3]. Barnett [5] reported a companion of (1.1) and the generalized trapezoid inequalites for various classes of functions, including functions of bounded variation, Lipschitzian, convex and absolutely continuous functions. Recently, Budak et. al [6] presented a new generalization of weighted Ostrowski's type inequality for mappings of bounded variation. Several further generalizations of (1.1) are provided in [7] - [16].

In [12], Qayyum et. al proved the following non-weighted generalization of Ostrowski's type integral inequality.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping. Then*

$$\begin{aligned}&\Bigg|\frac{1}{2(\alpha+\beta)}\left[\alpha(x-a)-\beta(b-x)\right]f''(x)-f(x)\\&+\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)dt+\frac{\beta}{b-x}\int_x^b f(t)dt\right]\Bigg|,\end{aligned}$$

$$\leq \begin{cases} \left[\alpha(x-a)^2+\beta(b-x)^2\right]\frac{\|f''\|_{\infty}}{6(\alpha+\beta)}, & f''\in L_{\infty}[a,b], \\ \frac{1}{(2q+1)^{\frac{1}{q}}}\left[\alpha^q(x-a)^{q+1} + \beta^q(b-x)^{q+1}\right]^{\frac{1}{q}}\frac{\|f''\|_p}{2(\alpha+\beta)}, & f''\in L_p[a,b],\ p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \left[\alpha(x-a)+\beta(b-x) + |\alpha(x-a)-\beta(b-x)|\right]\frac{\|f''\|_1}{4(\alpha+\beta)}, & f''\in L_1[a,b]. \end{cases} \tag{1.2}$$

where α and β are non-negative real numbers such that not both zero.

In this paper, motivated by the non-weighted case in [12], new general weighted Peano kernel has been defined. To obtain new general weighted inequality of Ostrowski's type that is more generalized and extended as compare to [12]. We consider a twice differentiable mapping f where, respectively, $f''\in L_{\infty}$, $f''\in L_p$

and $f'' \in L_1$. Moreover, we utilize Grüss type inequality to present the perturbed verion of our result. Finally, we investigate the new general weighted inequality in numerical integration.

Before we introduce our main result for a general weighted inequality of Ostrowski's type, we commence with the following definition and lemma.

Definition: *Let $\omega : (a,b) \to (0,\infty)$ be a non-negative weighted function (density) such that*

$$\int_a^b \omega(t)\,dt < \infty.$$

The domain of ω may be finite or infinite and may vanish at the boundary points. We denote the moments

$$\begin{aligned} m(a,b) &= \frac{1}{b-a}\int_a^b \omega(t)\,dt, \qquad M(a,b) = \frac{1}{b-a}\int_a^b t\omega(t)\,dt, \\ N(a,b) &= \frac{1}{b-a}\int_a^b t^2\omega(t)\,dt, \qquad \mu(a,b) = \frac{M(a,b)}{m(a,b)}, \\ \sigma^2(a,b) &= \frac{N(a,b)}{m(a,b)} - \mu^2(a,b). \end{aligned} \tag{1.3}$$

Furthermore, for a function $f : [a,b] \to \mathbb{R}$, we define the functional

$$S(f;a,b) = \frac{1}{b-a}\int_a^b f(t)\omega(t)\,dt. \tag{1.4}$$

Lemma 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping. Denote by $P_\omega : [a,b]^2 \to \mathbb{R}$ the weighted Peano kernel function that is given by*

$$P_\omega(x,t) = \begin{cases} \frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\int_a^t (t-u)\,\omega(u)du, & t \in [a,x] \\ \frac{\beta}{\alpha+\beta}\frac{1}{b-x}\int_b^t (t-u)\,\omega(u)du, & t \in (x,b], \end{cases} \tag{1.5}$$

where $\alpha, \beta \geq 0$ and not both zero. Then the following identity

$$\int_a^b P_\omega(x,t) f''(t)\,dt = F(f;\alpha,\beta), \tag{1.6}$$

where

$$\begin{aligned} F(f;\alpha,\beta) &= \frac{1}{\alpha+\beta}\{[\alpha m(a,x)(x-\mu(a,x)) \\ &\quad + \beta m(x,b)(x-\mu(x,b))] f'(x) \\ &\quad - [\alpha m(a,x) + \beta m(x,b)] f(x) \\ &\quad + \alpha S(f;a,x) + \beta S(f;x,b)\}, \end{aligned} \tag{1.7}$$

holds.

Proof: From (1.5), we have

$$\begin{aligned}
&\int_a^b P_\omega(x,t) f''(t)\,dt \\
&= \frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\int_a^x \left(\int_a^t (t-u)\,\omega(u)du\right) f''(t)\,dt \\
&\quad + \frac{\beta}{\alpha+\beta}\frac{1}{b-x}\int_x^b \left(\int_b^t (t-u)\,\omega(u)du\right) f''(t)\,dt \\
&= \frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\left[f'(x)\int_a^x (x-u)\,\omega(u)du \right. \\
&\quad \left. -f(x)\int_a^x \omega(u)du + \int_a^x f(t)\omega(t)dt\right] \\
&\quad + \frac{\beta}{\alpha+\beta}\frac{1}{b-x}\left[f'(x)\int_x^b (x-u)\,\omega(u)du \right. \\
&\quad \left. - f(x)\int_b^x \omega(u)du + \int_x^b f(t)\omega(t)dt\right].
\end{aligned}$$

After further simplification, the identity (1.6) can be obtained.

2. Main Results

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping in (a,b). Then $\forall x \in [a,b]$, we have*

$$|F(f;\alpha,\beta)| \le \begin{cases}
\begin{array}{l}\left\{\alpha m(a,x)\left[(x-\mu(a,x))^2+\sigma^2(a,x)\right]\right. \\ \left. +\,\beta m(x,b)\left[(x-\mu(x,b))^2+\sigma^2(x,b)\right]\right\} \\ \times\frac{\|f''\|_\infty}{2(\alpha+\beta)},\end{array} & f''\in L_\infty[a,b], \\[2ex]
\begin{array}{l}\left[\frac{\alpha^q}{(x-a)^q}\int_a^x (t-a)^{2q} m^q(a,t)dt\right. \\ \left. +\,\frac{\beta^q}{(b-x)^q}\int_x^b (b-t)^{2q} m^q(t,b)dt\right]^{\frac{1}{q}} \frac{\|f''\|_p}{2(\alpha+\beta)},\end{array} & \begin{array}{l} f''\in L_p[a,b], \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\end{array} \\[2ex]
\begin{array}{l}\max\left\{\alpha m(a,x)\left(x-\mu(a,x)\right),\right. \\ \left.\beta m(x,b)\left(\mu(x,b)-x\right)\right\}\frac{\|f''\|_1}{\alpha+\beta},\end{array} & f''\in L_1[a,b].
\end{cases} \tag{2.1}$$

where the functional $F(f;\alpha,\beta)$ is defined in (1.6).

Proof: Taking the modulus of the right hand side of (1.6), yields

$$\left|\int_a^b P_\omega(x,t) f''(t)\,dt\right| \le \int_a^b |P_\omega(x,t)|\,|f''(t)|\,dt. \tag{2.2}$$

Hence, for $f'' \in L_\infty[a,b]$

$$|F(f;\alpha,\beta)| \le \|f''\|_\infty \int_a^b |P_\omega(x,t)|\,dt. \tag{2.3}$$

Now, using (1.5) provids,

$$\begin{aligned}
&\int_a^b |P_\omega(x,t)|\,dt \\
&= \frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\left[\frac{x^2}{2}\int_a^x \omega(u)\,du\right. \\
&\left.- x\int_a^x u\omega(u)\,du + \frac{1}{2}\int_a^x t^2\omega(t)\,dt\right] \\
&+ \frac{\beta}{\alpha+\beta}\frac{1}{b-x}\left[\frac{x^2}{2}\int_x^b \omega(u)\,du\right. \\
&\left.- x\int_x^b u\omega(u)\,du + \frac{1}{2}\int_x^b t^2\omega(t)\,dt\right].
\end{aligned} \tag{2.4}$$

Thus, by combining (2.3) and (2.4), the first inequality of (2.1) results.

Further, from (2.2) and by using Hölder's integral inequality for $f'' \in L_p[a,b]$, we have

$$|F(f;\alpha,\beta)| \le \|f''\|_p \left(\int_a^b |P_\omega(x,t)|^q\,dt\right)^{\frac{1}{q}}, \tag{2.5}$$

where $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$. Now, by (1.5) and utilizing the weighted mean value theorem for integrals, we have

$$\begin{aligned}
&\int_a^b |P_\omega(x,t)|^q\,dt \\
&= \left(\frac{\alpha}{\alpha+\beta}\right)^q \frac{1}{(x-a)^q}\int_a^x \left(\int_a^t (t-u)\,\omega(u)du\right)^q dt
\end{aligned}$$

$$
\begin{aligned}
& +\left(\frac{\beta}{\alpha+\beta}\right)^q \frac{1}{(b-x)^q} \int_x^b \left(\int_b^t (t-u)\,\omega(u)du\right)^q dt \\
= & \left(\frac{\alpha}{\alpha+\beta}\right)^q \frac{1}{(x-a)^q} \int_a^x \left(\frac{t-a}{2} \int_a^t \omega(u)du\right)^q dt \\
& +\left(\frac{\beta}{\alpha+\beta}\right)^q \frac{1}{(b-x)^q} \int_x^b \left(\frac{b-t}{2} \int_t^b \omega(u)du\right)^q dt, \qquad (2.6)
\end{aligned}
$$

and so, by considering (2.5) and (2.6) the second inequality of (2.1) is obtained.

Finally, for $f'' \in L_1[a,b]$, we have from (1.6)

$$
F(f;\alpha,\beta) \le \sup_{t\in[a,b]} |P_\omega(x,t)| \, \|f''\|_1 , \qquad (2.7)
$$

where,

$$
\begin{aligned}
\sup_{t\in[a,b]} |P_\omega(x,t)| &= \max\left\{ \frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\int_a^x (x-u)\,\omega(u)du \right., \\
&\qquad \left. \frac{\beta}{\alpha+\beta}\frac{1}{b-x}\int_x^b (u-x)\,\omega(u)du \right\} \\
&= \frac{1}{\alpha+\beta}\max\left\{\alpha m(a,x)(x-\mu(a,x)\right. , \\
&\qquad \left.\beta m(x,b)(\mu(x,b)-x]\right\}. \qquad (2.8)
\end{aligned}
$$

Therefore, combining (2.7) and (2.8) gives the third inequality of (2.1), and so, the theorem is now completely proven.

Remark 2.1. *Setting $\omega(u)=1$ in Theorem (2.1) provids the corresponding non-weighted result (1.2) in [12]. For different weights, a variety of results can be obtained.*

Corollary 2.1. *Let the condition of Theorem* (2.1) *holds. Then*

$$\begin{aligned}
&\left|F\left(f;\alpha,\beta\right)\right| \\
\leq\ & \left\{\alpha m(a,x)\left[(x-\mu(a,x))^2+\sigma^2\left(a,x\right)\right]\right. \\
&\left.+\ \beta m(x,b)\left[(x-\mu(x,b))^2+\sigma^2\left(x,b\right)\right]\right\} \\
&\times\frac{\|f\,''\|_\infty}{2(\alpha+\beta)} \\
\leq\ & \left[\alpha\left(x-a\right)^2 m(a,x)+\beta\left(x-b\right)^2 m(x,b)\right] \\
&\times\frac{\|f\,''\|_\infty}{2(\alpha+\beta)}. \qquad (2.9)
\end{aligned}$$

Proof: From (1.5), we have

$$\begin{aligned}
&\int_a^b \left|P_\omega(x,t)\right| dt \\
=\ & \frac{1}{2}\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\int_a^x \left(x-t\right)^2\omega\left(t\right)dt \\
&+\frac{1}{2}\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\int_x^b \left(x-t\right)^2\omega\left(t\right)dt.
\end{aligned}$$

Now, by noting that

$$\begin{aligned}
\int_a^x \left(x-t\right)^2\omega\left(t\right)dt &\leq \sup_{t\in[a,x]}\left(x-t\right)^2\left[\left(x-a\right)m(a,x)\right] \\
&= \left(x-a\right)^3 m(a,x),
\end{aligned}$$

and

$$\begin{aligned}
\int_x^b \left(x-t\right)^2\omega\left(t\right)dt &\leq \sup_{t\in(x,b]}\left(x-t\right)^2\left[\left(b-x\right)m(x,b)\right] \\
&= \left(b-x\right)^3 m(x,b).
\end{aligned}$$

The desired second inequality of (2.9) can be obtained.

Corollary 2.2. *Setting* $\alpha=\beta$ *in Theorem* (2.1) *gives*

$$\begin{aligned}
&\left|\frac{1}{2}\left\{\left[m(a,x)(x-\mu(a,x))+m(x,b)(x-\mu(x,b))\right]f\,'(x)\right.\right. \\
&\left.\left.-\ \left(m(a,x)+m(x,b)\right)f(x)+S(f;a,x)+S(f;x,b)\right\}\right|
\end{aligned}$$

$$\leq \begin{cases} \left\{m(a,x)\left[(x-\mu(a,x))^2+\sigma^2(a,x)\right] \right. \\ \left. + m(x,b)\left[(x-\mu(x,b))^2+\sigma^2(x,b)\right]\right\} \\ \times\frac{\|f''\|_\infty}{4}, & f''\in L_\infty[a,b], \\ \\ \left[\frac{1}{(x-a)^q}\int\limits_a^x (t-a)^{2q} m^q(a,t)dt \right. \\ \left. + \frac{1}{(b-x)^q}\int\limits_x^b (b-t)^{2q} m^q(t,b)dt\right]^{\frac{1}{q}} \frac{\|f''\|_p}{4}, & f''\in L_p[a,b], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max\left\{m(a,x)(x-\mu(a,x)), \right. \\ \left. m(x,b)(\mu(x,b)-x)\right\}\frac{\|f''\|_1}{2}, & f''\in L_1[a,b]. \end{cases} \tag{2.10}$$

Remark 2.2. *Setting $\omega(u)=1$ in Corollary (2.2) gives the corresponding non-weighted result obtained by [12] for the case $\alpha=\beta$.*

Corollary 2.3. *Setting $x=(a+b)/2$ in Theorem (2.1) gives*

$$\left|\frac{1}{\alpha+\beta}\left\{\left[\alpha m\left(a,\frac{a+b}{2}\right)\left(\frac{a+b}{2}-\mu\left(a,\frac{a+b}{2}\right)\right)\right.\right.\right.$$
$$\left. + \beta m\left(\frac{a+b}{2},b\right)\left(\frac{a+b}{2}-\mu\left(\frac{a+b}{2},b\right)\right)\right] f'\left(\frac{a+b}{2}\right)$$
$$-\left[\alpha m\left(a,\frac{a+b}{2}\right)+\beta m\left(\frac{a+b}{2},b\right)\right] f\left(\frac{a+b}{2}\right)$$
$$\left.\left. + \alpha S\left(f;a,\frac{a+b}{2}\right)+\beta S\left(f;\frac{a+b}{2},b\right)\right\}\right|$$

$$\leq \begin{cases} \left\{\alpha m(a,\frac{a+b}{2})\left[(\frac{a+b}{2}-\mu(a,\frac{a+b}{2}))^2+\sigma^2\left(a,\frac{a+b}{2}\right)\right] \right. \\ \left. + \beta m(\frac{a+b}{2},b)\left[(\frac{a+b}{2}-\mu(\frac{a+b}{2},b))^2+\sigma^2\left(\frac{a+b}{2},b\right)\right]\right\} & f''\in L_\infty[a,b], \\ \times\|f''\|_\infty, \\ \\ \left[\alpha^q\int\limits_a^{\frac{a+b}{2}} (t-a)^{2q} m^q(a,t)dt \right. \\ \left. + \beta^q\int\limits_{\frac{a+b}{2}}^b (b-t)^{2q} m^q(t,b)dt\right]^{\frac{1}{q}} \frac{\|f''\|_p}{(b-a)(\alpha+\beta)}, & f''\in L_p[a,b], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max\left\{\alpha m(a,\frac{a+b}{2})\left(\frac{a+b}{2}-\mu(a,\frac{a+b}{2})\right), \right. \\ \left. \beta\, m(\frac{a+b}{2},b)\left(\mu(\frac{a+b}{2},b)-\frac{a+b}{2}\right)\right\}\frac{\|f''\|_1}{\alpha+\beta}, & f''\in L_1[a,b]. \end{cases} \tag{2.11}$$

Remark 2.3. *Setting $\omega(u) = 1$ in Corollary (2.3), gives the corresponding non-weighted result obtained by [12] when $x = (a+b)/2$.*

Corollary 2.4. *If (2.10) evaluated at $x = (a+b)/2$, then*

$$\Bigg|\frac{1}{2}\Bigg\{\Bigg[m(a,\frac{a+b}{2})(\frac{a+b}{2}-\mu(a,\frac{a+b}{2})) + m(\frac{a+b}{2},b)(x-\mu(\frac{a+b}{2},b))\Bigg]f'(\frac{a+b}{2}) - \left(m(a,\frac{a+b}{2})+m(\frac{a+b}{2},b)\right)f(\frac{a+b}{2}) + S(f;a,\frac{a+b}{2})+S(f;\frac{a+b}{2},b)\Bigg\}\Bigg|$$

$$\leq \begin{cases} \left\{m(a,\frac{a+b}{2})\left[(\frac{a+b}{2}-\mu(a,\frac{a+b}{2}))^2+\sigma^2\left(a,\frac{a+b}{2}\right)\right] + m(\frac{a+b}{2},b)\left[(\frac{a+b}{2}-\mu(\frac{a+b}{2},b))^2+\sigma^2\left(\frac{a+b}{2},b\right)\right]\right\} \times\frac{\|f''\|_\infty}{4}, & f''\in L_\infty[a,b], \\ \left[\int_a^{\frac{a+b}{2}}(t-a)^{2q}m^q(a,t)dt + \int_{\frac{a+b}{2}}^b (b-t)^{2q}m^q(t,b)dt\right]^{\frac{1}{q}}\frac{\|f''\|_p}{2(b-a)}, & f''\in L_p[a,b], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \max\left\{m(a,\frac{a+b}{2})\left(\frac{a+b}{2}-\mu(a,\frac{a+b}{2})\right), m(\frac{a+b}{2},b)\left(\mu(\frac{a+b}{2},b)-\frac{a+b}{2}\right)\right\}\frac{\|f''\|_1}{2}, & f''\in L_1[a,b]. \end{cases} \tag{2.12}$$

3. Perturbed Results

The Grüss inequality is as follows [13].

Theorem 3.1. *Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$ such that $\varphi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ $\forall x \in [a,b]$, where φ, Φ, γ, Γ are constants. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{3.1}$$

where the constant $\frac{1}{4}$ is sharp.

Now, the perturbed verions of the results in the pervious section may be obtained by using Grüss type inequalities involving the Čebyŝev functional [14],

$$T(f,g) = M(fg;a,b) - M(f;a,b)M(g;a,b), \tag{3.2}$$

where

$$M(f;a,b)=\frac{1}{b-a}\int_a^b f(t)dt,$$

is the integral mean of f over $[a,b]$.

Theorem 3.2. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping such that $\gamma\le f''(x)\le\Gamma\ \forall x\in[a,b]$ and $\alpha\ge 0,\beta\ge 0,\ \alpha+\beta\ne 0$. Then*

$$\begin{aligned}
&\Big|\digamma(f;\alpha,\beta)-\frac{k}{2(\alpha+\beta)}\left[\alpha m(a,x)\left[(x-\mu(a,x))^2+\sigma^2(a,x)\right]\right.\\
&\left.+\ \beta m(x,b)\left[(x-\mu(x,b))^2+\sigma^2(x,b)\right]\right]\Big|\\
\le\ &(b-a)N(x)\left[\frac{\|f''\|_2^2}{b-a}-k^2\right]^{\frac{1}{2}},\\
\le\ &(b-a)N(x)\frac{\Gamma-\gamma}{2},\\
\le\ &\frac{(b-a)(\Gamma-\gamma)}{4(\alpha+\beta)}\max\{\alpha m(a,x)(x-\mu(a,x)),\\
&\beta m(x,b)(\mu(x,b)-x)\},
\end{aligned}\tag{3.3}$$

where $\digamma(f;\alpha,\beta)$ is given by (1.7), $k=(f'(b)-f'(a))/(b-a)$, and

$$\begin{aligned}
N(x)\ =\ &\left\{\left[\left(\frac{\alpha}{x-a}\right)^2\int_a^x (t-a)^4 m^2(a,t)dt\right.\right.\\
&\left.+\left(\frac{\beta}{b-x}\right)^2\int_x^b (b-t)^4 m^2(t,b)dt\right]\frac{1}{4(\alpha+\beta)^2(b-a)}\\
&-\left\{\left[\alpha m(a,x)\left[(x-\mu(a,x))^2+\sigma^2(a,x)\right]\right.\right.\\
&\left.\left.\left.+\ \beta m(x,b)\left[(x-\mu(x,b))^2+\sigma^2(x,b)\right]\right]\frac{1}{2(\alpha+\beta)(b-a)}\right\}^2\right\}^{\frac{1}{2}}.
\end{aligned}\tag{3.4}$$

Proof: Replacing $f(t)$ by $P_\omega(x,t)$ and $g(t)$ by $f''(x)$ in (3.2) yields,

$$\begin{aligned}
T(P_\omega(x,t),f''(x))\ =\ &M(P_\omega(x,t)f''(x);a,b)\\
&-M(P_\omega(x,t);a,b)M(f''(x);a,b).
\end{aligned}\tag{3.5}$$

Now, by using both (1.6) and (2.4), we have

$$\begin{aligned}
T(P_\omega(x,t),f''(x))\ =\ &\frac{1}{b-a}\{\digamma(f;\alpha,\beta)\\
&-\frac{k}{2(\alpha+\beta)}\left[\alpha m(a,x)\left[(x-\mu(a,x))^2+\sigma^2(a,x)\right]\right.\\
&\left.+\ \beta m(x,b)\left[(x-\mu(x,b))^2+\sigma^2(x,b)\right]\right]\},
\end{aligned}\tag{3.6}$$

where k is the secant slope of f' over $[a,b]$ as given in (3.4). Moreover, by [3], we have

$$\begin{aligned}
T\left(P_\omega(x,t), f''(x)\right) &\leq T^{\frac{1}{2}}\left(P_\omega(x,t), P_\omega(x,t)\right) T^{\frac{1}{2}}\left(f''(x), f''(x)\right), \\
&\qquad \left(P_\omega(x,t),\ f''(x) \in L_2\left[a,b\right]\ \right) \\
&\leq \frac{\Gamma-\gamma}{2} T^{\frac{1}{2}}\left(P_\omega(x,t), P_\omega(x,t)\right), \\
&\qquad \left(\gamma \leq f''(x) \leq \Gamma,\ \forall x \in [a,b]\right) \\
&\leq \frac{1}{4}\left(\Phi-\varphi\right)\left(\Gamma-\gamma\right), \qquad (3.7) \\
&\qquad \left(\varphi \leq P_\omega(x,t) \leq \Phi,\ \forall\left(x,t\right) \in \left[a,b\right]^2\right).
\end{aligned}$$

But,

$$\begin{aligned}
0 &\leq T^{\frac{1}{2}}\left(f''(x), f''(x)\right) \\
&= \left[M(\left(f''(x)\right)^2; a,b) - M^2(f''(x); a,b)\right]^{\frac{1}{2}} \\
&= \left[\frac{1}{b-a}\int_a^b \left(f''(x)\right)^2 dx - \left(\frac{\int_a^b f''(x)dx}{b-a}\right)^2\right]^{\frac{1}{2}} \\
&= \left[\frac{1}{b-a}\left\|f''\right\|_2^2 - k^2\right]^{\frac{1}{2}} \\
&\leq \frac{\Gamma-\gamma}{2}, \qquad (3.8)
\end{aligned}$$

where $\gamma \leq f''(x) \leq \Gamma$, $\forall x \in [a,b]$.

Now, for $T^{\frac{1}{2}}\left(P_\omega(x,t), P_\omega(x,t)\right)$, we consider (1.5) as follows

$$\begin{aligned}
0 &\leq T^{\frac{1}{2}}\left(P_\omega(x,t), P_\omega(x,t)\right) \\
&= \left[M(\left(P_\omega(x,t)\right)^2; a,b) - M^2(P_\omega(x,t); a,b)\right]^{\frac{1}{2}} \\
&= \left[\frac{1}{b-a}\int_a^b \left(P_\omega(x,t)\right)^2 dx - \left(\frac{\int_a^b P_\omega(x,t)dx}{b-a}\right)^2\right]^{\frac{1}{2}} \\
&= N(x), \qquad (3.9)
\end{aligned}$$

where $N(x)$ is given in (3.4).

Therefore, by combining (3.6), (3.7), (3.8), and (3.9) gives the first and the second inequalities of (3.3). Further, to determine the values of φ and Φ for which $\varphi \leq P_\omega(x,t) \leq \Phi$, $\forall\left(x,t\right) \in \left[a,b\right]^2$, it may be noticed from the definition of $P_\omega(x,t)$ in (1.5) that for $\alpha,\beta \geq 0$, $\alpha+\beta \neq 0$, we have

$$\begin{aligned}\Phi &= \sup_{t\in[a,b]} P_\omega(x,t) \\ &= \frac{1}{\alpha+\beta}\max\left\{\alpha m(a,x)\left(x-\mu(a,x)\right), \beta m(x,b)\left(\mu(x,b)-x\right)\right\},\end{aligned}$$

$$\varphi = \inf_{t\in[a,b]} P_\omega(x,t) = 0. \tag{3.10}$$

Hence, from (3.6), (3.8), (3.10) and the last inequality in (3.7), we obtain the third inequality in (3.3) and the theorem is now completely proved.

4. Application in Numerical Integration

Let $a = \zeta_\circ < \zeta_1 < \cdots < \zeta_{n-1} < \zeta_n = x = \eta_\circ < \eta_1 < \cdots < \eta_{n-1} < \eta_n = b$ be a partition of the interval $[a,b]$, with $x_i \in [\zeta_i, \zeta_{i+1}]$ for $i = 0,1,.....,n-1$, $x_j^* \in [\eta_j, \eta_{j+1}]$ for $j = 0,1,.....,n-1$, $\delta = \zeta_{i+1}-\zeta_i$, and $\Delta = \eta_{j+1}-\eta_j$. Consider the following general quadrature rule.

$$\begin{aligned}&A\left(f,\zeta,\eta,x\right) \\ =\ & \alpha\sum_{i=0}^{n-1} m_i\left[f\ (x_i) - (x_i-\mu_i)\, f\,'(x_i)\right] \\ &+\beta\sum_{j=0}^{n-1} m_j^*\left[f\ (x_j^*) - \left(x_j^*-\mu_j^*\right) f\,'\left(x_j^*\right)\right].\end{aligned} \tag{4.1}$$

Theorem 4.1. *Let the conditions of Theorem* (2.1) *hold. The following weighted quadrature rule for weighted integral holds*

$$\begin{aligned}&\frac{\alpha}{\delta}\int_a^x f(t)\omega\left(t\right)dt + \frac{\beta}{\Delta}\int_x^b f(t)\omega\left(t\right)dt \\ =\ & A\left(f,\zeta,\eta,x\right) + R\left(f,\zeta,\eta,x\right),\end{aligned} \tag{4.2}$$

where $A(f,\zeta,\eta,x)$ is defined by (4.1)*, the remainder $R(f,\zeta,\eta,x)$ satisfies the estimate*

$$\begin{aligned}&R\left(f,\zeta,\eta,x\right) \\ \leq\ & \left\{\alpha\sum_{i=0}^{n-1} m_i\left[\left(x_i-\mu_i\right)^2+\sigma_i^2\right]\right. \\ &\left.+\ \beta\sum_{j=0}^{n-1} m_j^*\left[\left(x_j^*-\mu_j^*\right)^2+\sigma_j^{*2}\right]\right\}\frac{\|f\,''\|_\infty}{2},\end{aligned} \tag{4.3}$$

and the parameters m_i, μ_i, σ_i^2, m_j^*, μ_j^*, and σ_j^{*2} are given by

$$\begin{aligned} m_i &= m(\zeta_i,\zeta_{i+1}), \quad \mu_i = \mu\left(\zeta_i,\zeta_{i+1}\right), \quad \sigma_i^2 = \sigma^2\left(\zeta_i,\zeta_{i+1}\right), \\ m_j^* &= m\left(\eta_j,\eta_{j+1}\right), \quad \mu_j^* = \mu\left(\eta_j,\eta_{j+1}\right), \ \textit{and}\ \ \sigma_j^{*2} = \sigma^2\left(\eta_j,\eta_{j+1}\right).\end{aligned} \tag{4.4}$$

Proof: Applying the first inequality of (2.1) over the interval $[\zeta_i, \zeta_{i+1}]$ with $x = x_i \in [\zeta_i, \zeta_{i+1}]$ and over the interval $[\eta_j, \eta_{j+1}]$ with $x = x_j^* \in [\eta_j, \eta_{j+1}]$ gives

$$\begin{aligned} &\Big|\alpha m_i(x_i-\mu_i)f'(x_i)+\beta m_j^*(x_j^*-\ \mu_j^*)f'(x_j^*)-\alpha m_i f(x_i)+\beta m_j^* f(x_j^*) \\ &+\frac{\alpha}{\delta}\int_{\zeta_i}^{\zeta_{i+1}} f(t)\omega(t)\,dt+\frac{\beta}{\Delta}\int_{\eta_j}^{\eta_{j+1}} f(t)\omega(t)\,dt\Big| \\ \le\ & \left\{\alpha m_i\left[(x_i-\mu_i)^2+\ \sigma_i^2\right]+\beta m_j^*\left[(x_j^*-\mu_j^*)^2+\ \sigma_j^{*2}\right]\right\}\frac{\|f''\|_\infty}{2}, \end{aligned}$$

for all $i, j = 0, 1, ..., n-1$. Summing over i, j from 0 to $n-1$ and using the triangle inequality produces the desired result (4.2).

Theorem 4.2. *Let the conditions of Theorem* (2.1) *hold. The following weighted quadrature rule for weighted integral holds*

$$\begin{aligned} &\frac{\alpha}{\delta}\int_a^x f(t)\omega(t)\,dt+\frac{\beta}{\Delta}\int_x^b f(t)\omega(t)\,dt \\ =\ & A(f,\zeta,\eta,x)+R(f,\zeta,\eta,x), \end{aligned}$$

where $A(f,\zeta,\eta,x)$ *is defined by* (4.1), *the remainder* $R(f,\zeta,\eta,x)$ *satisfies the estimate*

$$\begin{aligned} &R(f,\zeta,\eta,x) \\ \le\ & \left[\frac{\alpha^q}{\delta^q}\int_{\zeta_i}^{\zeta_{i+1}}(t-\zeta_i)^{2q}m^q(\zeta_i,t)dt\right. \\ &\left.+\frac{\beta^q}{\Delta^q}\int_{\eta_j}^{\eta_{j+1}}(\eta_{j+1}-t)^{2q}m^q(t,\eta_{j+1})dt\right]^{\frac{1}{q}}\frac{\|f''\|_p}{2}. \end{aligned} \tag{4.5}$$

Proof: Applying the second inequality of (2.1) over the interval $[\zeta_i, \zeta_{i+1}]$ with $x = x_i \in [\zeta_i, \zeta_{i+1}]$ and over the interval $[\eta_j, \eta_{j+1}]$ with $x = x_j^* \in [\eta_j, \eta_{j+1}]$ gives

$$\begin{aligned} &\Big|\alpha m_i(x_i-\mu_i)f'(x_i)+\beta m_j^*(x_j^*-\ \mu_j^*)f'(x_j^*)-\alpha m_i f(x_i)+\beta m_j^* f(x_j^*) \\ &+\frac{\alpha}{\delta}\int_{\zeta_i}^{\zeta_{i+1}} f(t)\omega(t)\,dt+\frac{\beta}{\Delta}\int_{\eta_j}^{\eta_{j+1}} f(t)\omega(t)\,dt\Big| \\ \le\ & \left[\frac{\alpha^q}{\delta^q}\int_{\zeta_i}^{\zeta_{i+1}}(t-\zeta_i)^{2q}m^q(\zeta_i,t)dt\right. \\ &\left.+\frac{\beta^q}{\Delta^q}\int_{\eta_j}^{\eta_{j+1}}(\eta_{j+1}-t)^{2q}m^q(t,\eta_{j+1})dt\right]^{\frac{1}{q}}\frac{\|f''\|_p}{2}. \end{aligned}$$

for all $i, j = 0, 1, ..., n-1$. Summing over i, j from 0 to $n-1$ and using the triangle inequality produces the desired result (4.5).

Theorem 4.3. *Let the conditions of Theorem* (2.1) *hold. The following weighted quadrature rule for weighted integral holds*

$$\begin{aligned} & \frac{\alpha}{\delta}\int_{a}^{x} f(t)\omega(t)\,dt + \frac{\beta}{\Delta}\int_{x}^{b} f(t)\omega(t)\,dt \\ = \;& A(f,\zeta,\eta,x) + R(f,\zeta,\eta,x), \end{aligned}$$

where $A(f,\zeta,\eta,x)$ is defined by (4.1)*, the remainder $R(f,\zeta,\eta,x)$ satisfies the estimate*

$$\begin{aligned} & R(f,\zeta,\eta,x) \\ \leq \;& \max\left[\alpha m_i(x_i-\mu_i), \beta m_j^*\left(\mu_j^*-x_j^*\right)\right]\|f''\|_1 . \end{aligned} \tag{4.6}$$

Proof: Applying the third inequality of (2.1) over the interval $[\zeta_i, \zeta_{i+1}]$ with $x = x_i \in [\zeta_i, \zeta_{i+1}]$ and over the interval $[\eta_j, \eta_{j+1}]$ with $x = x_j^* \in [\eta_j, \eta_{j+1}]$ gives

$$\begin{aligned} & \Big|\alpha m_i(x_i-\mu_i)f'(x_i) + \beta m_j^*(x_j^*-\mu_j^*)f'(x_j^*) - \alpha m_i f(x_i) + \beta m_j^* f(x_j^*) \\ & + \frac{\alpha}{\delta}\int_{\zeta_i}^{\zeta_{i+1}} f(t)\omega(t)\,dt + \frac{\beta}{\Delta}\int_{\eta_j}^{\eta_{j+1}} f(t)\omega(t)\,dt\Big| \\ \leq \;& \max\left\{\alpha m_i(x_i-\mu_i), \beta m_j^*\left(\mu_j^*-x_j^*\right)\right\}\|f''\|_1 . \end{aligned}$$

for all $i, j = 0, 1, ..., n-1$. Summing over i, j from 0 to $n-1$ and using the triangle inequality produces the desired result (4.6).

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