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CONTROLLED $*$-G-FRAMES AND $*$-G-MULTIPLIERS IN HILBERT PRO-C^*-MODULES

ZAHRA AHMADI MOOSAVI[1,*] AND AKBAR NAZARI[2]

[1] *Department of Mathematics Faculty of Mathematics and computer Shahid Bahonar University of Kerman, 76169-14111, Kerman, Iran*

[2] *Department of Mathematics Faculty of Mathematics and computer Shahid Bahonar University of Kerman, 76169-14111, Kerman, Iran*

[*] *Corresponding author:* nazari@uk.ac.ir

Abstract. A generalization of multiplier, controlled g-frames and g-Bessel sequences to $*$-g-frames and $*$-g-Bessel sequences in Hilbert pro-C^*-modules is presented. It is demonstrated that controlled $*$-g-frames are equivalent to $*$-g-frames in Hilbert pro-C^*-modules.

1. Introduction

Frame theory is an application of harmonic analysis. This theory has been rapidly generalized to Hilbert spaces and Hilbert C^*-modules. In 2005, Sun [22] introduced the notion of g-frames as a generalization of frames for bounded operators on Hilbert spaces. Frank-Larson [5] have extended the theory for elements of C^*-algebras and (finitely or countably generated) Hilbert C^*-modules have been considered in [1].

It is well known that Hilbert C^*-modules are a generalization of Hilbert spaces where the inner product takes values in a C^*-algebra rather than in the field of complex numbers. The theory of Hilbert C^*-modules

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has applications in the study of locally compact quantum groups, complete maps between C^*-algebras, non-commutative geometry and KK-theory. Not all properties of Hilbert spaces hold in Hilbert C^*-modules. For instance, the Riesz representation theorem for continuous linear functionals on Hilbert spaces can not be extended to Hilbert C^*-modules [23] and there exist closed subspaces in Hilbert C^*-modules that have no orthogonal complement [16]. Moreover, as known, every bounded operator on a Hilbert space has an adjoint whereas there are bounded operators on Hilbert C^*-modules which do not have this property [17]. So, it is to be expected that frames and $*$-frames in Hilbert C^*-modules are more complicated than those in Hilbert spaces. The properties of g-frames for Hilbert C^*-modules have been widely investigated in the literature (see [1, 5, 12, 25], and the references therein).

The paper is organized as follows.In the next section, we give a brief survey of the fundamental definitions and notations of Hilbert pro-C^*-modules.

Section 3 is devoted to investigating $*$-g-frames with $\mathcal{A}$-valued bounds and analyzing their elementary properties. In Section 4 we define the concept of controlled $*$-g-frames and we show that a controlled $*$-g-frame is equivalent to a $*$-g-frame in Hilbert pro-C^*-modules. Finally, in section 5 we define multipliers of controlled $*$-g-frame operators in Hilbert pro-C^*-modules.

2. Preliminaries

In this section, we recall some of the basic definitions and properties of pro-C^*-algebras and Hilbert modules over them [7, 15, 18].

A pro-C^*-algebra is a complete Hausdorff complex topological $*$-algebra $\mathcal{A}$ whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_\lambda\}$ converges to 0 iff $\rho(a_\lambda) \to 0$ for any continuous C^*-seminorm ρ on $\mathcal{A}$ and we have:

(1) $\rho(ab) \leq \rho(a)\rho(b)$;

(2) $\rho(a^*a) = \rho(a)^2$;

for all C^*-seminorms ρ on $\mathcal{A}$ and $a, b \in \mathcal{A}$.

If the topology of pro-C^*-algebra is determined by only countably many C^*-seminorms, then it is called a σ-C^*-algebra.

Let $\mathcal{A}$ be a unital pro-C^*-algebra with unit $1_{\mathcal{A}}$ and let $a \in \mathcal{A}$. Then spectrum $sp(a)$ of $a \in A$ is the set $\{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - a$ is not invertible$\}$. If $\mathcal{A}$ is not unital, then the spectrum is taken with respect to its unitization $\tilde{A}$.

If $\mathcal{A}^+$ denotes the set of all positive elements of $\mathcal{A}$, then $\mathcal{A}^+$ is a closed convex C^*-seminorms on $\mathcal{A}$. We denote by $S(\mathcal{A})$, the set of all continuous C^*-seminorms on $\mathcal{A}$.

Example 2.1. *Every C^*-algebra is a pro-C^*-algebra.*

Example 2.2. *A sub-closed $*$-algebra of a pro-C^*-algebra is a pro-C^*-algebra.*

Proposition 2.1 ([6])**.** *Let $\mathcal{A}$ be a unital pro-C^*-algebra with an identity $1_{\mathcal{A}}$.Then for any $\rho \in S(\mathcal{A})$, we have:*

(1) $\rho(a) = \rho(a^*)$ *for all $a \in A$;*
(2) $\rho(1_{\mathcal{A}}) = 1$;
(3) *If $a, b \in \mathcal{A}^+$ and $a \leq b$, then $\rho(a) \leq \rho(b)$;*
(4) *If $1_{\mathcal{A}} \leq b$, then b is invertible and $b^{-1} \leq 1_{\mathcal{A}}$;*
(5) *If $a, b \in \mathcal{A}^+$ are invertible and $0 \leq a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$;*
(6) *If $a, b, c \in \mathcal{A}$ and $a \leq b$ then $c^*ac \leq c^*bc$;*
(7) *If $a, b \in \mathcal{A}^+$ and $a^2 \leq b^2$, then $0 \leq a \leq b$.*

Definition 2.1. *A pre-Hilbert module over pro-C^*-algebra $\mathcal{A}$, is a complex vector space E which is also a left $\mathcal{A}$-module compatible with the complex algebra structure, equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : E \times E \to \mathcal{A}$ which is $\mathbf{C}$-and $\mathcal{A}$-linear in its first variable and satisfies the following conditions:*

(1) $\langle x, y \rangle^* = \langle y, x \rangle$;
(2) $\langle x, x \rangle \geq 0$;
(3) $\langle x, x \rangle = 0$ *iff* $x = 0$;

for every $x, y \in E$. We say E is a Hilbert $\mathcal{A}$-module (or Hilbert pro-C^*-module over$\mathcal{A}$) If E is complete with respect to the topology determined by the family of seminorms

$$\overline{\rho}_E(x) = \sqrt{\rho(\langle x, x \rangle)} \qquad x \in E, \rho \in S(\mathcal{A}).$$

Let E be a pre-Hilbert $\mathcal{A}$-module.By [6], for $\rho \in S(\mathcal{A})$ and for all $x, y \in E$, the following Cauchy-Bunyakovskii inequality holds:

$$\rho(\langle x, y \rangle)^2 \leq \rho(\langle x, x \rangle)\rho(\langle y, y \rangle).$$

Consequently, for each $\rho \in S(\mathcal{A})$, we have:

$$\overline{\rho}_E(ax) \leq \rho(a)\overline{\rho}(x), \qquad a \in \mathcal{A}, x \in E.$$

Let $\mathcal{A}$ be a pro-C^*-algebra and E and F be two Hilbert $\mathcal{A}$-modules. An $\mathcal{A}$-module map $T : E \to F$ is said to bounded if for each $\rho \in S(\mathcal{A})$, there is $C_\rho > 0$ such that :

$$\overline{\rho}_F(Tx) \leq C\rho.\ \overline{\rho}_E(x) \qquad (x \in E),$$

where $\overline{\rho}_E$, respectively $\overline{\rho}_F$, are continuous seminorms on E, respectively F. A bounded $\mathcal{A}$-module map from E to F is called an operators from E to F . We denote the set of all operators from E to F by $Hom_A(E, F)$, and we set $Hom_A(E, F) = End_A(E)$

Proposition 2.2. *Let $T^* \in Hom_{\mathcal{A}}(E, F)$. We say T is adjointable if there exists an operator $T^* \in T \in Hom_{\mathcal{A}}(F, E)$ such that:*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

holds for all $x \in E, y \in F$.

We denote by $Hom^*_{\mathcal{A}}(E, F)$, the set of all adjointable operator from E to F and $End^*_{\mathcal{A}}(E) = Hom^*_{\mathcal{A}}(E, E)$

Proposition 2.3 ([6]). *Let $T : E \to F$ and $T^* : F \to E$ be two maps such that the equality*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

holds for all $x \in E,\ y \in F$. Then $T \in Hom^*_{A}(E, F)$.

It is easy to see that for any $\rho \in S(\mathcal{A})$, the map defined by:

$$\hat{\rho}_{E,F}(T) = \sup\{\overline{\rho}_F(T(x) :\ x \in E,\ \overline{\rho}_E(x) \leq 1\}, \qquad T \in Hom_{\mathcal{A}}(E, F),$$

is a seminorm on $Hom_{\mathcal{A}}(E, F)$.

Definition 2.2. *Let E and F be two Hilbert modules over pro-C^*-algebra $\mathcal{A}$. Then the operator $T : E \to F$ is called uniformly bounded (below), if there exists $C > 0$ such that:*

$$\overline{\rho}_F(Tx) \leq C\ \overline{\rho}_E(x). \tag{2.1}$$

$$(C\ \overline{\rho}_E(x) \leq \overline{\rho}_F(Tx)) \tag{2.2}$$

The number C in (2.1) is called an upper bound for T and we set :

$$\|T\|_\infty = \inf\{C : C \text{ is an upper bound for } T\}.$$

Clearly, in this case we have:

$$\hat{\rho}(T) \leq \|T\|_\infty, \qquad \forall \rho \in S(\mathcal{A}).$$

Let T be an invertible element in $End^*_{\mathcal{A}}(E)$ such that both are uniformly bounded. Then by [2, Proposition 3.2], for each $x \in E$ we have the inequality

$$\|T^{-1}\|^{-2}_\infty \langle x, x\rangle \leq \langle Tx, Tx\rangle \leq \|T\|^2_\infty \langle x, x\rangle. \tag{2.3}$$

The following proposition will be used in the next section.

Proposition 2.4 ([6]). *Let T be an uniformly bounded below operator in $Hom_{\mathcal{A}}(E, F)$. then T is closed(range) and injective.*

3. $*$-G-frames in Hilbert pro-C^*-modules

Throughout this section $\mathcal{A}$ is a pro-C*-algebra, U and V are two Hilbert $\mathcal{A}$-modules. also $\{V_j\}_{j\in J}$ is a countable sequence of closed submodules of V.

Definition 3.1. *A sequence $\Lambda = \{\Lambda_j \in Hom^*_{\mathcal{A}}(U, V_j)\}_{j\in J}$ is called a $*$- g-frame for U with respect to $\{V_j\}_{j\in J}$ if*

$$C\langle f, f\rangle C^* \leq \textstyle\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \leq D\langle f, f\rangle D^*$$

for all $f \in U$ and strictly nonzero elements $C, D \in \mathcal{A}$.

The number C and D are called $*$-g-frame bounds for Λ. The $*$-g-frame is called tight if $C = D$ and a Parseval if $C = D = 1$. If in the above we only have the upper bound, then Λ is called a $*$-g-Bessel sequence. Also if for each $j \in J, V_j = V$, we call Λ a $*$-g-frame for U with respect to V.

We mentioned that the set of all g-frames in Hilbert pro-C^*-modules are a subset of the family of $*$-g-frames. To illustrate this, let $\Lambda = \{\Lambda_j\}_{j\in J}$ be a g-frame for U with respect to $\{V_j\}_{j\in J}$. Note that for $f \in U$,

$$(\sqrt{C})1_{\mathcal{A}}\langle f, f\rangle(\sqrt{C})1_{\mathcal{A}} \leq \textstyle\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle(\sqrt{D})1_{\mathcal{A}}\langle f, f\rangle(\sqrt{D})1_{\mathcal{A}}$$

Therefore, every g-frame for U with real bounds C amd D is a $*$-g-frame for U with $\mathcal{A}$-valued $*$-g-frame bounds $(\sqrt{C})1_{\mathcal{A}}$ and $(\sqrt{D})1_{\mathcal{A}}$.

Example 3.1. *Let $\ell^2(\mathcal{A})$ be the set of all sequences $(a_n)_{n\in N}$ of elements of a pro-C^*-algebra $\mathcal{A}$ such that the series $\sum_{i\in N} a_i a_i^*$ is convergent in $\mathcal{A}$. Then, by [2, Example 3.2], $\ell^2(\mathcal{A})$ is a Hilbert module over $\mathcal{A}$ with respect to pointwise operations and inner product defined by:*

$$\langle (a_i)_{i\in N}, (b_i)_{i\in N}\rangle = \sum_{i\in N} a_i b_i^*.$$

Let $a = (a_i)_{i\in N}$ and $b = (b_i)_{i\in N}$ in $\ell^2(\mathcal{A})$. We define $ab = \{a_i b_i\}_{i\in N}$ and $\overline{\rho}(a) = \sqrt{\rho(\langle a, a\rangle)}$ and $a^* := \{\overline{a_i}\}_{i\in N}$ and $\langle a, b\rangle = ab^* = \sum_{i\in N} a_i b_i^*$.

Now, let $j \in J := N$ and define $f_j \in \ell^2(\mathcal{A})$ by $f_j = \{f_i^j\}_{i\in N}$ such that

$$f_i^j = \begin{cases} \frac{1}{i}1_{\mathcal{A}} & i = j; \\ 0 & i \neq j, \qquad \forall j \in N. \end{cases}$$

Set $\Lambda_j : \ell^2(\mathcal{A}) \to \mathcal{A}$ by $\Lambda_{f_j}(U) = \langle U, f_j\rangle$ for any $U \in \ell^2(\mathcal{A})$. We see that

$$\textstyle\sum_{j\in J}\langle \Lambda_{f_j}(U), \Lambda_{f_j}(U)\rangle \leq \langle U, U\rangle.$$

Thus$\{\Lambda_j\}_{j\in J}$ is a $*$-g-Bessel sequence .

Definition 3.2. *Let* $\Lambda = \{\Lambda_j \in End_{\mathcal{A}}^*(U, V_j)\}_{j\in J}$ *be a* $*$*-g-frame for* U *with respect to* $\{V_j\}_{j\in J}$ *with bounds* C *and* D*. We define the corresponding* $*$*-g-frame transform as follows:*

$$T_\Lambda : U \to \bigoplus_{j\in J} V_j \ , \qquad T_\Lambda f = \{\Lambda_j f : \ j \in J\}, \qquad \textit{for all } f \in U.$$

Since Λ *is a* $*$*-g-frame, hence for each* $f \in U$ *we have:*

$$C \ \langle f, f\rangle \ C^* \leq \sum_{j\in J} \langle \Lambda_j f, \Lambda_j f\rangle \leq D \ \langle f, f\rangle \ D^*,$$

So T_Λ *is well-defined. Also for any* $\rho \in S(\mathcal{A})$ *and* $f \in U$ *the following inequality is obtained:*

$$\rho(C)^2 \ \overline{\rho}_U(f) \leq \overline{\rho}_{\bigoplus_j V_j}(T_\Lambda f) \leq \rho(D)^2 \ \overline{\rho}_U(f).$$

From the above, it follows that the $*$*-g-frame transform is an uniformly bounded below operator in* $End_A(U, \bigoplus_{j\in J} V_j)$*. Thus by Proposition 2.4,* T_Λ *is closed and injective.*

Now, we define the synthesis operator for $*$*-g-frame* Λ *as follows:*

$$T_\Lambda^* : \bigoplus_{j\in j} V_j \to U, \qquad T_\Lambda^*(\{y_j\}_j) = \sum_{j\in J} \Lambda_j^*(y_j), \tag{3.1}$$

where Λ_j^* *is the adjoint operator of* Λ_j*.*

Proposition 3.1. *The synthesis operator defined by (3.1) is well-defined, uniformly bounded and the adjoint of the transform operator.*

Proof. Since $\Lambda = \{\Lambda_j : j \in J\}$ is a $*$-g-frame for U with respect to $\{V_j\}_{j\in J}$, there exist $C, D \in \mathcal{A}$ such that for any $f \in U$,

$$C \ \langle f, f\rangle \ C^* \leq \sum_{j\in J} \langle \Lambda_j f, \Lambda_j f\rangle \leq D \ \langle f, f\rangle \ D^*.$$

Let I be an arbitrary finite subset of J. Using the Cauchy-Bunyakovskii inequality and [24, Lemma 2.2], for any $\rho \in S(\mathcal{A})$ and $(y_j)_j \in \bigoplus_{j\in J} V_j$ we have:

$$\begin{aligned}
\overline{\rho}(\sum_{j\in I} \Lambda_j^*(y_j)) &= \sup\{\rho\langle \sum_{j\in I} \Lambda_j^*(y_j), f\rangle : \ f \in U \ , \ \overline{\rho}(f) \leq 1\} \\
&= \sup\{\rho(\sum_{j\in I} \langle y_j, \Lambda_j f\rangle) : \ f \in U \ , \ \overline{\rho}(f) \leq 1\} \\
&\leq \sup_{\overline{\rho}(f)\leq 1} \left(\rho(\sum_{j\in I} \langle y_j, y_j\rangle)\right)^{1/2} \left(\rho(\sum_{j\in I} \langle \Lambda_j f, \Lambda_j f\rangle)\right)^{1/2} \\
&\leq \sup_{\overline{\rho}(f)\leq 1} \ \rho(DD^*)^{1/2} \overline{\rho}(f) (\rho \sum_{j\in I} \langle y_j, y_j\rangle)^{1/2} \\
&\leq \left(\rho(D) \ (\rho \sum_{j\in I} \langle y_j, y_j\rangle)^{1/2}\right).
\end{aligned}$$

Now, since the series $\sum_{j\in J}\langle y_j, y_j\rangle$ converges in $\mathcal{A}$, the above inequality shows that $\sum_{j\in J}\Lambda_j^*(y_j)$ is convergent. Hence T_Λ^* is well-defined. On the other hand, for any $f\in U$ and $(y_j)_j\in\bigoplus_{j\in J}V_j$, we have:

$$\begin{aligned}\langle T_\Lambda(f),(y_j)_j\rangle &= \langle(\Lambda_j f)_j,(y_j)_j\rangle\\ &= \sum_{j\in J}\langle\Lambda_j f, y_j\rangle\\ &= \sum_{j\in J}\langle f,\Lambda_j^* y_j\rangle\\ &= \langle f,\sum_{j\in J}\Lambda_j^* y_j\rangle\\ &= \langle f, T_\Lambda^*(y_j)_{j\in J}\rangle.\end{aligned}$$

Therefore by Proposition 2.2 it follows that the synthesis operator is the adjoint of the transform operator. Also, for any $\rho\in S(\mathcal{A})$ we have:

$$\overline{\rho}_U(T_\Lambda^*(y))\leq\rho(D)\ \overline{\rho}_{\bigoplus_{j\in J}\ V_j}\ (y),\qquad y=(y_j)_j\in\bigoplus\nolimits_{j\in J}V_j.$$

Hence the synthesis operator is uniformly bounded. □

Let $\Lambda=\{\Lambda_j\ ,\ j\in J\}$ be a $*$-g-frame for U with repect to $\{V_j\}_{j\in J}$. Define the corresponding $*$-g-frame operator S_Λ as follows:

$$S_\Lambda=T_\Lambda^*T_\Lambda:U\to U\qquad S_\Lambda(f)=\sum\nolimits_{j\in J}\Lambda_j^*\Lambda_j f.$$

Since S_Λ is a combination of two bounded operators, it is a bounded operator.

Theorem 3.1. *Let $\Lambda=\{\Lambda_j\}_{j\in J}$ be a $*$-g-frame for U with respect to $\{V_j\}_{j\in J}$ and with bounds C,D. Then S_Λ is an invertible positive operator. Also it is a self-adjoint operator such that:*

$$CI_UC^*\leq S_\Lambda\leq DI_UD^*. \tag{3.2}$$

Here I_U is the identity function on U.

Proof. According to the definition of the transform operator, for any $f\in U$ we can write:

$$\langle T_\Lambda(f),T_\Lambda(f)\rangle=\langle\{\Lambda_j f\}_{j\in J},\{\Lambda_j f\}_{j\in J}\rangle=\sum\nolimits_{j\in J}\langle\Lambda_j f,\Lambda_j f\rangle.$$

Since Λ is a $*$-g-frame for U with bounds C and D, for each $f\in U$ it follows that

$$C\langle f,f\rangle C^*\leq\langle T_\Lambda(f),T_\Lambda(f)\rangle\leq D\langle f,f\rangle D^*.$$

On the other hand,

$$\langle S_\Lambda(f),f\rangle=\langle T_\Lambda^*T_\Lambda(f),f\rangle=\langle T_\Lambda(f),T_\Lambda(f)\rangle=\langle f,T_\Lambda^*T_\Lambda(f)\rangle=\langle f,S_\Lambda(f)\rangle.$$

Consequently, S_Λ is a self-adjoint operator. Also, for any $f\in U$, we obtain

$$C\langle f,f\rangle C^* \leq \langle S_\Lambda(f),f\rangle \leq D\langle f,f\rangle D^*.$$

It follows that $*$-g-frame operator is positive and (3.2) also holds. Moreover, since S_Λ is one-to-one it follows that S_Λ is invertible. □

According to (3.2) and Proposition 2.1 we have the following Lemma

Lemma 3.1.

$$D^{-1}I_U(D^{-1})^* \leq S_\Lambda^{-1} \leq C^{-1}I_U(C^{-1})^*.$$

Hence the $*$-g-frame operator and its inverse belong to $End_{\mathcal{A}}^*(U)$.

Theorem 3.2. *Let $\{\Lambda_j \in End_{\mathcal{A}}^*(U,V_j)\}_{j\in J}$ and $\sum_{j\in J}\langle\Lambda_j f,\Lambda_j f\rangle$ converge in the semi-norm for $f \in U$. Then $\Lambda = \{\Lambda_j\}_{j\in J}$ is a $*$-g-frame for U with respect to $\{V_j\}_{j\in J}$ if and only if there are two strictly nonzero elements $C, D \in \mathcal{A}$ such that for every $f \in U$,*

$$\begin{aligned}\rho(C^{-1})^{-1}\ \rho(\langle f,f\rangle)\rho(C^{*-1})^{-1} &\leq \rho(\sum_{j\in J}\langle\Lambda_j f,\Lambda_j f\rangle)\\ &\leq \rho(D)\ \rho(\langle f,f\rangle)\rho(D^*).\end{aligned} \tag{3.3}$$

Proof. If $\{\Lambda_j \in End_{\mathcal{A}}^*(U,V_j)\}_{j\in J}$ is a $*$-g-frame for U with respect to $\{V_j\}_{j\in J}$, then

$$(\langle f,f\rangle) \leq C^{-1}(\sum_{j\in J}\langle\Lambda_j f,\Lambda_j f\rangle)(C^*)^{-1})$$

and

$$(\sum_{j\in J}\langle\Lambda_j f,\Lambda_j f\rangle) \leq D\langle f,f\rangle D^*.$$

Therefore, by Proposition 2.1,

$$\begin{aligned}\rho(C^{-1})^{-1}\ \rho(\langle f,f\rangle)\rho(C^{*-1})^{-1} &\leq \rho(\sum_{j\in J}\langle\Lambda_j f,\Lambda_j f\rangle)\\ &\leq \rho(D)\ \rho(\langle f,f\rangle)\rho(D^*).\end{aligned} \tag{3.4}$$

For the converse, let (3.3) hold. Then we define a linear operator as follows:

$$M : U \to \bigoplus_{j\in J} V_j, \qquad M(f) = \{\Lambda_j f\}_{j\in J}, \qquad \forall f \in U,$$

$$\langle Mf, Mf\rangle = \sum_{j\in J}\langle\Lambda_j f,\Lambda_j f\rangle, \qquad \forall f \in U.$$

Hence, by (3.3), we have

$$\overline{\rho}_U(M(f)) \leq\ \rho(D)^{\frac{1}{2}}\ \overline{\rho}_U(f)\ \rho(D^*)^{\frac{1}{2}}.$$

This shows that M is uniformly bounded. We write $M^*M = K$. Then $\langle M(f), M(f)\rangle = \langle M^*M(f), f\rangle = \langle K(f), f\rangle$. Therefore, K is positive. As, $K^* = (M^*M), K$ is self-adjoint. On the other hand,

$$\langle K^{\frac{1}{2}} f, K^{\frac{1}{2}} f\rangle = \langle Kf, f\rangle = \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle.$$

Now, according to Proposition 2.4 and (3.3), $K^{\frac{1}{2}}$ is invertible and uniformly bounded; therefore, by [2, Proposition 3.2], we have:

$$\|K^{-\frac{1}{2}}\|_\infty^{-1}\langle f, f\rangle\|K^{-\frac{1}{2}}\|_\infty^{-1*} \leq \langle K^{\frac{1}{2}}(f), K^{\frac{1}{2}}(f)\rangle \leq \|K^{\frac{1}{2}}\|_\infty\langle f, f\rangle\|K^{\frac{1}{2}}\|_\infty$$

Hence $\{\Lambda_j\}_{j\in J}$ is a $*$-g-frame. □

4. Controlled $*$-G-frames in Hilbert pro-C^*-modules

In this section, we define the concept of multipliers for $*$-g-Bessel sequences and we show that controlled $*$-g-frames are equivalent to $*$-g-frames.

Let $\mathcal{A}$ be a pro-C^*-algebra, U and V be two Hilbert $\mathcal{A}$-modules. also, let $\{V_j\}_{j\in J}$ be a countable sequence of closed submodules of V, $L(U, V)$ and $L(U)$ the collection of all bounded linear operators from U into V and U respectively. $gl(U)$ the set of all bounded operators with a bounded inverse and $gl^+(U)$ be the set of positive operators in $gl(U)$.

Proposition 4.1. *Let $\Lambda = \{\Lambda_j \in L(U, V_j) : j \in J\}$ and $\theta = \{\theta_j \in L(U, V_j) : j \in J\}$ be $*$-g-Bessel sequences with bounds B_Λ and B_θ. If for $m = \{m_j\}_j \subseteq \ell^\infty(R)$, the operator*

$$M = M_{m,\Lambda,\theta} : U \to U$$

$$M(f) = \sum_j m_j\Lambda_j^*\theta_j f, \tag{4.1}$$

is well-defined, then M is called the $*$-g-multiplier of Λ, θ and m.

Proof. Let I be an arbitrary finite subset of J. Using the Cauchy-Bunyakovskii inequality and [24, Lemma 2.2], for any $\rho \in S(A)$ and $f \in U$ we have:

$$\begin{aligned}
\overline{\rho}(\sum_{j\in I} m_j\Lambda_j^*\theta_j f) &= \sup\{\rho\langle\sum_{j\in I} m_j\Lambda_j^*\theta_j f, g\rangle :\ g \in U\ ,\ \overline{\rho}(g) \leq 1\} \\
&= \sup\{\rho(\sum_{j\in I}\langle m_j\theta_j f, \Lambda_j g\rangle) :\ g \in U\ ,\ \overline{\rho}(g) \leq 1\} \\
&\leq \sup_{\overline{\rho}(g)\leq 1}\left(\rho(\sum_{j\in I}\langle m_j\theta_j f, m_j\theta_j f\rangle)\right)^{1/2}\left(\rho(\sum_{j\in I}\langle \Lambda_j g, \Lambda_j g\rangle)\right)^{1/2}.
\end{aligned}$$

Since

$$\begin{aligned}\sum_j \langle m_j\theta_j f, m_j\theta_j f\rangle &= \sum_j m_j\langle \theta_j f, \theta_j f\rangle m_j^* \\ &= \sum_j (\rho(m_j))^2\langle \theta_j f, \theta_j f\rangle \\ &\leq \|m\|_\infty^2 B_\theta\langle f, f\rangle B_\theta^*,\end{aligned}$$

so by Proposition 2.1 we have:

$$\rho(\textstyle\sum_j\langle m_j\theta_j f, m_j\theta_j f\rangle) \leq \|m\|_\infty^2(\overline{\rho}(f))^2\rho(B_\theta)^2.$$

Hence we have:

$$\overline{\rho}(\sum_{j\in I} m_j\Lambda_j^*\theta_j f) \leq \|m\|_\infty\ \overline{\rho}(f)\ \rho(B_\theta)\ \rho(B_\Lambda)$$

□

Definition 4.1. *Let $C, C' \in gl^+(U)$. The family $\Lambda = \{\Lambda_j \in L(U, V_j) : \ j \in J\}$ is called a (C, C')-controlled $*$-g-frame for U with respect to $\{V_j\}_{j\in J}$, if Λ is a $*$-g-Bessel sequence and*

$$A\langle f, f\rangle A^* \leq \sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C' f\rangle \leq B\langle f, f\rangle B^*, \tag{4.2}$$

for all $f \in U$ and strictly nonzero elements $A, B \in \mathcal{A}$.

A, B are called controlled $*$-g-frame bounds. If $C' = I$, we call $\Lambda = \{\Lambda_j\}_j$ a C-controlled $*$-g-frame for U with bounds A, B. If only the second part of the above inequality holds, it is called a (C, C')-controlled $*$-g-Bessel sequence with bound B.

Lemma 4.1 ([2]). *Let X be a Hilbert module over C^*-algebra $\mathcal{B}$, $S \geq 0$, i.e. this element is positive in C^*-algebra $L(U)$. Then for each $x \in X$,*

$$\langle Sx, x\rangle \leq \|S\|\langle x, x\rangle.$$

Proposition 4.2. *Let $C \in gl^+(H)$. The family*

$$\Lambda = \{\Lambda_j \in L(U, V_j) : \ j \in J\}$$

is a $*$-g-frame if and only if Λ is a C^2- controlled $*$-g-frame.

Proof. Let Λ be a C^2- controlled $*$-g-frame with bounds A, B. Then

$$A\langle f, f\rangle A^* \leq \sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j Cf\rangle \leq B\langle f, f\rangle B^*, \qquad \text{for } f \in U.$$

$$A\langle f, f\rangle A^* = A\langle CC^{-1}f, CC^{-1}f\rangle A^* \leq A\|C\|^2\langle C^{-1}f, C^{-1}f\rangle A^* \leq \|C\|^2\sum_{j\in J}\langle \Lambda_j CC^{-1}f, CC^{-1}f\rangle.$$

Hence

$$A\|C\|^{-1}\langle f,f\rangle A^*\|C\|^{-1} \leq \sum_{j\in J}\langle \Lambda_j f,\Lambda_j f\rangle.$$

On the other hand for every $f \in U$

$$\begin{aligned}\sum_{j\in J}\langle \Lambda_j f,\Lambda_j f\rangle &= \sum_{j\in J}\langle \Lambda_j CC^{-1}f, CC^{-1}f\rangle\\ &\leq B\langle C^{-1}f, C^{-1}f\rangle B^*\\ &\leq B\|C^{-1}\|^2\langle f,f\rangle B^*.\end{aligned}$$

These inequalities yield that Λ is a $*$-g-frame with bounds $A\|C^{-1}\|, B\|C^{-1}\|$. Conversely assume that Λ is a $*$-g-frame with bounds A', B'. Then for all $f \in U$,

$$A'\langle f,f\rangle {A'}^* \leq \sum_{j\in J}\langle \Lambda_j f,\Lambda_j f\rangle \leq B'\langle f,f\rangle {B'}^*.$$

So for $f \in U$,

$$\sum_{j\in J}\langle \Lambda_j Cf,\Lambda_j Cf\rangle \leq B'\langle Cf,Cf\rangle {B'}^* \leq B'\|C\|^2{B'}^*.$$

For the lower bound, since Λ is $*$-g-frame for any $f \in U$,

$$\begin{aligned}A'\langle f,f\rangle {A'}^* &= A'\langle C^{-1}Cf, C^{-1}Cf\rangle {A'}^*\\ &\leq A'\|C^{-1}\|^2\langle Cf,Cf\rangle {A'}^*\\ &\leq \|C^{-1}\|^2\sum_{j\in J}\langle \Lambda_j Cf,\Lambda_j Cf\rangle.\end{aligned}$$

Therefor Λ is a C^2-controlled $*$-g-frame with bounds $A'\|C^{-1}\|, B'\|C^{-1}\|$ □

5. Multipliers of controlled $*$-G-frames in Hilbert pro-C^*-modules

In this section, we define the multiplier of a controlled $*$-g-frame for C-controlled $*$-g-frames in Hilbert pro-C^*-modules. The definition of general case (C, C')-controlled $*$-g-frames is similar.

Lemma 5.1. *Let $C, C' \in gl^+(U)$ and $\Lambda = \{\Lambda_j \in L(U, V_j) : j \in J\}, \theta = \{\theta_j \in L(U, V_j) : j \in J\}$ be C'^2 and C^2-controlled $*$-g-Bessel sequences for U, respectively. Let $m = \ell^\infty$. Then*

$$M_{m,C,\theta,\Lambda,C'} : U \to U,$$

defined by

$$M_{m,C,\theta,\Lambda,C'}f := \sum_{j\in J} m_j C\theta_j^*\Lambda_j C'f,$$

is a well-defined bounded operator.

Proof. Let $\Lambda = \{\Lambda_j \in L(U,V_j) : \ j \in J\}, \theta = \{\theta_j \in L(U,V_j) : \ j \in J\}$ be C'^2 and C^2-controlled $*$-g-Bessel sequences for U, with bounds B, B', respectively. For any $f, g \in U$ and finite subset $I \subseteq J$,

$$\begin{aligned}\overline{\rho}(\sum_{j\in I} m_j C\theta_j^*\Lambda_j C'f) &\leq \sup\{\rho\langle\sum_{j\in I} m_j C\theta_j^*\Lambda_j C'f, g\rangle : \ g \in U \ , \ \overline{\rho}(g) \leq 1\} \\ &= \sup\{\rho(\sum_{j\in I}\langle m_j\Lambda_j C'f, \theta_j C^*g\rangle) : \ g \in U \ , \ \overline{\rho}(g) \leq 1\} \\ &\leq \sup_{\overline{\rho}(g)\leq 1} \left(\rho(\sum_{j\in I}\langle m_j\Lambda_j C'f, m_j\Lambda_j C'f\rangle)\right)^{1/2} \left(\rho(\sum_{j\in I}\langle \theta_j C^*g, \theta_j C^*g\rangle)\right)^{1/2},\end{aligned}$$

since

$$\begin{aligned}\sum_j \langle m_j\Lambda_j C'f, m_j\Lambda_j C'f\rangle &= \sum_j m_j\langle \Lambda_j C'f, \Lambda_j C'f\rangle m_j^* \\ &= \sum_j (\rho(m_j))^2\langle \Lambda_j C'f, \Lambda_j C'f\rangle \\ &\leq \|m\|_\infty^2 B\langle f, f\rangle B^*.\end{aligned}$$

So by Proposition 2.1 we have:

$$\begin{aligned}\rho(\sum_j \langle m_j\Lambda_j C'f, m_j\Lambda_j C'f\rangle) &= \rho(\sum_j m_j\langle \Lambda_j C'f, \Lambda_j C'f\rangle m_j^*) \\ &\leq \|m\|_\infty^2 (\overline{\rho}(f))^2\rho(B)^2.\end{aligned}$$

Hence

$$\overline{\rho}(\sum_{j\in I} m_j C\theta_j^*\Lambda_j C'f) \leq \|m\|_\infty \ \overline{\rho}(f) \ \rho(B) \ \rho(B)'.$$

This shows that $M_{m,C,\theta,\Lambda,C'}$ is well-defined and

$$\overline{\rho}(M_{m,C,\theta,\Lambda,C'}) \leq \|m\|_\infty \ \rho(B) \ \rho(B)'.$$ □

The above Lemma provides a motivation for the following definition.

Definition 5.1. *Let $C, C' \in gl^+(U)$ and $\Lambda = \{\Lambda_j \in L(U,V_j) : \ j \in J\}, \theta = \{\theta_j \in L(U,V_j) : \ j \in J\}$ be C'^2 and C^2-controlled $*$-g-Bessel sequences for U, respectively. Let $m = \ell^\infty$. The operator*

$$M_{m,C,\theta,\Lambda,C'} : U \to U,$$

defined by

$$M_{m,C,\theta,\Lambda,C'}f := \sum_{j\in J} m_j C\theta_j^*\Lambda_j C'f,$$

is called (C, C')-controlled multiplier operator with symbol m.

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