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$\mathfrak{c}$-ALGEBRABILITY OF PATHOLOGICAL SETS OF PRODUCT INTEGRABLE FUNCTIONS

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Abstract. In this paper we investigate linear algebraic structures in the set of product integrable matrix-valued functions and find $\mathfrak{c}$-generated algebras in $L([a,b],\mathbb{R}^{n\times n})\backslash L^*([a,b],\mathbb{R}^{n\times n})$ and $D([a,b],\mathbb{R}^{n\times n})\backslash L([a,b],\mathbb{R}^{n\times n})$.

1. Introduction

If X is a vector space, a subset M of X is called *lineable* if $M\cup\{0\}$ contains an infinite dimensional vector space. If X is a linear algebra and $M\subseteq X$, one calls M a κ-algebrable set if $M\cup\{0\}$ contains a κ-generated algebra, that is, an algebra which has a minimal system of generators of cardinality κ. These notions were coined by V.I. Guariy [1, 9] and then became a criterion for measuring how much large linear algebraic structures could be found in a set of functions with weird properties (see [2, 6–8]).

Another criterion is the concept of *strong algebrability* introduced by Glab and Bartoszewicz in [5]. Let κ be a cardinal number and X be a linear commutative algebra. A subset M of X is called *strongly κ-algebrable* if $M\cup\{0\}$ contains a κ-generated algebra isomorphic to a free algebra.

In this paper we seek a linear algebraic structures in the spaces of product integrable function. The notion of product integral has been introduced by Vito Volterra about the end of the 19th century, who studied

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linear systems of differential equations

$$W'(t) = A(t)W(t), t \in [a,b]$$
$$W(a) = I,$$

where I is the identity matrix, $A : [a,b] \to \mathbb{R}^{n\times n}$ is a given continuous function and $W : [a,b] \to \mathbb{R}^{n\times n}$ is the unknown function (see [17]). Later, Ludwig Schlesinger introduced the definition of the Riemann product integral as follows: Given a tagged partition of an interval $[a,b]$, which is a collection of point-interval pairs $D = (\xi_i, [t_{i-1}, t_i])_{i=1}^{m}$, where $a = t_0 \le t_1 \le ... \le t_m = b$ and $\xi_i \in [t_{i-1}, t_i]$ for every $i \in \{1, 2, ..., m\}$. We refer to $t_0, t_1, ..., t_m$ as the division points of D, while $\xi_1, \xi_2, ..., \xi_m$ are the tags of D.

Remark 1.1. *If we replace $\xi_i \in [t_{i-1}, t_i]$ by $\xi_i \in [a,b]$, then the collection D is called a free tagged partition. Given a function $\delta : [a,b] \to \mathbb{R}^{+}$ (called a gauge on $[a,b]$), a free tagged partition is called δ-fine if*

$$[t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \ , \ i = \{1, 2, ..., m\}.$$

Now consider a matrix function $A : [a,b] \to \mathbb{R}^{n\times n}$ with entries $\{a_{ij}\}_{i,j=1}^{n}$. Put

$$\Delta t_i = t_i - t_{i-1} \ , \ i = 1, 2, ..., m \ , \ \upsilon(D) = \max_{1\le i\le m} \Delta t_i,$$

and define

$$P(A, D) = \prod_{i=1}^{m} (I + A(\xi_i)\Delta t_i)$$
$$= (I + A(\xi_1)\Delta t_1)(I + A(\xi_2)\Delta t_2)....(I + A(\xi_m)\Delta t_m).$$

In case the limit $\lim_{\upsilon(D)\to 0} P(A, D)$ exists, it is called the *Riemann product integral* of the function A on the interval $[a,b]$ and is denoted by the symbol $(I + A(t)dt)\prod_{a}^{b}$.

In this paper $R([a,b], \mathbb{R}^{n\times n})$ denotes the set of all Riemann product integrable functions.

Utilizing step functions Schlesinger generalized this definition and introduced the Lebesgue product integral (see [11, 12, 16]). Let us recall some facts that will be needed:

1. A function $A : [a,b] \to \mathbb{R}^{n\times n}$ is called a *step function* if there exists numbers $a = t_0 < t_1 < ... < t_m = b$ such that A is constant function on every interval (t_{k-1}, t_k), $k = 1, 2, ..., m$.
2. For $A \in \mathbb{R}^{n\times n}$ we will use the operator norm $\|A\| = \sup\{\|Ax\| : \|x\| \le 1\}$, where $\|Ax\|$ and $\|x\|$ denote the Euclidean norms of vectors Ax, $x \in \mathbb{R}^n$.
3. A sequence of functions $\{A_k : [a,b] \to \mathbb{R}^{n\times n}\}_{k\in\mathbb{N}}$ is called *uniformly bounded* if there exists a number $M \in \mathbb{R}$ such that $\|A_k(x)\| \le M$ for all $k \in \mathbb{N}$ and all $x \in [a,b]$.

Theorem 1.1. *[16, Lemma 3.5.4 and Theorem 3.5.5] Let $A_k : [a,b] \to \mathbb{R}^{n\times n}$, $k \in \mathbb{N}$, be a uniformly bounded sequence of step functions such that $\lim_{k\to\infty} A_k(x) = A(x)$ a.e. on $[a,b]$. Then*

$$\lim_{k\to\infty} \|A_k - A\|_1 = \lim_{k\to\infty} \int_a^b \|A_k(x) - A(x)\| dx = 0,$$

and the limit $\lim\limits_{k\to\infty} (I + A_k(x)dx) \prod\limits_a^b$ exists and is independent of the choice of the sequence $\{A_k\}$.

Definition 1.2. *[16, Definiton 3.5.6] Consider the function $A : [a,b] \to \mathbb{R}^{n\times n}$. Assume there exists a uniformly bounded sequence of step functions $A_k : [a,b] \to \mathbb{R}^{n\times n}$ such that $\lim_{k\to\infty} A_k(x) = A(x)$ a.e. on $[a,b]$, then the function A is called Lebesgue product integrable and we define*

$$(I + A(x)dx) \prod_a^b = \lim_{k\to\infty} (I + A_k(x)dx) \prod_a^b .$$

The symbole $L^*([a,b], \mathbb{R}^{n\times n})$ denotes the set of all Lebesgue product integrable functions. It is easy to show that

$$L^*([a,b], \mathbb{R}^{n\times n}) = \{A : [a,b] \to \mathbb{R}^{n\times n} \ : \ A \text{ is measurable and bounded}\}.$$

Let us recall that a function $A : [a,b] \to \mathbb{R}^{n\times n}$ is called Bochner intagrable if there is a sequence of simple functions $A_k : [a,b] \to \mathbb{R}^{n\times n}$, $k \in \mathbb{N}$ such that $\lim\limits_{k\to\infty} A_k(t) = A(t)$ a.e. on $[a,b]$ and

$$\lim_{k\to\infty} \|A_k - A\|_1 = 0.$$

Thus by Theorem 1.1 and Definition 1.2, each $A \in L^*([a,b], \mathbb{R}^{n\times n})$ is Bochner intagrable.

After that Schlesinger extended the definition of $L^*([a,b], \mathbb{R}^{n\times n})$ to all matrix functions with Lebesgue integrable (not necessarily bounded) entries and used the next symbole:

$$L([a,b], \mathbb{R}^{n\times n}) = \{A : [a,b] \to \mathbb{R}^{n\times n} \ : (L) \int_a^b \|A(x)\| dx < \infty\}.$$

The symbole (L) estands for the Lebesgue integral. Taking account of Theorem 1.1 it is natural to state the following definition.

Definition 1.3. *[16, Definiton 3.8.1] A function $A : [a,b] \to \mathbb{R}^{n\times n}$ is called product integrable if there exists a sequence of step functions $\{A_k\}$ such that*

$$\lim_{k\to\infty} \|A_k - A\|_1 = 0.$$

We define

$$(I + A(x)dx) \prod_a^b = \lim_{k\to\infty} (I + A_k(x)dx) \prod_a^b$$

Remark 1.2. *Since step functions belong to the complete space $L([a,b],\mathbb{R}^{n\times n})$, every product integrable function also belongs to $L([a,b],\mathbb{R}^{n\times n})$. Moreover, step functions form a dense subset in this space, and hence $(I + A(x)dx)\prod_a^b$ exists if and only if $A \in L([a,b],\mathbb{R}^{n\times n})$, i.e., the Lebesgue integral $\int_a^b \|A(t)\|dt$ is finite.*

Concerning the above definitions of product integral we have the following chain of strict inclusions:

$$R([a,b],\mathbb{R}^{n\times n}) \subset L^*([a,b],\mathbb{R}^{n\times n}) \subset L([a,b],\mathbb{R}^{n\times n}).$$

2. The exponential function and the product integral

Recall that for every $A \in \mathbb{R}^{n\times n}$ the matrix exponential is defined by $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

Theorem 2.1. *[16, Theorem 3.2.2] Consider a Riemann integrable function $A : [a,b] \to \mathbb{R}^{n\times n}$. Then*

$$\lim_{\upsilon(D)\to 0} \prod_{k=1}^{m} e^{A(\xi_k)\Delta t_k} = \lim_{\upsilon(D)\to 0} \prod_{k=1}^{m} (I + A(\xi_k)\Delta t_k) = (I + A(t)dt)\prod_a^b,$$

where partitions are as in introduction.

Remark 2.1. *If $A \in L^*([a,b],\mathbb{R}^{n\times n})$ and $\{A_k\}_{k=1}^{\infty}$ is a uniformly bounded sequence of step functions in $L^*([a,b],\mathbb{R}^{n\times n})$ such that $A_k \to A$ a.e. on $[a,b]$, then by [16, Theorem 3.6.3] we have $(I+A(x)dx)\prod_a^b = \lim_{k\to\infty}(I+A_k(x)dx)\prod_a^b$. Now every function A_k is associated with a partition*

$$D_k : a = t_0^k < t_1^k < ... < t_{m(k)}^k = b$$

such that

$$A_k(x) = A_j^k \ , \ x \in (t_{j-1}^k, t_j^k),$$

and

$$\lim_{k\to\infty} \upsilon(D_k) = 0.$$

So by the definition of Lebesgue product integrable functions,

$$(I + A(x)dx)\prod_a^b = \lim_{k\to\infty}(I + A_k(x)dx)\prod_a^b = \lim_{k\to\infty}\prod_{j=1}^{m(k)} \exp(A_j^k \Delta t_j^k).$$

Moreover Schlesinger in [16, p. 485-486] proved the product integral might be also calculated as

$$(I + A(x)dx)\prod_a^b = \lim_{k\to\infty}\prod_{j=1}^{m(k)} (I + (A_j^k \Delta t_j^k).$$

We remark that each $A \in L^*([a,b],\mathbb{R}^{n\times n})$ is Bochner integrable and hence the product integrals $\prod_a^b \exp(A(t)dt$ and $\prod_a^b (I + A(t)dt)$ exist and equal to each other; see [13, Theorem 14, Theorem 16]. Thus according to the previous discussion, Theorem 2.1 holds for all $A \in L^*([a,b],\mathbb{R}^{n\times n})$.

Now cosider a function $A \in L([a,b],\mathbb{R}^{n\times n})$. By the definition 1.3 there exists a sequence of step functions $\{A_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \|A_k - A\|_1 = 0 \text{ and } (I + A(t)dt)\prod_a^b = \lim_{k\to\infty} (I + A_k(t)dt)\prod_a^b .$$

Thus Theorem 2.1 does also hold for $A \in L([a,b],\mathbb{R}^{n\times n})$. So we can state the next theorem.

Theorem 2.2. *Let $A : [a,b] \to \mathbb{R}^{n\times n}$ be a matrix function and $A \in L([a,b],\mathbb{R}^{n\times n})$, then $\exp \circ A$ is product integrable.*

3. Lebesgue product integrable functions

The next definition and theorem provide important tools for proving the existence of infinitely generated algebras in the family of real or complex functions.

Definition 3.1 ([3]). *We say that a function $f : \mathbb{R} \to \mathbb{R}$ is an exponential-like function (of rank m) whenever f is given by $f(x) = \sum_{i=1}^{m} a_i e^{b_i x}$ for some distinct nonzero real numbers $b_1, b_2, ..., b_m$ and some nonzero real numbers $a_1, a_2, ..., a_m$.*

Theorem 3.2 ([3,4]). *Let $\mathcal{F} \subset \mathbb{R}^{[0,1]}$ and assume that there exists a function $F \in \mathcal{F}$ such that $foF \in \mathcal{F}\backslash\{0\}$ for every exponential-like function $f : \mathbb{R} \to \mathbb{R}$. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable. More exactly, if $H \subset \mathbb{R}$ is a set of cardinality $\mathfrak{c}$ and linearly independent over the rationals $\mathbb{Q}$, then $\exp \circ(rF)$, $r \in H$, are free generators of an algebra contained in $\mathcal{F} \cup \{0\}$.*

Note that in all proofs we apply Theorem 3.2

Theorem 3.3. *The set of Riemann real valued integrable functions is strongly $\mathfrak{c}$-algebrable.*

Proof. Volterra in [17] showed that the Riemann integrable functions are product integrable, thus by Theorem 2.1 and Theorem 3.2 the proof follows. □

Theorem 3.4. *The set of real valued Lebesgue integrable functions is strongly $\mathfrak{c}$-algebrable.*

Proof. Schlesinger in [12, 16] showed the product integrability of Lebesgue integrable functions. So by Theorem 2.2 and Theorem 3.2, the proof is complete. □

Theorem 3.5. *The set $L([a,b],\mathbb{R}^{n\times n})\backslash L^*([a,b],\mathbb{R}^{n\times n})$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Let $A : [0,1] \to \mathbb{R}^{n\times n}$ be given by $A(x) = (a_{ij}(x))_{i,j=1}^{n}$ such that for each $i, j = 1, 2, .., n$,

$$a_{ij}(x) = \begin{cases} \frac{1}{\sqrt{x}} & x \in (0,1] \\ 0 & x = 0 \end{cases}.$$

So for some $y \in \mathbb{R}^{n\times 1}$ and $\|y\| \leq 1$,

$$A(x)y = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ \vdots \\ a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{pmatrix},$$

$$\|A(x)y\| \geq \sqrt{\frac{n}{x}(y_1 + \cdots + y_n)^2} \geq \frac{1}{x}\ , \ x \in (0, 1].$$

Thus A is not bounded and so A and $exp \circ (A)$ are not in $L^*([0, 1], \mathbb{R}^{n\times n})$. Now let $A_m : [0, 1] \to \mathbb{R}^{n\times n}$ be given by $A_m(x) = (\overset{(m)}{b_{ij}}(x))_{i,j=1}^n$ such that for each $i, j = 1, 2, .., n$,

$$\overset{(m)}{b_{ij}}(x) = \begin{cases} 0 & x \in [0, \frac{1}{m}] \\ \frac{1}{\sqrt{x}} & x \in (\frac{1}{m}, 1] \end{cases}.$$

Given an arbitrary i and j, and note that for $m \geq 2$, $\overset{(m)}{b_{ij}}(x)$ is Lebesgue integrable on $[0, 1]$. Since $\lim_{m\to\infty} \overset{(m)}{b_{ij}}(x) = a_{ij}(x)$ for each $x \in [0, 1]$, so by the Monotone Convergence Theorem $a_{ij}(x)$ is Lebesgue integrable. Thus A and $\exp \circ(A)$ are in $L([0, 1], \mathbb{R}^{n\times n})$ so $f \circ (A)$ is in $L([0, 1], \mathbb{R}^{n\times n})$, for every exponential-like function f, and the proof is complete by Theorem 3.2. □

Theorem 3.6. *The set of* $L([a, b], \mathbb{R}^{n\times n})\backslash R([a, b], \mathbb{R}^{n\times n})$ *is* $\mathfrak{c}$*-algebrable.*

Proof. Since $L([a, b], \mathbb{R}^{n\times n})\backslash L^*([a, b], \mathbb{R}^{n\times n}) \subseteq L([a, b], \mathbb{R}^{n\times n})\backslash R([a, b], \mathbb{R}^{n\times n})$, the preceding theorem implies that $L([a, b], \mathbb{R}^{n\times n})\backslash R([a, b], \mathbb{R}^{n\times n})$ is $\mathfrak{c}$-algebrable. □

4. Product integrability of Denjoy integrable matrix-valued functions

The following definition generalizes the concept of Denjoy product integration.

Definition 4.1. *Consider the function* $A : [a, b] \to \mathbb{R}^{n\times n}$ *and let* $[c, d] \subset [a, b]$. *The oscilation of* A *on the interval* $[c, d]$ *is the number*

$$osc(A, [c, d]) = \sup \{\|A(\xi_1) - A(\xi_2)\| : \xi_1, \xi_2 \in [c, d]\}.$$

The abbreviations AC, BV and ACG stand for "absolutely continuous", "bounded variations" and "generalized absolutely continiuous", respectively.

Definition 4.2. *Let* $A : [a, b] \to \mathbb{R}^{n\times n}$ *and* $E \in [a, b]$.

1. *The strong variation of* F *on* E *is defined by*

$$V_*(F, E) = \sup \left\{ \sum_{i=1}^{n} osc(F, [c_i, d_i]) \right\},$$

where the supremum is taken over all finite collections $\{[c_i, d_i] : 1 \leq i \leq n\}$ *of non-overlapping intervals that have endpoints in* E.

2. *The function F is of bounded variation in the restricted sense on E (briefely A is BV_* on E) if $V_*(F,E)$ is finite.*
3. *The function A is absolutely continuous in the restricted sense on E (briefely A is AC_* on E) if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^{n} osc(A,[c_i,d_i]) < \varepsilon$, whenevere $\{[c_i,d_i] : 1 \le i \le n\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}(d_i - c_i) < \delta$.*
4. *The function A is generalized absolutely continuous in the restricted sense on E (briefely A is ACG_* on E) if $A\big|_E$ is continuous on E and E can be written as a countable union sets on each of which A is AC_*.*

Note that in general, $V(F,E) \le V_*(F,E)$ and hence A is $BV(AC, BVG, ACG)$ on E if it is $BV_*(AC_*, BVG_*, ACG_*)$ on E.

Definition 4.3. *The function $A : [a,b] \to \mathbb{R}^{n\times n}$ is Denjoy integrable on $[a,b]$ if there exists an ACG_* function $\mathcal{A} : [a,b] \to \mathbb{R}^{n\times n}$ such that $\mathcal{A}' = A$ a.e. on $[a,b]$.*

Theorem 4.4. *[15, Theorem 6.2] Let $F : [a,b] \to \mathbb{R}^{n\times n}$ and $E \subseteq [a,b]$.*

(1) *If F is $AC(ACG, AC_*, ACG_*)$ on E, then F is $BV(BVG, BV_*, BVG_*)$ on E.*
(2) *If F is BV_* on E, then F is BV_* on $\overline{E}$.*
(3) *Suppose that E is closed with $a, b \in E$ and let G be the linear extension of F to $[a,b]$. If F is $BV(AC)$ on E, then G is $BV(AC)$ on $[a,b]$.*

Remark 4.1. *Let P be a perfect set. A perfect portion of P is a set of the form $P \cap [c,d]$ where $P \cap (c,d) \neq \varnothing$, $c, d \in P$, and $P \cap [c,d]$ is a perfect set.*

Theorem 4.5. *[15, Theorem 6.10] Suppose that $F : [a,b] \to \mathbb{R}^{n\times n}$ is $ACG(ACG_*)$ on $[a,b]$ and let $E \subset [a,b]$ be a perfect set. Then there is a perfect portion $E \cap [c,d]$ of E such that F is $AC(AC_*)$ on $E \cap [c,d]$.*

(Note that in this case, each subinterval of $[a,b]$ contains an interval on which the function F is $AC(AC_*)$. The endpoints of all the intervals on which F is $AC(AC_*)$ form a dence set in $[a,b]$).

We recall that the next Lemma and proposition are mentioned in [15] as exercises.

Lemma 4.1. *Let $F : [a,b] \to \mathbb{R}^{n\times n}$, and E be a closed set with bounds a and b, and let $[a,b] - E = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Suppose that G is the linear extension of F from E to $[a,b]$ and $c \in E$. Then $\frac{G(x)-G(c)}{x-c}$ is between $\frac{F(a_n)-F(c)}{a_n-c}$ and $\frac{F(b_n)-F(c)}{b_n-c}$ for each $x \in (a_n, b_n)$. In particular, if c is two-sided limit point of E and F is differentiable at c, then G is differentiable at c and $G'(c) = F'(c)$.*

Proof. First we note that $G = F$ on E and G is linear on each of the intervals contiguous to E. For each $x \in [a_n, b_n]$, we have

$$G(x) = \frac{F(b_n) - F(a_n)}{b_n - a_n}(x - a_n) + F(a_n),$$

and hence an easy calculation completes the proof. □

Proposition 4.1. *Suppose that $A : [a, b] \to \mathbb{R}^{n\times n}$ is Denjoy integrable on $[a, b]$. Then $[a, b] = \cup_{n=1}^{\infty} E_n$ where each E_n is closed and A is Lebesgue integrable on each E_n.*

Proof. By the hypothesis, there exists an ACG_* function $\mathcal{A} : [a, b] \to \mathbb{R}^{n\times n}$ such that $\mathcal{A}' = A$ a.e. on $[a, b]$, and we can write $[a, b] = \cup_{n=1}^{\infty} E_n$, where $\mathcal{A}$ is AC_* on each E_n. By Theorem 4.4 we can assume that each E_n is closed. Then by Theorem 4.5 there exists a perfect portion $E_n \cap [c, d]$ of E_n for $n \in \mathbb{N}$, such that $\mathcal{A}$ is AC_* on $E_n \cap [c, d]$. Let $G : [c, d] \to \mathbb{R}^{n\times n}$ be the linear extension of $\mathcal{A}\big|_{E_n\cap[c,d]}$ to $[c, d]$. By part 3 of Theorem 4.4, G is AC on $[c, d]$. So the function G' exists a.e. and is Lebesgue integrable on $[c, d]$. But by Lemma 4.1 $\mathcal{A}' = G' = A$ a.e. on $E_n \cap [c, d]$, so the function A is Lebesgue integrable. □

Theorem 4.6. *Let $A : [a, b] \to \mathbb{R}^{n\times n}$ be Denjoy integrable on $[a, b]$, then it is product integrable.*

Proof. Let $D([a, b], \mathbb{R}^{n\times n})$ be endowed by the norm $\|A\| = (D)\int_a^b \|A(t)\|dt$, where (D) stands for the Denjoy integral. By Proposition 4.1 there exists subsets E_n such that $[a, b] = \cup_{n=1}^{\infty} E_n$ where for each $n \in \mathbb{N}$, E_n is non-overlapping, closed and A is Lebesgue integrable on E_n. Let A_n be the restriction of A to E_n for each $n \in \mathbb{N}$. Then each A_n is Lebesgue integrable and so product integrable and hence for each A_n there exists a sequense of step functions $\{A_{n_k}\}_{k=1}^{\infty}$ such that $A_{n_k} : E_n \to \mathbb{R}^{n\times n}$ and

$$\lim_{k\to\infty} \|A_{n_k} - A_n\|_{E_n} = \lim_{k\to\infty} \int_{E_n} \|A_{n_k}(x) - A_n(x)\|dx = 0$$

For each n, put $a_n = infE_n$ and $b_n = supE_n$, so both a_n, b_n are in E_n. Thus for each E_n there exist $t_0, t_1, ..., t_n$ such that

$$t_0 = a_n \le t_1 \le ... \le t_n = b_n,$$

and A_{n_k} is constant on (t_{k-1}, t_k) for $k = 1, \dots, n$. Now let $\{B_k\}_{k=1}^{\infty}$ be a sequence of step functions on $[a, b]$ such that $[a, b] = \bigcup_{n=1}^{\infty} E_n$ and $B_k = A_{n_k}$ on each E_n. Then by Dominated Convergence Theorem we have the followings:

$$\begin{aligned}\lim_{k\to\infty} \|B_k - A\|_1 &= \lim_{k\to\infty} \int_a^b \|B_k(x) - A(x)\|dx \\ &= \lim_{k\to\infty} \sum_{n=1}^{\infty} \int_{E_n} \|A_{n_k}(x) - A_n(x)\|dx = 0,\end{aligned}$$

i.e., B_k converges to A also in the norm of space $D([a,b],\mathbb{R}^{n\times n})$ and hence by [16, Theorem 3.5.5] $\lim_{k\to\infty} (I + B_k(x))dx \prod_a^b$ exists. So the proof is complete. □

5. $\mathfrak{c}$-algebrability of the set of Denjoy product integrable

In this section, some pathological properties (more precisely algebrability) of sets of product integrable functions contained in $D([a,b],\mathbb{R}^{n\times n})\backslash L([a,b],\mathbb{R}^{n\times n})$ are investigated. First we note that a matrix $A = \{a_{ij}\}_{i,j=1}^n$ is called regular if it has a nonzero determinant.

Definition 5.1. *A function $A : [a,b] \to \mathbb{R}^{n\times n}$ is called Perron product integrable if there is a regular matrix $B \in \mathbb{R}^{n\times n}$ such that for every $\varepsilon > 0$ there is a function $\delta : [a,b] \to (0,\infty)$ such that $\|B - P(A,D)\| < \varepsilon$ for every δ-fine partition D of [a,b].*

Theorem 5.2. *Consider the function $A : [a,b] \to \mathbb{R}^{n\times n}$ in $D([a,b],\mathbb{R}^{n\times n})$. Then*

$$\prod_a^b e^{A(t)dt} = (I + A(t)dt)\prod_a^b .$$

Proof. By [10, Theorem 2.12] and [15, Theorem 11.2], the proof is clear. □

Corollary 5.1. *If $A : [a,b] \to \mathbb{R}^{n\times n}$ is product integrable function, then $exp \circ (A)$ is product integrable function.*

Theorem 5.3. *The set of product integrable functions is strongly $\mathfrak{c}$-algebrable.*

Proof. By Corollary 5.1 and Theorem 3.2 the proof follows. □

Proposition 5.1. *[15, Theorem 7.11] Suppose that $f : [a,b] \to \mathbb{R}$ is Denjoy integrable on each subinterval $[c,d] \subseteq (a,b)$. If $\int_c^d f$ converges to a finite limit as $c \to a^+$ and $d \to b^-$, then f is Denjoy integrable on $[a,b]$ and $\int_a^b f = \lim_{\substack{c\to a^+\\ d\to b^-}} \int_c^d f$.*

Theorem 5.4. *The set of $D([a,b],\mathbb{R}^{n\times n})\backslash L\,([a,b],\mathbb{R}^{n\times n})$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Let $\sum_{n=1}^\infty c_n$ be a nonabsolutely convergent series of real numbers and let $I_n = \left(2^{-n}, 2^{-n+1}\right)$, $n \in \mathbb{N}$. Define the function $A : [0,1] \to \mathbb{R}^{n\times n}$ by $A(x) = (a_{ij}(x))_{i,j=1}^n$, such that for each $i,j = 1,2,...,n$

$$(a_{ij}(x)) = \begin{cases} 2^n c_n & x \in I_n \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\int_0^1 |a_{ij}(x)|dx = \sum_{n=1}^\infty \int_{I_n} |a_{ij}(x)|dx = \sum_{n=1}^\infty \left|2^{-n}c_n 2^n\right| = \sum_{n=1}^\infty |c_n| = \infty.$$

Hence neither a_{ij} nor A is Lebesgue integrable on $[0,1]$. Now we are going to show that A is Denjoy integrable on $[0,1]$. For each $0<\alpha<1$ both of functions a_{ij} and A are bounded on $[\alpha,1]$, so they are Lebesgue integrable on $[\alpha,1]$. Let $B(x)=\int\limits_x^1 a_{ij}$ for each $x\in(0,1]$. The function B is linear on each I_n. It follows that $B(x)$ is between $B(2^{-n})$ and $B(2^n)$ for each $x\in I_n$. Now $B(2^{-n})=\sum\limits_{k=1}^{n}c_k$ and $\lim\limits_{n\to\infty}B\left(2^{-n}\right)=\sum\limits_{k=1}^{\infty}c_k$. Therefore $\lim\limits_{x\to 0^+}B\left(x\right)=\sum\limits_{n=1}^{\infty}c_n$ and according to Proposition 5.1, a_{ij} is Denjoy integrable on $[0,1]$ for each $i,j=1,2,...,n$. Thus for each $a_{ij}(x)$ there exists an ACG_* function f_{ij} such that $f'_{ij}(x)=a_{ij}(x)$ a.e. on $x\in[0,1]$. Now put $F(x)=(f_{ij}(x))_{i,j=1}^n$ for each $x\in[0,1]$. So

$$F'(x)=(f'_{ij}(x))_{i,j=1}^n=(a_{ij}(x))_{i,j=1}^n=A(x) \text{ a.e. on } [0,1].$$

Hence A is Denjoy integrable on $x\in[0,1]$. One can see easily that $exp\circ a_{ij}$ is Denjoy integrable and so is $\exp\circ A$. Thus by Theorem 3.2 the proof is complete. □

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