International Journal of Analysis and Applications
Volume 16, Number 4 (2018), 472-483
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-16-2018-472

A NEW FIXED POINT THEOREM IN MODULAR METRIC SPACES

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Abstract. In this article, we first give a new fixed point theorem which is main theorem of our study in modular metric spaces. After that, by using this theorem, we express some interesting results. Moreover, we characterize completeness in modular metric spaces via this theorem. Finally, we use our main result to show the existence of solution for a specific problem in dynamic programming.

1. Introduction

The fixed point theory is used in many different fields of mathematics such as topology, analysis, nonlinear analysis and operator theory. Moreover, it can be applied to different disciplines such as statistics, economy, engineering, etc. In literature, studies of fixed point theory cover a wide range. The most basic and famous fixed point theorem is Banach fixed point theorem which was introduced in 1922 [6]. It guarantees the existence and uniqueness of solution of a functional equation. Besides Banach, many different fixed point theorems were introduced such as Kannan, Caristi, Coupled, Suzuki, etc [1, 2, 7, 8, 13–16, 19, 23, 24].

In 1950, Nakano introduced modular spaces [21]. Then Chistyakov introduced the concept of modular metric spaces, which have a physical interpretation, via F-modulars [9] in 2008 and he further developed

Received 2018-03-20; accepted 2018-05-09; published 2018-07-02.
2010 *Mathematics Subject Classification.* 46A80, 47H10, 54H25.
Key words and phrases. modular metric space; fixed point theorem; complete modular metric.

the theory of these spaces in 2010 [10]. Then many authors made various studies on this structures, e.g. [3–5, 11, 12, 17, 18, 20].

In this paper, we first give a new fixed point theorem which is main theorem of our study. After that, by using this theorem, we express some interesting results. Moreover, we characterize completeness in modular metric spaces via this theorem. Finally, we use our main theorem to show the existence of solution for a specific problem in dynamic programming.

2. Modular Metric Spaces

Here, we express a series of definitions of some basic concepts related to modular metric spaces.

Definition 2.1. *[22] Let X be a linear space on $\mathbb{R}$. If a functional $\rho : X \to [0, \infty]$ satisfies the following conditions, we call that ρ is a modular on X:*

(1) $\rho(0) = 0$;

(2) *If* $x \in X$ *and* $\rho(\alpha x) = 0$ *for all numbers* $\alpha > 0$, *then* $x = 0$;

(3) $\rho(-x) = \rho(x)$, *for all* $x \in X$;

(4) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *for all* $\alpha, \beta \geq 0$ *with* $\alpha + \beta = 1$ *and* $x, y \in X$.

Let $X \neq \emptyset$ and $\lambda \in (0, \infty)$. Generally, a function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is denoted as $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $x, y \in X$ and $\lambda > 0$.

Definition 2.2. *[10] Let $X \neq \emptyset$. A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$, which satisfies the following conditions for all $x, y, z \in X$, is called a metric modular on X:*

(m1) $\omega_\lambda(x, y) = 0$ *for all* $\lambda > 0 \Leftrightarrow x = y$;

(m2) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ *for all* $\lambda > 0$;

(m3) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ *for all* $\lambda, \mu > 0$.

If $0 < \mu < \lambda$, from properties of metric modular, we obtain that

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y)$$

for all $x, y \in X$.

From [10, 11], we know that for a fixed $x_0 \in X$, the two sets

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X^*_\omega = X^*_\omega(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x, x_0) < \infty\}$$

are said to be modular spaces.

It is known [10, 11] that if ω is a metric modular on a nonempty set X, then the modular space X_ω can be equipped with a metric, generated by ω and given by

$$d_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq \lambda\}$$

for all $x, y \in X_\omega$. The pair (X_ω, d_ω) is called a modular metric space.

Definition 2.3. [18] Let X_ω be a modular metric space, $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_ω and $C \subseteq X_\omega$. Then
(1) $\{x_n\}_{n\in\mathbb{N}}$ is called a modular convergent sequence such that $x_n \to x,\ x \in X_\omega$, if for $\lambda > 0$

$$\omega_\lambda(x_n, x) \to 0 \text{ as } n \to \infty.$$

(2) $\{x_n\}_{n\in\mathbb{N}}$ is called a modular Cauchy sequence, if for $\lambda > 0$

$$\omega_\lambda(x_n, x_m) \to 0 \text{ as } m, n \to \infty.$$

(3) C is called closed, if the limit of a modular convergent sequence in C always belongs to C.
(4) C is called complete modular, if every modular Cauchy sequence $\{x_n\}$ in C is modular convergent in C.
(5) C is called bounded, if

$$\delta_\omega(C) = \sup\{\omega_\lambda(x, y) :\ x, y \in C,\ \lambda > 0\} < \infty.$$

3. Main Results

Let $\omega : (0, \infty) \times X \times X \to [0, \infty]$ be a metric modular on X, X_ω be a modular metric space, $C \subseteq X_\omega$ and $\psi : C \to \mathbb{R}^+$ be a function on C. ψ is called lower semi-continuous (l.s.c.) on C if

$$\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0 \Rightarrow \psi(x) \leq \lim_{n\to\infty} \inf(\psi(x_n))$$

for all $\{x_n\} \subseteq C$ and $\lambda > 0$.

Theorem 3.1. *Let ω be a metric modular on X, X_ω be a complete modular metric space, $\psi : X_\omega \to \mathbb{R}^+$ be a lower semi-continuous function on X_ω and $T : X_\omega \to X_\omega$ be a mapping satisfying the inequality*

$$\omega_\lambda(x, Tx) \leq \psi(x) - \psi(Tx) \tag{3.1}$$

for all $x \in X_\omega$ and $\lambda > 0$. Then T has a fixed point in X_ω.

Proof. For any $x \in X_\omega$ denote,

$$P(x) = \{y \in X_\omega : \omega_\lambda(x, y) \leq \psi(x) - \psi(y) \text{ for all } \lambda > 0\},$$

$$\alpha(x) = \inf\{\psi(y) : y \in P(x)\}.$$

As $x \in P(x)$, $P(x)$ is not empty and $0 \leq \alpha(x) \leq \psi(x)$. Let $x \in X_\omega$ be an arbitrary point. Now, we construct a sequence $\{x_n\}$ in X_ω as follows: Let $x_1 = x$ and when $x_1, x_2, ..., x_n$ have been chosen, choose $x_{n+1} \in P(x_n)$ such that $\psi(x_{n+1}) \leq \alpha(x_n) + \frac{1}{n}$, for all $n \in \mathbb{N}$. By doing so, we get a sequence $\{x_n\}$ satisfying the conditions

$$\begin{aligned} \omega_\lambda(x_n, x_{n+1}) &\leq \psi(x_n) - \psi(x_{n+1}), \\ \alpha(x_n) &\leq \psi(x_{n+1}) \leq \alpha(x_n) + \tfrac{1}{n} \end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N}$ and $\lambda > 0$. Then $\{\psi(x_n)\}$ is a nonincreasing sequence in $\mathbb{R}$ and it is bounded from below by zero. So, the sequence $\{\psi(x_n)\}$ is convergent to a number $M \geq 0$. By virtue of (3.2), we get

$$M = \lim_{n\to\infty} \psi(x_n) = \lim_{n\to\infty} \alpha(x_n). \tag{3.3}$$

Now, let $k \in \mathbb{N}$ be arbitrary. From (3.2) and (3.3), there exists at least a number N_k such that $\psi(x_n) < M + \frac{1}{k}$ for all $n \geq N_k$. Since $\psi(x_n)$ is monotone, we get

$$M \leq \psi(x_m) \leq \psi(x_n) < M + \frac{1}{k}$$

for $m \geq n \geq N_k$. It follows that

$$\psi(x_n) - \psi(x_m) < \frac{1}{k} \quad \text{for all} \quad m \geq n \geq N_k. \tag{3.4}$$

Preserving the generality, suppose that $m > n$ and $m, n \in \mathbb{N}$. From (3.2), we get

$$\omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) \leq \psi(x_n) - \psi(x_{n+1})$$

for $\frac{\lambda}{m-n} > 0$. Now, we obtain that

$$\begin{aligned} \omega_\lambda(x_n, x_m) &\leq \omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) + \omega_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}) + \cdots + \omega_{\frac{\lambda}{m-n}}(x_{m-1}, x_m) \\ &\leq \psi(x_n) - \psi(x_{n+1}) + \psi(x_{n+1}) - \psi(x_{n+2}) + \cdots + \psi(x_{m-1}) - \psi(x_m) \\ &= \psi(x_n) - \psi(x_m) \end{aligned} \tag{3.5}$$

for all $m, n \geq N_k$. Then by (3.4),

$$\omega_\lambda(x_n, x_m) < \frac{1}{k} \quad \text{for all} \quad m \geq n \geq N_k. \tag{3.6}$$

Letting k or m and n tend to infinity in (3.6), we conclude that

$$\lim_{m,n\to\infty} \omega_\lambda(x_n, x_m) = 0.$$

Then $\{x_n\}_{n\in\mathbb{N}}$ is a modular Cauchy sequence. Since X_ω is complete modular, there exists a point $u \in X_\omega$ such that $x_n \to u$ as $n \to \infty$. Since ψ is lower semi-continuous, using (3.5), we have

$$\begin{aligned} \psi(u) &\leq \lim_{m\to\infty} \inf \psi(x_m) \\ &\leq \lim_{m\to\infty} \inf(\psi(x_n) - \omega_\lambda(x_n, x_m)) \\ &= \psi(x_n) - \omega_\lambda(x_n, u) \end{aligned}$$

and hence

$$\omega_\lambda(x_n, u) \leq \psi(x_n) - \psi(u).$$

Thus, $u \in P(x_n)$ for all $n \in \mathbb{N}$ and hence $\alpha(x_n) \leq \psi(u)$. Then by (3.3), we get $M \leq \psi(u)$. Moreover, using lower semi-continuity of ψ and (3.3), we have

$$\psi(u) \leq \lim_{n\to\infty} \inf \psi(x_n) = M.$$

So, $\psi(u) = M$. From (3.1), we know that $Tu \in P(u)$. Since $u \in P(u)$ for $n \in \mathbb{N}$, we have

$$\begin{aligned}\omega_\lambda(x_n, Tu) &\leq \omega_{\frac{\lambda}{2}}(x_n, u) + \omega_{\frac{\lambda}{2}}(u, Tu)\\ &\leq \psi(x_n) - \psi(u) + \psi(u) - \psi(Tu)\\ &= \psi(x_n) - \psi(Tu).\end{aligned}$$

Then $Tu \in P(x_n)$ and this implies $\alpha(x_n) \leq \psi(Tu)$. Hence, we obtain $M \leq \psi(Tu)$. From (3.1), we get $\psi(Tu) \leq \psi(u)$. As $\psi(u) = M$, we have

$$\psi(u) = M \leq \psi(Tu) \leq \psi(u).$$

Therefore, $\psi(Tu) = \psi(u)$. Then from (3.1), we get

$$\omega_\lambda(u, Tu) \leq \psi(u) - \psi(Tu) = 0.$$

Thus, $Tu = u$. □

Theorem 3.2. *Let ω be a metric modular on X and X_ω be a complete modular metric space and $\psi : X_\omega \to \mathbb{R}$ be a lower semi-continuous function on X_ω. If ψ is bounded below, then there exists a point $u \in X_\omega$ such that*

$$\psi(u) < \psi(x) + \omega_\lambda(u, x)$$

for each $x \in X_\omega$, $x \neq u$ and $\lambda > 0$.

Proof. Following the proof Theorem 3.1, we obtain a sequence $\{x_n\}$ that converges to some $u \in X_\omega$. Under the same notations, for any $u \in X_\omega$, define

$$\begin{aligned}&P(u) = \{x \in X_\omega : \omega_\lambda(u, x) \leq \psi(u) - \psi(x) \text{ for all } \lambda > 0\}\\ &\alpha(u) = \inf\{\psi(x) : x \in P(u)\}.\end{aligned}$$

We will show that $u \notin P(u)$ as $x \neq u$. Suppose the contrary, that is, we get $v \in P(u)$ for some $v \neq u$. Then $0 < \omega_\lambda(u, v) \leq \psi(u) - \psi(v)$ implies $\psi(v) < \psi(u) = M$. Since

$$\begin{aligned}\omega_\lambda(x_n, v) &\leq \omega_{\frac{\lambda}{2}}(x_n, u) + \omega_{\frac{\lambda}{2}}(u, v)\\ &\leq \psi(x_n) - \psi(u) + \psi(u) - \psi(v)\\ &= \psi(x_n) - \psi(v)\end{aligned}$$

for all $\lambda > 0$, $v \in P(x_n)$. So,

$$\alpha(x_n) \leq \psi(v) \text{ for all } n \in \mathbb{N}.$$

Letting n tends to infinity, we get $M \leq \psi(v)$. This equation contradicts with $\psi(v) < M = \psi(u)$. Therefore, for each $x \in X_\omega$, $x \neq u$ implies $x \notin P(u)$, that is

$$x \neq u \Rightarrow \omega_\lambda(u, x) > \psi(u) - \psi(x).$$

□

Theorem 3.3. Let X_ω and Y_ω be complete modular metric spaces and the mapping $T : X_\omega \to X_\omega$ be arbitrary. Assume that there exists a closed mapping $S : X_\omega \to Y_\omega$, a lower semi-continuous mapping $\psi : S(X_\omega) \to \mathbb{R}^+$ and a constant $c > 0$ such that for any $x \in X_\omega$ and $\lambda > 0$

$$\begin{aligned}\omega_\lambda(x, Tx) &\leq \psi(Sx) - \psi(STx) \text{ and}\\ c \cdot \omega_\lambda(Sx, STx)\} &\leq \psi(Sx) - \psi(STx).\end{aligned} \tag{3.7}$$

Then the mapping T has a fixed point.

Proof. For any $x \in X_\omega$, we set

$$\begin{aligned}P(x) &= \{z \in X_\omega : \omega_\lambda(x, z) \leq \psi(Sx) - \psi(Sz) \text{ and}\\ &\qquad c \cdot \omega_\lambda(Sx, Sz)\} \leq \psi(Sx) - \psi(Sz) \text{ for all } \lambda > 0\}\\ \alpha(x) &= \inf\{\psi(Sz) : z \in P(x)\}.\end{aligned}$$

As $x \in P(x)$, it is clear that $P(x) \neq \emptyset$ and $0 \leq \alpha(x) \leq \psi(Sx)$. Similar to the proof of Theorem 3.1, choose a sequence $\{x_n\}$ in X_ω: $x_1 = x$, $x_{n+1} \in P(x_n)$ such that

$$\psi(Sx_{n+1}) \leq \alpha(x_n) + \frac{1}{n}$$

for all $n \geq 1$. Thus we obtain that

$$\begin{aligned}\omega_\lambda(x_n, x_{n+1}) &\leq \psi(Sx_n) - \psi(Sx_{n+1}),\\ c \cdot \omega_\lambda(Sx_n, Sx_{n+1})\} &\leq \psi(Sx_n) - \psi(Sx_{n+1})\end{aligned} \tag{3.8}$$

and

$$\psi(Sx_{n+1}) - \frac{1}{n} \leq \alpha(x_n) \leq \psi(Sx_{n+1}). \tag{3.9}$$

From (3.8), $\{\psi(Sx_n)\}$ is a nonincreasing and bounded sequence on $\mathbb{R}$. So, $\{\psi(Sx_n)\}$ is a modular convergent sequence. Therefore, by (3.9) there is a number $M \geq 0$ such that

$$M = \lim_{n\to\infty} \alpha(x_n) = \lim_{n\to\infty} \psi(Sx_n). \tag{3.10}$$

Now, let $k \in \mathbb{N}$ be an arbitrary point. From (3.10), there exists some N_k such that $\psi(Sx_n) < M + \frac{1}{k}$ for all $n \geq N_k$. Thus, by monotonicity of $\{\psi(Sx_n)\}$, for all $m \geq n \geq N_k$ we have

$$M \leq \psi(Sx_m) \leq \psi(Sx_n) \leq M + \frac{1}{k}.$$

So,

$$\psi(Sx_n) - \psi(Sx_m) \leq \frac{1}{k}. \tag{3.11}$$

Preserving the generality, suppose that $m > n$ and $m, n \in \mathbb{N}$. From (3.8), we easily obtain that

$$\begin{aligned} \omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) &\leq \psi(Sx_n) - \psi(Sx_{n+1}) \text{ and} \\ c \cdot \omega_{\frac{\lambda}{m-n}}(Sx_n, STx_{n+1})\} &\leq \psi(Sx_n) - \psi(Sx_{n+1}) \end{aligned} \tag{3.12}$$

for $\frac{\lambda}{m-n} > 0$. From (3.8), (3.12) and condition (M3) of modular metric, we have

$$\begin{aligned} \omega_\lambda(x_n, x_m) &\leq \omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) + \omega_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}) + \cdots + \omega_{\frac{\lambda}{m-n}}(x_{m-1}, x_m) \\ &\leq \psi(Sx_n) - \psi(Sx_{n+1}) + \psi(Sx_{n+1}) - \psi(Sx_{n+2}) \\ &\quad + \cdots + \psi(Sx_{m-1}) - \psi(Sx_m) \\ &= \psi(Sx_n) - \psi(Sx_m) \\ c \cdot \omega_\lambda(Sx_n, Sx_m) &\leq c \cdot \omega_{\frac{\lambda}{m-n}}(Sx_n, Sx_{n+1}) + c \cdot \omega_{\frac{\lambda}{m-n}}(Sx_{n+1}, Sx_{n+2}) \\ &\quad + \cdots + c\omega_{\frac{\lambda}{m-n}}(Sx_{m-1}, Sx_m) \\ &\leq \psi(Sx_n) - \psi(Sx_{n+1}) + \psi(Sx_{n+1}) - \psi(Sx_{n+2}) \\ &\quad + \cdots + \psi(Sx_{m-1}) - \psi(Sx_m) \\ &= \psi(Sx_n) - \psi(Sx_m). \end{aligned} \tag{3.13}$$

From (3.11), we get

$$\omega_\lambda(x_n, x_m) < \frac{1}{k} \text{ and } c \cdot \omega_\lambda(Sx_n, Sx_m) < \frac{1}{k}$$

for all $m \leq n \leq N_k$ and $k \in \mathbb{N}$. Therefore, $\{x_n\}_{n\in\mathbb{N}}$ is a modular Cauchy sequence in X_ω and $\{Sx_n\}_{n\in\mathbb{N}}$ is a modular Cauchy sequence in Y_ω. By completeness of X_ω and Y_ω, there exist $p \in X_\omega$ and $q \in Y_\omega$ such that

$x_n \to p$ and $Sx_n \to q$. The fact that, S is a closed mapping implies $Sp = q$. Since ψ is lower semi-continuous, using equation (3.13), we have

$$\begin{aligned}\psi(q) = \psi(Sp) \le \lim_{m\to\infty} \inf \psi(Sx_m) &\le \lim_{m\to\infty} \inf(\psi(Sx_n) - \omega_\lambda(x_n, x_m)) \\ &= \psi(Sx_n) - \omega_\lambda(x_n, p).\end{aligned}$$

Then we obtain

$$\omega_\lambda(x_n, p) \le \psi(Sx_n) - \psi(Sp)$$

for $\lambda > 0$. Similarly, we get

$$c \cdot \omega_\lambda(x_n, p) \le \psi(Sx_n) - \psi(Sp).$$

Thus, $p \in P(x_n)$ for all $n \in \mathbb{N}$. Then $\alpha(x_n) \le \psi(Sp)$. So, by (3.10), we get $M \le \psi(Sp)$. On the other hand, using lower semi-continuity of ψ and (3.10), we have

$$\psi(q) = \psi(Sp) = \lim_{m\to\infty} \alpha(x_n) = M.$$

Therefore, $\psi(Sp) = M$. By benefiting from the proof of Theorem 3.2, we obtain that $x \ne p$ implies $x \notin P(p)$. From (3.7), it's clear that $Tp \in P(p)$, then we have $Tp = p$. □

Corollary 3.1. *Theorem 3.3 holds with inequality*

$$max\{\omega_\lambda(x, Tx), c \cdot \omega_\lambda(Sx, STx)\} \le \psi(Sx) - \psi(STx)$$

in the place of inequality (3.7).

Example 3.1. *Let $X = \mathbb{R}$. We define the mapping $\omega : (0, \infty) \times \mathbb{R} \times \mathbb{R} \to [0, \infty]$ by $\omega_\lambda(x, y) = \frac{|x-y|}{1+\lambda}$ for all $x, y \in \mathbb{R}$ and $\lambda > 0$. Then it is obvious that $\mathbb{R}_\omega$ is a complete modular metric space. Define $T : \mathbb{R}_\omega \to \mathbb{R}_\omega$ by $Tx = \frac{x}{4}$ and $\psi : \mathbb{R}_\omega \to [0, \infty]$ by $\psi(x) = 3|x|$. Then for all $x, y \in \mathbb{R}$ and $\lambda > 0$, we have*

$$\omega_\lambda(x, Tx) = \frac{|x - Tx|}{1 + \lambda} = \frac{|x - \frac{x}{4}|}{1 + \lambda} = \frac{3|x|}{4(1 + \lambda)} \le \frac{3}{4}|x|$$

and

$$\psi(x) - \psi(Tx) = 3|x| - \frac{3|x|}{4} = \frac{9}{4}|x|.$$

Hence, $\omega_\lambda(x, Tx) \le \psi(x) - \psi(Tx)$. From Theorem 3.1, the mapping T has a fixed point. Moreover, it is $0 \in \mathbb{R}_\omega$.

4. Characterization of Completeness

We now prove a new theorem, which together with Theorem 3.1 characterizes completeness in modular metric spaces.

Theorem 4.1. *Let X_ω be a modular metric space which is not complete modular. Then there exists a fixed point free function $T : X_\omega \to X_\omega$ and a lower semi-continuous mapping $\psi : X_\omega \to \mathbb{R}^+$ such that*

$$\omega_\lambda(x, Tx) \leq \psi(x) - \psi(Tx)$$

for all $x \in X_\omega$ and $\lambda > 0$.

Proof. Let $\{x_n\} \subset X_\omega$ be a modular Cauchy sequence, which has no limit. We define a function $\phi : X_\omega \to \mathbb{R}^+$ by

$$\phi(u) = \lim_{m\to\infty} \omega_\lambda(u, x_m),\ u \in X_\omega \text{ for all } \lambda > 0.$$

Given $x \in X_\omega$ and let n denote the smallest positive integer such that

$$0 < \frac{1}{2}\omega_\lambda(x, x_n) \leq \phi(x) - \phi(x_n) \text{ for all } \lambda > 0. \tag{4.1}$$

Note that $\phi(x_n) \to 0$ as $\phi(x) > 0$. With n so determined, we define function $T : X_\omega \to X_\omega$ as $Tx = x_n$. Let $\psi(x) = 2\phi(x)$. Then from (4.1), we obtain that

$$\omega_\lambda(x, Tx) \leq \psi(x) - \psi(Tx).$$

□

5. Application

Let X_ω be a complete modular metric space, Y be a Banach space, $M \subseteq X_\omega$, $S \subseteq Y$ and $\theta : M \times S \to M$, $H : M \times S \times \mathbb{R} \to \mathbb{R}$ be two functions. Using Theorem 3.1, we show the existence of a bounded solution for the following problem in dynamic programming:

We take a $g \in B(M)$ such that

$$g(t) = \sup_{s\in S}\{H(t, s, g(\theta(t, s)))\} \tag{5.1}$$

where $t \in M$ and $B(M)$ is a Banach space which consists of all bounded real functionals on M with the norm $\|g\| = \sup_{t\in M} |g(t)|$. We define a complete modular metric on $B(M)$ with

$$\omega_\lambda(g, k) = \sup_{t\in Z}\left\{\left|\frac{g(t) - k(t)}{1 + \lambda}\right|\right\} \tag{5.2}$$

for all $g, k \in B(M)$ and $\lambda > 0$. If we take a Cauchy sequence $\{g_n\}_{n\in\mathbb{N}}$ in $B(M)$, then from completeness of X_ω, there exists a function $u \in B(M)$ such that the sequence $\{g_n\}_{n\in\mathbb{N}}$ is convergent to u.

Theorem 5.1. *Let $\theta : M \times S \to M$, $H : M \times S \times \mathbb{R} \to \mathbb{R}$ be bounded and $\psi : B(M) \to \mathbb{R}^+$ be lower semi continuous on X_ω and define by $\psi(g) = \|g\|$. We define a operator $T : B(M) \to B(M)$ by*

$$T(g)(t) = \sup_{s \in S}\{H(t, s, g(\theta(t, s)))\}$$

for all $g \in B(M)$ and $t \in M$. If

$$\sup_{t \in M}\left|\frac{g(t) - H(t, s, g(\theta(t, s)))}{\lambda}\right| \leq \psi(g) - \psi(T(g)) \tag{5.3}$$

for all $\lambda > 0$, $g, k \in B(M)$ and $s \in S$, then the functional equation (5.1) has a bounded solution.

Proof. Let $t \in M$ and $g \in B(M)$. Then there exists $s \in S$ and $\epsilon > 0$ such that

$$T(g)(x) < H(t, s, g(\theta(t, s))) + \epsilon \tag{5.4}$$

and

$$T(g)(x) > H(t, s, g(\theta(t, s))). \tag{5.5}$$

On the other hand, it is obvious that

$$g(t) < g(t) + \epsilon \tag{5.6}$$

and

$$g(t) > g(t) - \epsilon. \tag{5.7}$$

for all $\epsilon > 0$. By using the inequalities (5.5) and (5.6) we obtain that

$$\begin{aligned} g(t) - T(g)(t) &< g(t) - H(t, s, g(\theta(t, s))) + \epsilon \\ &\leq |g(t) - H(t, s, g(\theta(t, s)))| + \epsilon. \end{aligned} \tag{5.8}$$

Similarly, by using the inequalities (5.4) and (5.7) we obtain that

$$\begin{aligned} T(g)(t) - g(t) &< H(t, s, g(\theta(t, s))) - g(t) + 2\epsilon \\ &\leq |H(t, s, g(\theta(t, s))) - g(t)| + 2\epsilon. \end{aligned} \tag{5.9}$$

Therefore, from the inequalities (5.8) and (5.9), we get

$$|g(t) - T(g)(t)| < |g(t) - H(t, s, g(\theta(t, s)))| + 2\epsilon. \tag{5.10}$$

If we divide both sides of the inequality (5.10) by $1 + \lambda$, we get

$$\left|\frac{g(t) - T(g)(t)}{1 + \lambda}\right| < \left|\frac{g(t) - H(t, s, g(\theta(t, s)))}{1 + \lambda}\right| + \frac{2\epsilon}{1 + \lambda} \tag{5.11}$$

for all $\lambda > 0$. Since $\frac{2\epsilon}{1+\lambda} > 0$ in the inequality (5.11), we can ignore the contrary $\frac{2\epsilon}{1+\lambda}$. Then we have

$$\left|\frac{g(t) - T(g)(t)}{1 + \lambda}\right| < \left|\frac{g(t) - H(t, s, g(\theta(t, s)))}{1 + \lambda}\right|$$

for all $\lambda > 0$. Then from property of supremum, we get

$$\sup_{t\in Z}\left|\frac{g(t)-T(g)(t)}{1+\lambda}\right| < \sup_{t\in Z}\left|\frac{g(t)-H(t,s,g(\theta(t,s)))}{1+\lambda}\right|.$$

Then from inequalities (5.2) and (5.3) we obtain that

$$\omega_\lambda(g,T(g)) < \psi(g)-\psi(T(g)).$$

Therefore, from Theorem 3.1, T has a fixed point $u \in B(Z)$. Then the functional equation (5.1) has a bounded solution. □

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