International Journal of Analysis and Applications
Volume 16, Number 4 (2018), 528-541
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-16-2018-528

ON LEFT ALMOST SEMIHYPERRINGS

SHAH NAWAZ[1], INAYATUR REHMAN[2], MUHAMMAD GULISTAN[1,*]

[1]*Department of Mathematics, Hazara University, Mansehra, Pakistan*

[2]*Department of Mathematics and Science, College of Arts and Applied Sciences, Dhofar University, Salalah, Oman*

[*]*Corresponding author:* gulistanmath@hu.edu.pk

Abstract. The purpose of this article is to introduce the notion of left almost semihyperrings which is a generalization of left almost semirings. We investigate the basic properties of left almost semihyperrings. By using the concept of hyperideal and regular relations we prove some useful results on it.

1. Introduction

Kazim and Naseeudin [10] studied left almost semigroup (abbreviated as LA-semigroup). They generalized some handy sequal of semigroup theroy. Mushtaq and others [14–16] added many useful result of theory of LA-semigroups, also see [2, 8, 9]. LA-semigroup is the midway structure between a commutative semigroup and a groupoid. On the other hand it posses many interesting properties which we usully find in commutativ and associative algebric structure.

Hyperstructures were introduced in 1934, when Marty [13] defined hypergroups, began to study their properties, and applied them to groups. A number of papers and several book have been written on hyperstructure theory; see [3,19]. Currently a book published on hyperstructure [4] points out on its applications in rough set theory, cryptography, automata, codes automata, probability, geometry, lattices, binary relations, graphs, and hypergraphs. Hila and Dine [7] introduced the notion of left almost semihypergroups. Yaqoob

Received 2018-02-18; accepted 2018-04-27; published 2018-07-02.

2010 *Mathematics Subject Classification.* 20N25.

Key words and phrases. LA-semihypergroups; LA-semihyperrings; regular relations.

et al. [20] extended the work of Hila and Dine and characterized intra-regular left almost semihypergroups by their hyperideals using pure left identity. Gulistan et al. [20] introduced the notion of H_v-LA-semigroups. Rehman et al. [20] studied hyperideals and hypersystems in LA-hyperrings. Yaqoob and Gulistan [21] introduced the concept of partial ordering on left almost semihypergroups. Yaqoob et al. [22] introduced the idea of left almost polygroups. Fuzzy set theory and rough set theory are also applied to LA-semihypergroups, see [1,6,12,23,24,26]. In [25], Yusuf extended the concept of LA-groups to a non-associative structure with respect to the binary operations '+' and '×' namely left almost ring (LA-ring). Further M. Shah and T. Shah studied some basic properties of LA-ring in [18]. Kellil [11] studied left almost semirings.

The purpose of this article is to introduce a new and more gerenal class of left almost semihyperrings which is of cource a generalization of left almost semiring. We have investigated the basic properties of left almost semihyperrings. By using the concept of hyperideal and regular relations we have proved some useful results on it.

2. LA-semihypergroups

In this section we recalled some basic ideas from litrature about an LA-semihypergroup which helped in further development of this article.

Definition 2.1. *A map $\circ : X \times X \longrightarrow \mathcal{P}^*(X)$ is called an hyperoperation or join operation on the set X, where X is a non-empty set and $\mathcal{P}^*(X) = \mathcal{P}(X)\backslash\{\emptyset\}$ shows the all non empty subset of X. A hypergroupoid is a set X with the binary operation is a hyperoperation.*

If A and B be two non-empty subsets of X then we write product as fallow

$$A \circ B = \bigcup_{x_1 \in A, x_2 \in B} x_1 \circ x_2,\ x_1 \circ A = \{x_1\} \circ A \ \text{ and } x_1 \circ B = \{x_1\} \circ B.$$

Definition 2.2. *[7,20] A hypergroupoid $(X, \circ)$ is called an LA-semihypergroup if for all $x_1, x_2, x_3 \in X$,*

$$(x_1 \circ x_2) \circ x_3 = (x_3 \circ x_2) \circ x_1,$$

the law $(x_1 \circ x_2) \circ x_3 = (x_3 \circ x_2) \circ x_1$ is known as left invertive law.

Every LA-semihypergroup satisfies the law

$$(x_1 \circ x_2) \circ (x_3 \circ x_4) = (x_1 \circ x_3) \circ (x_2 \circ x_4)$$

for all $x_1, x_2, x_3, x_4 \in X$. This law is known as medial law (cf. [7]).

Lemma 2.1. *[20] Let X be an LA-semihypergroup with pure left idetity e, then $x_1 \circ (x_2 \circ x_3) = x_2 \circ (x_1 \circ x_3)$ holds for all $x_1, x_2, x_3 \in X$.*

Lemma 2.2. *[20] Let X be an LA-semihypergroup with pure left identity e, then $(x_1 \circ x_2) \circ (x_3 \circ x_4) = (x_4 \circ x_2) \circ (x_3 \circ x_1)$ holds for all $x_1, x_2, x_3, x_4 \in X$.*

The law $(x_1 \circ x_2) \circ (x_3 \circ x_4) = (x_4 \circ x_2) \circ (x_3 \circ x_1)$ is called a paramedial law.

3. Left Almost Semihyperrings

In this section we define the notion of left almost semihyperrings and provided some examples with some basic properties.

Definition 3.1. *An algebraic hyperstructure $(R, \oplus, \otimes)$ is said to be an LA-semihyperring if it satisfies the following axioms:*

(1) $(R, \oplus)$ is an LA-semihypergroup;

(2) $(R, \otimes)$ is an LA-semihypergroup, with an absorbing element $0 \in R$, that is, $0 \otimes x = x \otimes 0 = 0$ for all $x \in R$.

(3) The operation $\otimes$ is distributive with respect to the hyperoperation $\oplus$, that is

$$\begin{aligned} x_1 \otimes (x_2 \oplus x_3) &= (x_1 \otimes x_2) \oplus (x_1 \otimes x_3), \\ (x_1 \oplus x_2) \otimes x_3 &= (x_1 \otimes x_3) \oplus (x_2 \otimes x_3), \end{aligned}$$

for all $x_1, x_2, x_3 \in R$.

Example 3.1. *Let $R = \{0, a, b\}$ be a set with the hyperoperations $\oplus$ and $\otimes$ defined as follow:*

$\oplus$	0	a	b
0	0	R	R
a	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
b	R	R	R

$\otimes$	0	a	b
0	0	0	0
a	0	R	b
b	0	R	R

Then $(R, \oplus, \otimes)$ is an LA-semihyperring.

Definition 3.2. *Let R be an LA-semihyperring. An element $e \in R$ is called*

(i) Left identity (resp., pure left identity) if for all $x \in R$, $x \in e \otimes x$ (resp., $x = e \otimes x$),

(ii) Right identity (resp., pure right identity) if for all $x \in R$, $x \in x \otimes e$ (resp., $x = x \otimes e$),

(iii) Identity (resp., pure identity) if for all $x \in R$, $x \in e \otimes x \cap x \otimes e$ (resp., $x = e \otimes x \cap x \otimes e$).

Lemma 3.1. *If an LA-semihypering R has a pure left identity e, then it is unique.*

Proof. Let us consider that there exist another pure left identity f_L such that $e \otimes f_L = f_L$ and $f_L \otimes e = e$. Consider

$$\begin{aligned} e &= e \otimes e = (f_L \otimes e) \otimes e \\ &= (e \otimes e) \otimes f_L \\ &= e \otimes f_L = f_L. \end{aligned}$$

□

Definition 3.3. *An element $x \in (R, \oplus, \otimes)$ is called additively idempotent if $x \in x \oplus x$.*

Set of all additively idempotent element is denoted by $I^{\oplus}(R)$. If every element of R is additively idempotent then R is said to be additively idempotent.

Definition 3.4. *An element $x \in (R, \oplus, \otimes)$ is called multiplicatively idempotent if $x \in x \otimes x$.*

Set of all multiplicatively idempotent element is denoted by $I^{\otimes}(R)$.

Lemma 3.2. *In an LA-semihyperring the following laws hold .*

(i)

$$(x \otimes y) \otimes (z \otimes w) = (x \otimes z) \otimes (y \otimes w), \forall x, y, z, w \in R,$$

known as medial law.

(ii) With left identity of an LA-semihyperring R,

$$(x \otimes y) \otimes (z \otimes w) = (w \otimes y) \otimes (z \otimes x), \forall x, y, z, w \in R,$$

known as paramedial law.

Proof. (i) Consider

$$\begin{aligned} (x \otimes y) \otimes (z \otimes w) &= ((z \otimes w) \otimes y) \otimes x \\ &= ((y \otimes w) \otimes z) \otimes x \\ &= (x \otimes z) \otimes (y \otimes w). \end{aligned}$$

(ii) Let $e \in R$ be the left identity. Consider

$$\begin{aligned}
(x \otimes y) \otimes (z \otimes w) &= (e \otimes (x \otimes y)) \otimes (z \otimes w) \\
&= ((z \otimes w) \otimes (x \otimes y)) \otimes e \\
&= ((z \otimes x) \otimes (w \otimes y)) \otimes e \\
&= (e \otimes (w \otimes y)) \otimes (z \otimes x) \\
&= (w \otimes y) \otimes (z \otimes x).
\end{aligned}$$

□

Remark 3.1. *By medial law we have*

$$\begin{aligned}
(x \otimes y) \otimes (z \otimes w) &= (w \otimes y) \otimes (z \otimes x) \\
&= (w \otimes z) \otimes (y \otimes x).
\end{aligned}$$

Theorem 3.1. *Let R is an LA-semihyperring with pure left identity e then $x \otimes (y \otimes z) = y \otimes (x \otimes z)$ for all $x, y, z \in R$.*

Proof. Consider

$$\begin{aligned}
x \otimes (y \otimes z) &= (e \otimes x) \otimes (y \otimes z) \\
&= (e \otimes y) \otimes (x \otimes z) \text{ by medial law} \\
&= y \otimes (x \otimes z),
\end{aligned}$$

for all $x, y, z, w \in R$. □

Theorem 3.2. *An LA-semihyperring R is said to be semihyperring if and only if $x \otimes (y \otimes z) = (z \otimes y) \otimes x$ for all $x, y, z \in R$.*

Proof. Let R is semihyperring then

$$(x \otimes y) \otimes z = x \otimes (y \otimes z),$$

but

$$(x \otimes y) \otimes z = (z \otimes y) \otimes x,$$

so

$$x \otimes (y \otimes z) = (z \otimes y) \otimes x \text{ for all } x, y, z \in R.$$

Conversely let

$$x \otimes (y \otimes z) = (z \otimes y) \otimes x \text{ for all } x, y, z \in R.$$

Since R is LA-semihyperring so

$$\begin{aligned}(x \otimes y) \otimes z &= (z \otimes y) \otimes x \\ &= x \otimes (y \otimes z).\end{aligned}$$

□

Lemma 3.3. *In an LA-semihyperring R, with pure left identity e, $x \otimes y = z \otimes w \Rightarrow y \otimes x = w \otimes z$ for all $x, y, z, w \in R$.*

Proof. Consider

$$\begin{aligned}y \otimes x &= (e \otimes y) \otimes x \\ &= (x \otimes y) \otimes e \\ &= (z \otimes w) \otimes e \text{ since } x \otimes y = z \otimes w \\ &= (e \otimes w) \otimes z \text{ by left invertive law} \\ &= w \otimes z \text{ for all } x, y, z, w \in R.\end{aligned}$$

□

Definition 3.5. *Let $\emptyset \neq K \subseteq R$, then K is called sub LA-semihyperring if K itself form the LA-semihyperring.*

Proposition 3.1. *Let $\emptyset \neq K \subseteq R$, then K is called sub LA-semihyperring if $\forall\ x, y \in K$ we get $x \oplus y \in K$ and $x \otimes y \in K$.*

Proof. Straightforward. □

Definition 3.6. *Let $\emptyset \neq A \subseteq R$, then A is called a left hyperideal (resp., right hyperideal) of an LA-semihyperring R if $A \oplus A \subseteq A$ and $R \otimes A \subseteq A$ (resp., $A \otimes R \subseteq A$).*

If A is both a left and a right hyperideal of R, then it is called a hyperideal of R.

Definition 3.7. *Let $\emptyset \neq B \subseteq R$, then B is called a bi-hyperideal of an LA-semihyperring R if $B \oplus B \subseteq B$, $B \otimes B \subseteq B$ and $(B \otimes R) \otimes B \subseteq B$.*

Definition 3.8. *Let $\emptyset \neq I \subseteq R$, then I is called an interior hyperideal of an LA-semihyperring R if $I \oplus I \subseteq I$ and $(R \otimes I) \otimes R \subseteq I$.*

Definition 3.9. *Let $\emptyset \neq Q \subseteq R$, then Q is called a quasi-hyperideal of an LA-semihyperring R if $Q \oplus Q \subseteq Q$ and $Q \otimes R \cap R \otimes Q \subseteq Q$.*

Definition 3.10. *A hyperideal A of an LA-semihyperring R is called prime hyperideal of R, if for hyperideals I and J of R satisfying, $I \otimes J \subseteq A$, implies, either $I \subseteq A$, or $J \subseteq A$.*

Definition 3.11. *For any non-empty subsets X, Y of an LA-semihyperring $(R, \oplus, \otimes)$ we define*

$$X \oplus Y = \bigcup_{\substack{l_1 \in X \\ l_2 \in Y}} (l_1 \oplus l_2) \text{ and } X \otimes Y = \bigcup_{\substack{l_{1i} \in X \\ l_{2i} \in Y}} \left(\sum_{i=1}^{n} l_{1i} \otimes l_{2i} \right).$$

Proposition 3.2. *Let X and Y be any two hyperideals of an LA-semihyperring $(R, \oplus, \otimes)$ then $X \otimes Y \subseteq X \cap Y$.*

Proof. Let $x \in X \otimes Y = \bigcup_{\substack{l_1 \in X \\ l_2 \in Y}} \left(\sum_{=1}^{n} l_1 \otimes l_2 \right) \Rightarrow x \in \bigcup_{\substack{l_1 \in X \\ l_2 \in Y}} l_3$, where $l_3 = \sum_{=1}^{n} l_1 \otimes l_2$ for each $l_1 \in X$ and $l_2 \in Y$. Now since X is an hyperideal so $l_1 \otimes l_2 \in X$ for $l_2 \in Y \subseteq R$. This implies that $\sum_{=1}^{n} l_1 \otimes l_2 \subseteq X$. we $x \in X$. Similarly by using the fact that Y is also hyperideal we get $x \in Y$, and thus $x \in X \cap Y$. Hence $X \otimes Y \subseteq X \cap Y$. □

Proposition 3.3. *Any left hyperideal of an LA-semihyperring R is a sub LA-semihyperring.*

Proof. Let I is a left hyperideal of an LA-semihyperring R. Then obviously $\forall\, x, y \in I$ we get $x \oplus y \in I$. Also for any $l_1, x \in I$ and since $l_1 \in I \subseteq R$ so we have $l_1 \otimes x \in I$. Thus I is a sub LA-simihyperring. □

In particular every right hyperideal becomes sub LA-semihyperring and so does the hyperideal.

Theorem 3.3. *Intersection of any family of hyperideal of an LA-semihyperring R is an hyperideal of R.*

Proof. Let $\{I_i\}_{i \in \wedge}$ be a family of hyperideals of an LA-semihyperring R and we have to show that $\bigcap_{i \in \wedge} I_i$ is also an hyperideal of R. Let $x, y \in \bigcap_{i \in \wedge} I_i$, then $x, y \in I_i$. Now since each I_i is an hyperideal so $x \oplus y \in I_i$ for all $i \in \wedge$. Thus $x \oplus y \in \bigcap_{i \in \wedge} I_i$. Again let $x \in \bigcap_{i \in \wedge} I_i$ and $l_1 \in R$. From $x \in \bigcap_{i \in \wedge} I_i$ we have $x \in I_i$ for all $i \in \wedge$. Since each I_i is an hyperideal so $l_1 \oplus x \in I_i$ for all $i \in \wedge$. Which implies that $l_1 \oplus x \in \bigcap_{i \in \wedge} I_i$. Thus $\bigcap_{i \in \wedge} I_i$ is an left hyperideal of R. Similarly it can easily be proved for right hyperideals and hence $\bigcap_{i \in \wedge} I_i$ is an hyperideal of R. □

Corollary 3.1. *Intersection of any family of sub LA-semihyperring of an LA-semihyperring R is again sub LA-semihyperring of R.*

Proof. Straightforward. □

Theorem 3.4. *If I and J are hyperideals of an LA-semihyperring R, then $I \oplus J$ are hyperideals of R. Moreover it is the smallest hyperideal of R containing both I and J.*

Proof. Let us define $I \oplus J = \bigcup_{\substack{l_1 \in I \\ l_2 \in J}} (l_1 \oplus l_2)$. Let $x, y \in I \oplus J$ then $\exists\ l_{11}, l_{12} \in x$ and $l_{21}, l_{22} \in J$ such that $x \in l_{11} \oplus l_{21}$ and $y \in l_{12} \oplus l_{22}$. Consider $x \oplus y \subseteq (l_{11} \oplus l_{21}) \oplus (l_{12} \oplus l_{22}) = (l_{11} \oplus l_{12}) \oplus (l_{21} \oplus l_{22})$ by medial law. Since I and J are hyperideals so $(l_{11} \oplus l_{12}) \subseteq I$ and $(l_{21} \oplus l_{22}) \subseteq J$. Thus $x \oplus y \subseteq (l_{11} \oplus l_{12}) \oplus (l_{21} \oplus l_{22}) \subseteq I \oplus J$. Hence $x \oplus y \subseteq I \oplus J$ for all $x \in I$, and $y \in J$.

Again consider $x \in I \oplus J$ and $r \in R$. For $x \in I \oplus J$ there exist some $l_1 \in I$ and $l_2 \in J$ such that $x \in l_1 \oplus l_2$. Now $r \otimes x \in r \otimes (l_1 \oplus l_2) = (r \otimes l_1) \oplus (r \otimes l_2)$ by distributive law. Since I and J are hyperideals so $(r \otimes l_1) \subseteq I$ and $(r \otimes l_2) \subseteq J$ for any $r \in R$, $l_1 \in I$ and $l_2 \in J$. Thus $(r \otimes l_1) \oplus (r \otimes l_2) \subseteq I \oplus J$ and hence $r \otimes x \in I \oplus J$ for $x \in I \oplus J$ and $r \in R$. Which shows that $I \oplus J$ is left hyperideal of R. Similarly it can easily be proved for right ideals and thus $I \oplus J$ is an hyperideal of R.

Now we will show that $I \oplus J$ contains both I and j i.e. $I \cup J \subseteq I \oplus J$. Let $x \in I \cup J \Rightarrow x \in I$ or $x \in J$. Since I and J are hyperideals so $0 \in I$ and $0 \in J$. Now if $l_1 \in I$, since $x = x \oplus 0 \subseteq I \oplus J$. And if $x \in J$, since $x = 0 \oplus x \subseteq I \oplus J$. Hence $x \in I \oplus J$ and thus $I \cup J \subseteq I \oplus J$.

Next we will show that $I \oplus J$ is the smallest hyperideal. Let M be any other hyperideal containing both I and J and we have to show that $I \oplus J \subseteq M$. For this let $x \in I \oplus J$ then there exist $l_1 \in I$ and $l_2 \in J$ such that $x \in l_1 \oplus l_2$. Since $l_1 \in I \subseteq I \cup J \subseteq M$ and $l_2 \in J \subseteq I \cup J \subseteq M$. Which implies that $l_1, l_2 \in M$, but M is the hyperideal so $l_1 \oplus l_2 \subseteq M$. Thus $x \in M$, and so $I \oplus J \subseteq M$. Hence $I \oplus J$ is the smallest hyperideal of R containing both I and J. □

Proposition 3.4. *Let $\emptyset \neq K \subseteq R$ and $\emptyset \neq I \subseteq R$ such that K is sub LA-semihyperring and I is an hyperideal then*

(*i*) *$K \oplus I$ is a sub LA-semihyperring of R.*

(*ii*) *$K \cap I$ is a hyperideal of R.*

Proof. (*i*) let us define $K \oplus I = \bigcup_{\substack{l_1 \in K \\ l_2 \in I}} (l_1 \oplus l_2)$. Since $0 \in K$ and $0 \in I$. Therefore we have $\{0\} = 0 \oplus 0 \subseteq K \oplus I$. thus $K \oplus I$ is non-empty. Let $x, y \in K \oplus I$ then there exist $l_{11}, l_{12} \in K$ and $l_{21}, l_{22} \in I$ such that $x \in l_{11} \oplus l_{21}$ and $y \in l_{12} \oplus l_{22}$. Consider $x \oplus y \subseteq (l_{11} \oplus l_{21}) \oplus (l_{12} \oplus l_{22}) = (l_{11} \oplus l_{12}) \oplus (l_{21} \oplus l_{22})$ by medial law. Since K is sub LA-semihyperring and I is an hyperideal so $(l_{11} \oplus l_{12}) \subseteq K$ and $(l_{21} \oplus l_{22}) \subseteq I$. Thus $x \oplus y \subseteq (l_{11} \oplus l_{12}) \oplus (l_{21} \oplus l_{22}) \subseteq K \oplus I$. Hence $x \oplus y \subseteq K \oplus I$ for all $x, y \in K \oplus I$. And again consider $x \otimes y \subseteq (l_{11} \oplus l_{21}) \otimes (l_{12} \oplus l_{22}) = (l_{11} \otimes l_{12}) \oplus (l_{11} \otimes l_{22}) \oplus (l_{21} \otimes l_{12}) \oplus (l_{21} \otimes l_{22})$ by distributive law. Now since K is sub LA-semihyperring so $(l_{11} \otimes l_{12}) \in K$, and I is an hyperideal so $((l_{11} \otimes l_{22}) \oplus (l_{21} \otimes l_{12}) \oplus (l_{21} \otimes l_{22})) \in I$. Eventually $x \otimes y \subseteq K \oplus I$. Hence $K \oplus I$ is a sub LA-semihyperring of R.

(*ii*) Let $x, y \in K \cap I \Rightarrow x, y \in K$ and $x, y \in I$. Since K is sub LA-semihyperring and I is an hyperideal so $x \oplus y \in K$ and $x \oplus y \in I$. Which implies that $x \oplus y \in K \cap I$. Now again let $x \in K \cap I$ and $l_1 \in R$. From $x \in K \cap I$ we have $x \in K$ and $x \in I$. Since K is sub LA-semihypergroup so $l_1 \otimes x \in K$ and I is an hyperideal so $l_1 \oplus x \in I$. Which implies that $l_1 \oplus x \in K \cap I$. Thus $K \cap I$ is an left hyperideal of R. Similarly it can easily be proved for right hyperideals and hence $K \cap I$ is an hyperideal of R. □

Theorem 3.5. *Let R be an LA-semihyperring with pure left identity then right distributive implies left distributive.*

Proof. Let R is right distributive, then $(l_1 \oplus l_2) \otimes l_3 = (l_1 \otimes l_3) \oplus (l_2 \otimes l_3)$.

Consider

$$\begin{aligned} ((l_1 \otimes l_3) \oplus (l_2 \otimes l_3)) \otimes e &= ((l_1 \otimes l_3) \otimes) e \oplus ((l_2 \otimes l_3) \otimes e) \text{ by right distributive law} \\ &= ((e \otimes l_3) \otimes l_1) \oplus ((e \otimes l_3) \otimes l_2) \text{ by left invertive law} \\ &= (l_3 \otimes l_1) \oplus (l_3 \otimes l_2) \text{ as } e \text{ is the left identity} \\ &= l_3 \otimes (l_1 \oplus l_2) \end{aligned}$$

which shows that it is left distributive. □

Theorem 3.6. *Let $e \in R$ be an LA-semihyperring with pure left identity then every right hyperideal is also a left hyperideal.*

Proof. Let l_1 be any right hyperideal of R, then it is a sub LA-semihyperring of R. Now let $l_1 \in X$ and $h \in R$ then

$$\begin{aligned} h \otimes l_1 &= (e \otimes h) \otimes l_1 \\ &= (l_1 \otimes h) \otimes e \in X. \end{aligned}$$

which shows that X is left hyperideal of R and hence hyperideal of R. □

Lemma 3.4. *Let X is an right hyperideal of R with pure left identity e then $X \otimes X$ is an hyperideal of R.*

Proof. Let $x \in X \otimes X$ then $l_1 = l_2 \otimes l_3$ where $l_2, l_3 \in X$. Consider

$$\begin{aligned} l_1 \otimes h &= (l_2 \otimes l_3) \otimes h \\ &= (h \otimes l_3) \otimes l_2 \in X \otimes X, \text{for all } h \in R. \end{aligned}$$

Hence $X \otimes X$ is an right hyperideal of R. As e is the left identity of R so by theorem 3.6 ,$X \otimes X$ is left hyperideal of R and hence hyperideal of R. □

Lemma 3.5. *Let R is an LA-semihyperring with pure left identity e. If X is a proper ideal of R then $e \notin X$.*

Proof. Suppose that $e \in X$. Let $h \in R$ and consider

$$h = e \otimes h \in R \otimes X \subseteq X \text{ for all } h \in R.$$

which implies that $R \subseteq X$. But $X \subseteq R$ is obvious and thus $X = R$. Which is contradiction to the fact that X is a proper ideal of R. Hence $e \notin X$. □

4. Homomorphisms on LA-semihyperrings

Definition 4.1. *A map $\gamma : R_1 \to R_2$ where both R_1 and R_2 are LA-semihyperring is called inclusion homorphism if*

(i) $\gamma(x \oplus y) \subseteq \gamma(x) \oplus \gamma(y)$

(ii) $\gamma(x \otimes y) \subseteq \gamma(x) \otimes \gamma(y)$ *for all* $x, y \in R_1$.

Definition 4.2. *A map $\gamma : R_1 \to R_2$ where both R_1 and R_2 are LA-semihyperring is called strong homorphism if*

(i) $\gamma(x \oplus y) = \gamma(x) \oplus \gamma(y)$

(ii) $\gamma(x \otimes y) = \gamma(x) \otimes \gamma(y)$ *for all* $x, y \in R_1$.

Definition 4.3. *Let σ be an equivalence relation on R, then σ is said to be left regular if for $x, y, z \in R$ such that $(x, y) \in \sigma$, then $(l_5, l_6) \in \sigma$ for all $l_5 \in z \oplus x, l_6 \in z \oplus y$ and $(z \otimes x, z \otimes y) \in \sigma$. σ is said to be right regular if for $x, y, z \in R$ such that $(x, y) \in \sigma$, then $(l_5, l_6) \in \sigma$ for all $l_5 \in x \oplus z, l_6 \in y \oplus z$ and $(x \otimes z \in y \otimes z) \in \sigma$. σ is said to be regular if for $x, y, z, w \in R$ such that $(x, y) \in \sigma$ and $(z, w) \in \sigma$, then $(l_5, l_6) \in \sigma$ for all $l_5 \in x \oplus z, l_6 \in y \oplus z$ and $(x \otimes z, y \otimes z) \in \sigma$.*

Proposition 4.1. *Let $(R, \oplus, \otimes)$ an LA-semihyperring and σ is an equivalence relation on R. Then σ is a regular relation on R if and only if σ is a left and right regular respectively.*

Proof. Suppose that σ is a regular relation on R. Let $x, y, z \in R$ such that $(x, y) \in \sigma$ and $(z, z) \in \sigma$ then $(l_5, l_6) \in \sigma$ for $l_5 \in z \oplus x, l_6 \in z \oplus y$ and $(z \otimes x, z \otimes y) \in \sigma$. Hence σ is a left regular relation on R. Similarly σ is a right regular relation on R. Conversely let σ is a left and right regular respectively, and let for $x, y, z, w \in R$ such that $(x, y) \in \sigma$ and $(z, w) \in \sigma$. Then by left regularity we have $(l_5, l_6) \in \sigma$ for $l_5 \in x \oplus z, l_6 \in x \oplus w$ and $(x \otimes z, x \otimes w) \in \sigma$. Now by right regularity we have $(l_6, l_4) \in \sigma$ for $l_6 \in x \oplus w, l_4 \in y \oplus w$ and $(x \otimes w, y \otimes w) \in \sigma$. Now by using transitivity of σ we have $(l_5, l_4) \in \sigma$ for $l_5 \in x \oplus z, l_4 \in y \oplus w$ and $(x \otimes z, y \otimes w) \in \sigma$. Thus σ is a regular relation on R. □

Proposition 4.2. *Let $\gamma : R_1 \to R_2$ where both R_1 and R_2 are LA-semihyperring is called inclusion hommorphism, then this inclusion hommorphism defines a regular relation σ on R_1 given by $(h_1, h_2) \in \sigma$ if and only if $\gamma(h_1) = \gamma(h_2)$ for all $h_1, h_2 \in R_1$.*

Proof. Straightforward. □

Definition 4.4. *Let x be an hyperideal of LA-semihyperring. Define a relation ϱ on R as $(x, y) \in \varrho$ if and only if $x = y$ or $x \in A$ and $y \in A$, Then ϱ is a regular relation on R and it is known as Rees regular relation.*

Lemma 4.1. *ϱ is a regular relation on R.*

Proof. It is obviously an equivalence relation. Now let $(x, y) \in \varrho$ and $z \in R$.

Case(i) If $x = y$ then $z \oplus x = z \oplus y$ and $z \otimes x = z \otimes y$ so $(l_5, l_5) \in \varrho$ for $l_5 \in z \oplus x$ and $(z \otimes x, z \otimes y) \in \varrho$. So ϱ is left regular relation on R. Similarly ϱ is right regular relation on R. Hence ϱ is a regular relation on R.

Case(ii) If both x and $y \in A$ then for $z \in R, l_5 \in z \oplus x \subseteq I$ and $l_6 \in z \oplus y \subseteq I$ and $z \otimes x \in I$ and $z \otimes y \in I$ so we have $(l_5, l_6) \in \varrho$ for $l_5 \in z \oplus x$, $l_6 \in z \oplus y$ and $(z \otimes x, z \otimes y) \in \varrho$. So ϱ is left regular relation on R. Similarly ϱ is right regular relation on R. Hence ϱ is a regular relation on R. □

Lemma 4.2. *Let σ be a regular relation on LA-semihyperring R, then $\{\sigma(l_3) : l_3 \in \sigma(l_1) \oplus \sigma(l_2) \in \sigma(l_1 \oplus l_2)\}$ and $\sigma(l_1) \otimes \sigma(l_2) = \sigma(l_1 \otimes l_2)$ for all $l_1, l_2 \in R$.*

Proof. Straightforward. □

Theorem 4.1. *Let σ be a regular relation on LA-semihyperring R, then $(R/\sigma, \oplus, \otimes)$ is an LA-semihyperring with the mapping $\oplus : R/\sigma \times R/\sigma \to \mathcal{P}^*(R/\sigma)$ and $\otimes : R/\sigma \times R/\sigma \to R/\sigma$ by $\sigma(l_1) \oplus \sigma(l_2) = \{\sigma(l_3) : l_3 \in \sigma(l_1) \oplus \sigma(l_2) \in \sigma(l_1 \oplus l_2)\}$ and $\sigma(l_1) \otimes \sigma(l_2) = \sigma(l_1 \otimes l_2)$ for all $\sigma(l_1), \sigma(l_2) \in R/\sigma$.*

Proof. Indeed by Proposition 4.2, the hyperoperation $\oplus$ and binary operation $\otimes$ are well defined. Now let $\sigma(x), \sigma(y), \sigma(z) \in R/\sigma$

then

(1)

$$\begin{aligned}
(\sigma(x) \oplus \sigma(y)) \oplus \sigma(z) &= (\{\sigma(w) : w \in \sigma(x) \oplus \sigma(y) \in \sigma(x \oplus y)\}) \oplus \sigma(z) \\
&= \{\sigma(e) : e \in (\sigma(x) \oplus \sigma(y)) \oplus \sigma(z) \in \sigma((x \oplus y) \oplus z)\} \\
&= \{\sigma(z_{51}) : e \in (\sigma(z) \oplus \sigma(y)) \oplus \sigma(x) \in \sigma((z \oplus y) \oplus x)\} \\
&= (\sigma(z) \oplus \sigma(y)) \oplus \sigma(x).
\end{aligned}$$

(2)

$$\begin{aligned}
(\sigma(x) \otimes \sigma(y)) \otimes \sigma(z) &= \sigma(x \otimes y) \otimes \sigma(z) \\
&= \sigma((x \otimes y) \otimes z) \\
&= \sigma((z \otimes y) \otimes x) \\
&= \sigma(z \otimes y) \otimes \sigma(x) \\
&= (\sigma(z) \otimes \sigma(y)) \otimes \sigma(x).
\end{aligned}$$

(3)

$$\begin{aligned}
\sigma(x) \otimes (\sigma(y) \oplus \sigma(z)) &= \sigma(x) \otimes (\{\sigma(w) : w \in \sigma(y) \oplus \sigma(z) \in \sigma(y \oplus z)\}) \\
&= \sigma(x \otimes w) \\
&= \sigma(x \otimes (y \oplus z)) \\
&= \sigma((x \otimes y) \oplus (x \otimes z)) \\
&= \{\sigma(e) : e \in \sigma(x \otimes y) \oplus \sigma(x \otimes z) \in \sigma((x \otimes y) \oplus (x \otimes z))\} \\
&= \{\sigma(e) : e \in ((\sigma(x) \otimes \sigma(y)) \oplus (\sigma(x) \otimes \sigma(z)))\} \\
&= (\sigma(x) \otimes \sigma(y)) \oplus (\sigma(x) \otimes \sigma(z))),
\end{aligned}$$

which shows that R/σ is left distributive and similarly it is right distributive.

Hence $(R/\sigma, \oplus, \otimes)$ is an LA-semihyperring. □

Theorem 4.2. *$(R/\varrho, \oplus, \otimes)$ is an LA-semihyperring.*

Proof. It follows from the proof of the Theorem, 4.1. □

Proposition 4.3. *Let $(R, \oplus, \otimes)$ be an LA-semihyperring and $\emptyset \neq N \subseteq R$. If we define a well defined hyperoperation $\boxplus$ and binary operation $\boxtimes$ on $R/N = \{N(x)|x \in R\}$ as $(N(x)) \boxplus (N(y)) = \{N(n)|n \in x \oplus y\}$, and $(N(x)) \boxtimes (N(y)) = N(x \otimes y)$ $\forall$ $x, y \in R$. Then $(R/N, \boxplus, \boxtimes)$ is an LA-semihyperring.*

Proof. Let $(N(x)), (N(y)), (N(z)) \in R/N$, $\forall$ $x, y, z \in R$.

(1)Consider

$$\begin{aligned}
(N(x) \boxplus N(y)) \boxplus N(z) &= (\{N(n) | n \in x \oplus y\}) \boxplus (N(z)) \\
&= \{N(z) | z \in n \oplus z\} \\
&= \{N(z) | z \in (x \oplus y) \oplus z\} \\
&= \{N(z) | z \in (z \oplus y) \oplus x\} \\
&= \{N(z) | z \in n \oplus x\} \\
&= (\{N(n) | n \in z \oplus y\}) \boxplus (N(x)) \\
&= ((N(z)) \boxplus (N(y))) \boxplus (N(x)) .
\end{aligned}$$

Hence $(R/N, \boxplus)$ is an LA-semihypergroup.

(2) Consider for $(N(x)), (N(y)), (N(z)) \in R/N$, $\forall\ x, y, z \in R$, we have

$$\begin{aligned}
(N(x) \boxtimes N(y)) \boxtimes N(z) &= (N(x \otimes y)) \boxtimes (N(z)) \\
&= N((x \otimes y) \otimes z) \\
&= N((z \otimes y) \otimes x) \\
&= (N(z \otimes y)) \boxtimes (N(x)) \\
&= (N(z) \boxtimes N(y)) \boxtimes N(x).
\end{aligned}$$

Hence $(R/N, \boxtimes)$ is an LA-semigroup.

(3) Now let $N(x), N(y), N(z) \in R/N$, $\forall\ x, y, z \in R$, then consider

$$\begin{aligned}
N(x) \boxtimes (N(y) \boxplus N(z)) &= N(x) \boxtimes (\{N(n) | n \in y \oplus z\}) \\
&= N(x \otimes n) \\
&= N(x \otimes (y \oplus z)) \\
&= N((x \otimes y) \oplus (x \otimes z)) \\
&= N(x \otimes y) \boxplus N(x \otimes z) \\
&= (N(x) \boxtimes N(y)) \boxplus (N(x) \boxtimes N(z)),
\end{aligned}$$

and similarly

$$(N(x) \boxplus N(y)) \boxtimes N(z) = (N(x) \boxtimes N(z)) \boxplus (N(y) \boxtimes N(z)).$$

Thus the operation $\boxtimes$ is distributive with respect to the hyperoperation $\boxplus$ for all $N(x), N(y), N(z) \in R/N$.

Hence $(R/N, \boxplus, \boxtimes)$ is an LA-semihyperring. $\square$

References

[1] M. Azhar, M. Gulistan, N. Yaqoob and S. Kadry, On fuzzy ordered LA-semihypergroups, Int. J. Anal. Appl., 16(2) (2018) 276-289.

[2] J. R. Cho, J. Jezek and T. Kepka, Paramedial groupoids, Czechoslovak Math. J., 49(2) (1999) 277-290.

[3] P. Corsini, Prolegomena of Hypergroup Theory, Aviani Editore, (1993).

[4] P. Corsini and V. Leoreanu, Applications of Hyperstructure Theory, Kluwer Academic, (2003).

[5] M. Gulistan, N. Yaqoob and M. Shahzad, A note on H_v-LA-semigroups, UPB Sci. Bull., Ser. A, 77(3) (2015) 93-106.

[6] M. Gulistan, M. Khan, N. Yaqoob and M. Shahzad, Structural properties of cubic sets in regular LA-semihypergroups, Fuzzy Inf. Eng., 9 (2017) 93-116.

[7] K. Hila and J. Dine, On hyperideals in left almost semihypergroups, ISRN Algebra, Article ID 953124 (2011) 8 pages.

[8] P. Holgate, Groupoids satisfying a simple invertive law, Math. Stud., 61(1-4) (1992) 101-106.

[9] M. Iqbal, and I. Ahmad, On further study of CA-AG-groupoids, Proc. Pakistan Acad. Sci., 53(3) (2016) 325–337.

[10] M.A. Kazim and M. Naseeruddin, On almost semigroups, Aligarh Bull. Math., 2 (1972) 1-7.

[11] R. Kellil, On inverses of left almost semirings and strong left almost semirings, J. Math. Sci., Adv. Appl., 26 (2014) 29-39.

[12] M. Khan, M. Gulistan, N. Yaqoob and F. Hussain, General cubic hyperideals of LA-semihypergroups, Afrika Mat., 27(3-4) (2016) 731-751.

[13] F. Marty, Sur une generalization de la notion de groupe, 8^{im_5z} Congres Mathematicians Scandinaves Tenua Stockholm, (1934) 45-49.

[14] Q. Mushtaq and S.M. Yusuf, On LA-semigroups, Aligarh Bull. Math., 8 (1978) 65-70.

[15] Q. Mushtaq and S. M. Yousaf, On locally associative LA-semigroups, J. Nat. Sci. Math., 19 (1979), 57-62.

[16] Q. Mushtaq and M. S. Kamran: On left almost groups, Proc. Pakistan Acad. Sci., 331 (1996) 53-55.

[17] I. Rehman, N. Yaqoob and S. Nawaz, Hyperideals and hypersystems in LA-hyperrings, Songklanakarin J. Sci. Techn., 39(5) (2017) 651-657.

[18] M. Shah and T. Shah, Some basic properties of LA-ring, Int. Math. Forum, 6(44) (2011) 2195-2199.

[19] T. Vougiouklis, Hyperstructures and Their Representations, Hadronic Press, Palm Harbor, Flarida, USA, (1994).

[20] N. Yaqoob, P. Corsini and F. Yousafzai, On Intra-regular left almost semihypergroups with pure left identity, J. Math., 2013 (2013) Article ID 510790, 10 pages.

[21] N. Yaqoob and M. Gulistan, Partially ordered left almost semihypergroups, J. Egyptian Math. Soc., 23 (2015) 231-235.

[22] N. Yaqoob, I. Cristea, M. Gulistan and S. Nawaz, Left almost polygroups, Italian J. Pure Appl. Math., 39 (2018) 465-474.

[23] N. Yaqoob, M. Gulistan, V. Leoreanu-Fotea and K. Hila, Cubic hyperideals in LA-semihypergroups, J. Intell. Fuzzy Syst., 34(4) (2018) 2707-2721.

[24] N. Yaqoob, Applications of rough sets to Γ-hyperideals in left almost Γ-semihypergroups, Neural Comput. Appl., 21(1) (2012) 267-273.

[25] S. M. Yusuf, On Left Almost Ring, Proc. of 7th International Pure Math. Conference, Islamabad, (2006).

[26] J. Zhan, N. Yaqoob and M. Khan, Roughness in non-associative po-semihyprgroups based on pseudohyperorder relations, J. Multiple-Valued Logic Soft Comput., 28(2-3) (2017) 153-177.